THE CAPTURE HYPOTHESIS OF T. J. J. SEE.[1]

In the opinion of Mr. See,[2] the planets were not formed from fragments of the solar nebula, nor did the moon originate from a piece of that of the earth. He believes that the planets had a cosmic origin outside of the solar nebula; that they are foreign bodies captured by the sun while passing near it in their journey; and that in the same way the moon was captured by the earth at a certain remote time.

How was this phenomenon accomplished? Mr. See thinks that the sun was formerly surrounded by a vast atmosphere and that the capture took place as the result of a resistance created by this atmosphere.

Let us therefore study the effect of the resistance of the medium on the motion of a planet.[3] If there were no resistance the motion would be Keplerian, the orbit would be an ellipse of any eccentricity whatever. The density of the resisting medium being by hypothesis very small, this orbit would vary slowly. We shall study the variations of this orbit by the method of the variation of constants.

First let us recall some formulas pertaining to the elliptical motion of planets.

Calling the radius vector r and the true anomaly v, the equation of the orbit is

$$r=\frac{p}{1+e\cos v}, \tag{1}$$

e denoting the eccentricity, and

$$p=a(1-e^2) \tag{2}$$

denoting the parameter of the elliptical orbit whose major axis is $2a$. We have also the equation of the areas

$$r^2\frac{dv}{dt}=C,$$

the constant C of the areas having the value

$$C=\sqrt{Mp},$$

in which M represents the mass of the sun. (We disregard the

[1] Translated by Lydia G. Robinson from the author's *Leçons sur les hypothèses cosmogoniques,* Chaps. VI and XIII. Paris, Hermann, 1911.

[2] T. J. J. See, *Researches on the Evolution of the Stellar Systems,* Vol. II, "The Capture Theory of Cosmical Evolution." Lynn, Mass., Nichols & Sons; Paris, Hermann, 1910.

[3] See *loc. cit.,* Chap. VII, pp. 134-158.

mass of the planet compared to the sun's mass.) The mean motion n is connected with half the major axis a by Kepler's third law.

(3) $$n^2a^3=\text{M}.$$

Finally the equation of the *vis viva* gives

$$\text{T}-\frac{\text{M}}{r}=-\frac{\text{M}}{2a},$$

in which T is half the *vis viva.*

Differentiating equation (1) with reference to time, we have

$$\begin{aligned}\frac{dr}{dt}&=\frac{pe\sin v}{(1+e\cos v)^2}\frac{dv}{dt}\\ &=\frac{pe\sin v}{(1+e\cos v)^2}\frac{\text{C}}{r^2}\\ &=\frac{pe\sin v}{(1+e\cos v)^2}\frac{\text{C}}{p^2}(1+e\cos v)^2\\ &=\frac{\text{C}}{p}e\sin v.\end{aligned}$$

Now dr/dt is the component of velocity in the direction of the radius vector. The component perpendicular to this radius vector has for its value

$$\begin{aligned}r\frac{dv}{dt}&=\frac{\text{C}}{r}\\ &=\frac{\text{C}}{p}(1+e\cos v).\end{aligned}$$

From the two components of the velocity V, we derive the square of this velocity,

$$\text{V}^2=\frac{\text{C}^2}{p^2}(1+2\,e\cos v+e^2).$$

In short, if we put

$$\rho^2=1+2\,e\cos v+e^2,$$

we shall have

$$\begin{aligned}\text{V}&=\frac{\text{C}}{p}\rho\\ &=\rho\sqrt{\frac{\text{M}}{p}}.\end{aligned}$$

The above formulas belong to Keplerian motion.

Now let us suppose that there is an atmospheric medium with a resistance R directly opposed to the velocity and function of the value V of that velocity. The constant of the *vires vivae* $-\mathrm{M}/2a$ during the time dt will undergo a variation

$$\frac{\mathrm{M}}{2a^2}da;$$

this variation will equal the work of the resistance R which is

$$-\mathrm{RV}dt.$$

Hence we have

$$\frac{\mathrm{M}}{2a^2}\frac{da}{dt}=-\mathrm{RV}$$

$$=-\mathrm{R}\rho\sqrt{\frac{\mathrm{M}}{p}},$$

whence we derive

$$\frac{da}{dt}=-\frac{2\mathrm{R}\rho a^2}{\sqrt{\mathrm{M}p}};$$

replacing M and p by their values (2) and (3) in this last equation, we obtain

$$(4)\qquad \frac{da}{dt}=-\frac{2\mathrm{R}\rho}{n\sqrt{1-e^2}}.$$

This is the equation which gives the variation of the major axis; the second member is necessarily negative. Hence the effect of the resistance of the medium is always to diminish a and consequently according to equation (3) to increase n. The angular velocity of the planet increases[4] at the same time that its mean distance from the sun diminishes.

W shall now study the effect of resistance of the medium on the eccentricity of the orbit.

First of all the derivative $d\mathrm{C}/dt$ of the areal constant C would be equal to the momentum of the disturbing force R, with reference to the center of attraction. Now this force R opposed to the velocity has for its components:

in the direction of the vector ray

$$-\mathrm{R}\frac{\frac{dr}{dt}}{\mathrm{V}},$$

perpendicular to the vector ray

[4] Formula (3) even shows that na increases as a diminishes, whence we have the curious result that resistance of the medium causes an increase in the linear velocity of the planet.

$$-R\frac{r\frac{dv}{dt}}{V};$$

and the momentum of the force R with reference to the sun is

$$-R\frac{r^2\frac{dv}{dt}}{V}=-R\frac{C}{V}.$$

Hence we have

(5) $$\frac{dC}{dt}=-\frac{RC}{V}.$$

Remember that

$$C=\sqrt{Mp}$$
$$=M^{\frac{1}{2}}a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}.$$

Taking the logarithmic derivatives of the two extreme members, we have

$$\frac{dC}{C}=\frac{1}{2}\left(\frac{da}{a}-\frac{2ede}{1-e^2}\right).$$

This equation makes it possible for us to obtain de since da and dC have been computed. We find

$$\frac{2e}{1-e^2}\frac{de}{dt}=\frac{1}{a}\frac{da}{dt}-\frac{2}{C}\frac{dC}{dt},$$

an equation which may be written by replacing da/dt and dC/dt by their values (4) and (5),

(6) $$\frac{2e}{1-e^2}\frac{de}{dt}=-\frac{2R\rho}{na\sqrt{1-e^2}}+\frac{2R}{V}.$$

Let us now transform the second member of this equation. We have previously found (page 461)

$$V=\rho\sqrt{\frac{M}{p}}$$
$$=\rho\frac{na}{\sqrt{1-e^2}};$$

hence the second member may assume the form

$$-\frac{2R}{na\sqrt{1-e^2}}\left[\rho-\frac{1-e^2}{\rho}\right],$$

or again, by restoring the value of ρ^2, this other form

$$-\frac{2R}{na\sqrt{1-e^2}}\cdot\frac{2e\cos v+2e^2}{\rho}.$$

Finally equation (6) then gives

$$(7)\qquad \frac{de}{dt}=-\frac{2R\sqrt{1-e^2}}{nap}(e+\cos v).$$

This is the equation which gives the variation of the eccentricity of the orbit.

Formulas (4) and (7) make it possible to compute at any instant the variations of the major axis and of the eccentricity. But here it is only desirable to obtain their *secular* variations, and in order to do this, to compute the value of da and de during the time of a complete revolution.

Taking as an independent variable the true anomaly v we shall have

$$(8)\qquad \begin{cases} \dfrac{da}{dv}=\dfrac{da}{dt}\dfrac{dt}{dv}, \\ \dfrac{de}{dv}=\dfrac{de}{dt}\dfrac{dt}{dv}. \end{cases}$$

Now the equation of the areas

$$(9)\qquad \frac{dt}{dv}=\frac{r^2}{C}$$

$$=\frac{p^2}{C}(1+e\cos v)^{-2}.$$

Formulas (4), (7) and (9) therefore make it possible to write the values (8) of da/dv and de/dv which, integrated between 0 and 2π will give the variations of half the major axis and the eccentricity during one revolution.

We may here offer certain hypotheses on medial resistance R. This resistance increases as the velocity; we shall suppose it proportional to a certain power of the velocity V. It varies directly as the distance r from the sun, for the density, and consequently the resistance, of the sun's atmosphere increases inversely as the distance; let us suppose R proportional to a certain power (negative) of r. In short let us put

$$10)\qquad R=hV^{\alpha}r^{-\beta},$$

h, α and β being positive constants. Since V is proportionate to ρ, and r to $1/(1+e\cos v)$, we can write formula (10) as follows:

$$R=k\rho^{\alpha}(1+e\cos v)^{\beta},$$

k being a new positive constant.

In view of these hypotheses on R, the values (8) of da/dv and

de/dv, computed by means of the formulas (4), (7) and (9), may be written

$$(11)\qquad \begin{cases} \dfrac{da}{dv}=-a\mathrm{H}(1-e^2)^{-\frac{1}{2}}\rho^{\alpha+1}(1+e\cos v)^{\beta-2}, \\ \dfrac{de}{dv}=-\ \mathrm{H}(1-e^2)^{\frac{1}{2}}\ \rho^{\alpha-1}\ (1+e\cos v)^{\beta-2}\ (e+\cos v); \end{cases}$$

where H denotes the positive constant

$$\mathrm{H}=\frac{2p^2k}{na\mathrm{C}};$$

bear in mind that in these values (11)

$$\rho=(1+2e\cos v+e^2)^{\frac{1}{2}}.$$

In order to study the secular variations of a and e we must develop the second members of the values (11) in trigonometric series according to the cosines of the multiples of v, and integrate between $v=0$ and $v=2\pi$. By integration all the cosines will be 0; therefore we are interested in the constant terms of these trigonometric developments and especially the sign of these constant terms.

We already know that da/dv is necessarily negative, since da/dt is always negative. Therfore we shall work only with de/dv. We must develop in a trigonometric series the expression

$$\rho^{\alpha-1}(1+e\cos v)^{\beta-2}(e+\cos v).$$

Now if we first develop the product of the two first terms we obtain:

$$(12)\qquad \rho^{\alpha-1}(1+e\cos v)^{\beta-2}=\mathrm{A}_0+\mathrm{A}_1\cos v+\mathrm{A}_2\cos 2v+\dots.$$

We observe that A_0 is necessarily positive because it is the mean value of the first member both of whose terms are always positive. Then multiplying the two members of formula (12) by $(e+\cos v)$ we have

$$\rho^{\alpha-1}(1+e\cos v)^{\beta-2}(e+\cos v)=\left(\mathrm{A}_0e+\frac{\mathrm{A}_1}{2}\right)+\dots,$$

all the unwritten terms of the second member having their mean value 0.

The second formula (11) therefore gives for the mean value of de/dv during one revolution

$$(13)\qquad \frac{de}{dv}=-\mathrm{H}(1-e^2)^{\frac{1}{2}}\left(\mathrm{A}_0e+\frac{\mathrm{A}_1}{2}\right).$$

Since the second member of equation (13) is generally negative we conclude from it that the medial resistance has the effect of diminishing the eccentricity of the orbit. This would be the case particularly whenever A_1 is positive. Now according to formula (12) we have

$$A_1=\frac{2}{\pi}\int_0^{\pi}(1+2e\cos v+e^2)^{\frac{\alpha-1}{2}}(1+e\cos v)^{\beta-2}\cos v\,dv.$$

If at the same time

$$\alpha>1, \qquad \beta>2,$$

A_1 will be positive, for of two elements of the integral corresponding to the two values v' and $\pi-v'$ of the variable of integration, one is positive and the other negative, but the positive element possesses a greater absolute value than the negative.

In an analogous way we know that if the two inequalities

$$\alpha>1, \qquad \alpha+2\beta>5,$$

are satisfied, we shall likewise have

$$A_1>0.$$

If we suppose the eccentricity e to be so small that we can disregard its square e^2 we shall find more general conditions. The second formula (11) is reduced to

$$\frac{de}{dv}=-H[1+(\alpha-1)e\cos v+(\beta-2)e\cos v](e+\cos v);$$

whence by retaining only the mean value of the second member we derive

$$\frac{de}{dv}=-H\left(e+\frac{\alpha+\beta-3}{2}e\right)$$
$$=-\frac{He}{2}(\alpha+\beta-1).$$

Then in order to diminish the eccentricity it is sufficient that

$$\alpha+\beta>1.$$

In this case even if $\beta=0$ (that is, if the resistance R does not vary with the distance r from the sun) we need only have

$$\alpha>1,$$

that is to say, R increasing more rapidly than the simple power of the velocity. Now we often grant as an approximation that a medial resistance is proportionate to the square of the velocity.

This diminution of the eccentricity because of a medial resistance might have been foreseen in general and without calculation in the following manner. Suppose the resistance is not felt except in the vicinity of the perihelion P (Fig. 1). In that case the planet undergoes at this point P a sudden diminution of velocity which results in a decrease in the major axis. Since the perihelion remains the same and the aphelion approaches it, it is clear that the eccentricity is lessened. On the other hand, if resistance acts only at the moment of the aphelion, the new orbit would have the same aphelion as the former one, but its perihelion would be nearer that of the sun, and the eccentricity would be increased. In fact the resistance is felt all along the orbit, but two reasons combine to make it felt more strongly at the perihelion: in the first place the velocity is greatest at that point, since the atmosphere which is generally denser nearer the sun offers a greater resistance near the perihelion.

To sum up, the effect of medial resistance on a Keplerian orbit is to diminish both the major axis and the eccentricity.[5] Therefore if we agree with Mr. See that a resisting atmosphere originally extended for vast distances around the sun, we can conceive that a

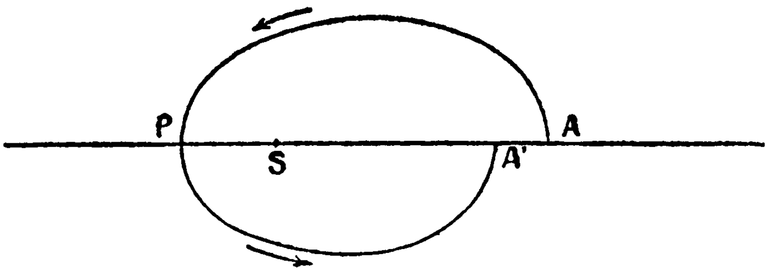

Fig. 1.

body of cosmical origin when passing into the sun's sphere of influence might be able to modify its trajectory. Whether it was parabolic or hyperbolic it now becomes elliptical, because the medial resistance continues to diminish the major axis and the eccentricity of the orbit which approaches the circular form. The resisting atmosphere is gradually absorbed by the sun, and when it finally disappears the smaller body continues to revolve around the sun in its orbit which is almost a circle. Such, according to Mr. See, is the history of all the planets.

Just as the planets have been captured by the sun so also, according to Mr. See, have the satellites been captured by their respective planets.[6]

In order to study this capture we shall take up the comparatively simple case called the restricted problem. The sun S and a planet J

[5] It is easy to recognize that this resistance does not produce any secular effect (at least at the first approximation) on the longitude of the perihelion. To be sure it does not modify the plane of the orbit which retains the same inclination and the same line of nodes with reference to a fixed plane.

[6] *Loc. cit.*, Chap. VIII, pp. 159-182; X, pp 211-236.

(e. g., Jupiter) each revolve around their common center of gravity G in a circular orbit with a constant angular velocity ω (Fig. 2). It is required to study the motion of a small planet P whose mass is negligible with reference to that of the principal planet J and which consequently will not affect the motion of the latter. We will take as origin the center of gravity G, of the system S – J; as plane of the coordinates xy, the plane in which S and J describe their circular orbits; and in this plane rectangular movable axes, the axis of x being the straight line SGJ which connects the sun with Jupiter; the axis of z is the perpendicular to the plane of the orbit at G. The forces acting actually upon the point P (x, y, z) are the attraction of the sun and of Jupiter. These two forces are derived respectively from the two functions of forces[7]

$$U_1=\frac{M}{\rho_1},\quad U_2=\frac{M_2}{\rho_2},$$

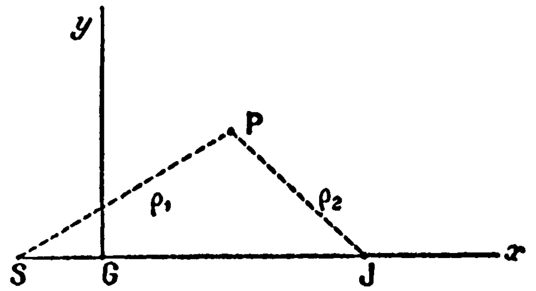

Fig. 2.

M_1, M_2 being the masses of the sun and Jupiter, ρ_1, ρ_2 their distances from P. Since the axes are movable we must add to these forces the centrifugal force and the compound centrifugal force. The components of the centrifugal force are

$$\omega^2 x,\qquad \omega^2 y,\qquad 0.$$

The components of the compound centrifugal force are

$$2\,\omega\frac{dy}{dt},\qquad -2\,\omega\frac{dx}{dt},\qquad 0.$$

Hence the equations of the motion of the planet P with relation to the movable axes are

$$\frac{d^2x}{dt^2}=\frac{dU_1}{dx}+\frac{dU_2}{dx}+\omega^2 x+2\omega\frac{dy}{dt},$$

$$\frac{d^2y}{dt^2}=\frac{dU_1}{dy}+\frac{dU_2}{dy}+\omega^2 y-2\omega\frac{dx}{dt},$$

[7] We assume the mass m of the small planet P to be equal to unity. More exactly, since this mass m is a factor in every case we shall not write it in the formulas.

$$\frac{d^2z}{dt^2}=\frac{dU_1}{dz}+\frac{dU_2}{dz}.$$

If we multiply these three equations

$$dx=\frac{dx}{dt}dt,\qquad dy=\frac{dy}{dt}dt,\qquad dz=\frac{dz}{dt}dt,$$

respectively, and add the results, we obtain a combination immediately integrable which brings us to the following integral

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]=\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)-C,$$

known by the name of the integral of Jacobi.

Since the first member of this last equation is positive, the coordinates x, y, z of the point P will satisfy the inequality

$$\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)-C>0.$$

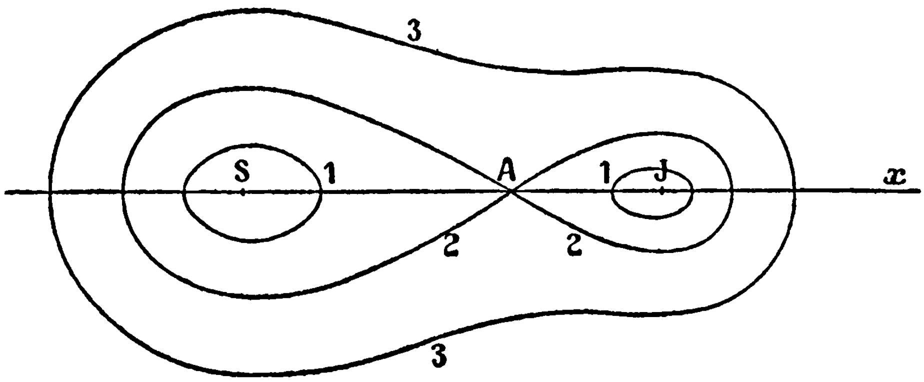

Fig. 3.

Hence the projection (x, y) of the point P on the plane of xy will be within the curve

$$\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)=C;$$

in this equation ρ_1 and ρ_2 denoting the distances of this projection of the point P from the points S and J. For very great values of the constant C this curve comprises two rings (denoted by 1 on Fig. 3) surrounding the points S and J respectively. As C diminishes, these two rings become dilated and finally unite at a double point A (Curve 2). Then when C is further diminished they finally make only one curve (Curve 3) surrounding at the same time both S and J.[8] Hence when the constant C is not too great the small planet is obliged to remain within Curve 3 but still is free to travel in the proximity either of the sun or of Jupiter. On the contrary

[8] We pay no attention here to certain portions of curves which are very far removed from the origin.

if the constant C is very great the small planet will remain within one of the rings 1; it will be a satellite either of the sun or of Jupiter.

Now the effect of a passive resistance like that of a medium is to increase the constant C of the second member of Jacobi's integral. Hence the curve encircling the small planet constantly contracts. If it was originally Curve 3 at a definite moment it will become Curve 2 with the double point. If at this moment the planet is near the sun it will never return to the proximity of Jupiter; it is captured by the sun. If on the contrary it is in the neighborhood of Jupiter it will never return to that of the sun; it will be captured by Jupiter and from that moment will become one of his satellites.

The theory of Mr. See accounts for the smallness of the eccentricities of the orbits of planets and satellites.[9] But why are the movements of almost all the heavenly bodies in a straight line, and why have their orbits such small mutual inclinations? In the hypothesis of Mr. See these two questions remain without any satisfactory answer. To try to explain the smallness of the inclinations we may suppose that the resisting atmosphere of the sun is of a greatly flattened lenticular form; hence a body whose orbit is greatly inclined to the plane of this disk suffers a resistance much smaller than a body moving in the very plane of the disk. The first body has therefore much less tendency to be captured than the second, and is in the plane of the disk in which the captures of the planets are made.

We may also suppose that the resisting medium itself revolves. It will then tend not to counteract the velocity of the planet revolving within it but to impose upon this planet a certain velocity. Since the resistance is no longer directly opposed to the velocity, the plane of the orbit could vary and tend to diminish its inclination to the equatorial plane of the solar atmosphere.

FORMATION OF SPIRAL NEBULAS.

In the work previously referred to,[10] Mr. See is concerned with the formation of nebulas, especially with the origin of spiral nebulas.

Let us imagine two masses of cosmical vapor N and N′, almost equal in size and traveling in opposite directions (Fig. 4*a*). As they

[9] The diminution of the eccentricity because of a resisting medium is of first importance not only in the theory of Mr. See; it is taken into consideration also in the theories of Faye and of Du Ligondès.

[10] *Op. cit.*, Chap. XIX.

approach each other their adjacent extremities will be prolonged each in the direction of the other by mutual attraction (Fig. 4*b*) and may even end in uniting to form a single body (Fig. 4*c*) near whose center attraction combined with friction will tend to produce a condensation, a sort of central nucleus. The two masses of vapor N and N′ will turn in the directions of the arrows around this center like two arms of a windmill.

Such, according to Mr. See, would be the origin of the spiral nebulas. The central nucleus would tend to enlarge more and more

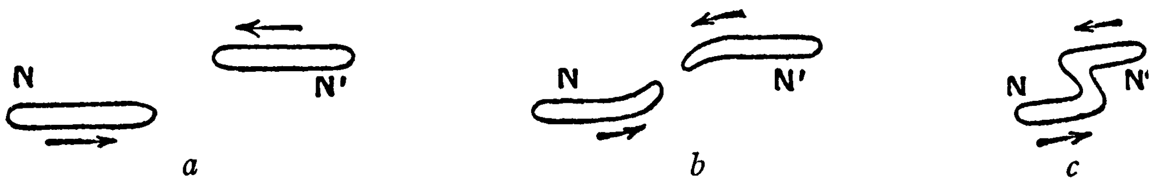

Fig. 4.

at the expense of the matter in the two spiral branches N and N′. Hence we see that in the opinion of Mr. See the motion of the matter in the two arms of the spiral nebula contrary to the usual view would be centripetal and not centrifugal. Moreover whether the motion is convergent or divergent the law of areas accounts equally in both cases for the slowness of the arm's revolution around its pivot, that is to say, the spiral form of both arms.

It may happen that the ends of the two masses of vapor N and N′ do not join as they approach each other, but are merely deviated by attraction. Then the phase following phase 2 of Fig. 4. is not

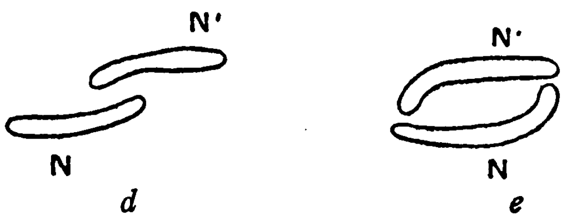

Fig. 5.

phase *c* but phase *d* (Fig. 5) after which it assumes phase *e*. In such a case we have the origin of an annular nebula like that of Lyra. In the two diametrically opposed light portions seen in the ring of Lyra, Mr. See finds an argument for the application of this theory in that adjacent ends of the two masses of vapor N and N′ would not be perfectly united.

Hence Mr. See thinks that an annular nebula is formed by the same mechanical process as spiral nebulas of which it thus proves to be in some sense a particular case. But the annular form is very rare because the conditions for the formation of a perfect ring are not often realized.

One great objection may be offered to this theory. The two arms of a spiral nebula are usually almost symmetrical. In the ordinary hypothesis in which the movement of the arms is assumed to be divergent this symmetry may be explained by the common origin of the two arms. In the hypothesis of Mr. See there is no way to account for it, for the two masses of cosmical vapor N and N′ which give rise to the nebula and which have met accidentally will not usually be equal. They ought then to give birth to an unsymmetrical nebula.

Mr. See thinks that originally the solar system was a spiral nebula of vast extent. The matter at its center first became agglomerated into particles which with the help of the resistance of the medium were condensed into asteroids, according to the process explained above, and then into planets, which are further increased by bombardment.[11]

Mr. See is led by analogy to believe that the spiral nebulas which are less advanced in their evolution than the solar system are composed of a vast number of very small bodies like the planets or even the moon. If we can not analyze these nebulas it will be because of the extremely small size of their component parts and not because these celestial objects are so excessively remote. Mr. Bohlin has tried to measure the parallax of the nebula of Andromeda (which is a spiral nebula of a continuous spectrum) and he has found it equal to 0″, 17, so that this nebula would be comparatively very near us. But considering how little accuracy the points on the nebulas admit of, can we regard this observation as conclusive and certain?

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NOTES ON THE CONSTRUCTION OF MAGIC SQUARES OF ORDERS IN WHICH n IS OF THE FORM $8p+2$.

Referring to the article in the last issue of *The Monist* by Messrs. Andrews and Frierson, under the above heading, it was shown that the minimum series to be used in constructing this class of squares is selected from the series 1, 2, 3, ... $(n+3)^2$, by

[11] Mr. See sees in the lunar craters signs of a bombardment produced at the surface of the moon by the fall of a large number of little satellites. He compares these craters to the marks left by great drops of rain in the mud (*op. cit.*, p. 342, plate XII).