On the dynamics of the electron: Introduction, §§1, 9

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Introduction.

It seems at first that the aberration of light and related optical and electrical phenomena will provide us with a means of determining the absolute motion of the Earth, or rather its motion with respect to the ether, as opposed to its motion with respect to other celestial bodies. Fresnel pursued this idea, but soon recognized that the Earth's motion does not alter the laws of refraction and reflection. Analogous experiments, like that of the water-filled telescope, and all those considering terms no higher than first order relative to the aberration, yielded only negative results; the explanation was soon discovered. But Michelson, who conceived an experiment sensitive to terms depending on the square of the aberration, failed in turn.

It appears that this impossibility to detect the absolute motion of the Earth by experiment may be a general law of nature; we are naturally inclined to admit this law, which we will call the *Postulate of Relativity* and admit without restriction. Whether or not this postulate, which up to now agrees with experiment, may later be corroborated or disproved by experiments of greater precision, it is interesting in any case to ascertain its consequences.

An explanation was proposed by Lorentz and FitzGerald, who introduced the hypothesis of a contraction of all bodies in the direction of the Earth's motion and proportional to the square of the aberration. This contraction, which we will call the *Lorentzian contraction*, would explain Michelson's experiment and all others performed up to now. The hypothesis would become insufficient, however, if we were to admit the postulate of relativity in full generality.

Lorentz then sought to extend his hypothesis and to modify it in order to obtain perfect agreement with this postulate. This is what he succeeded in doing in his article entitled *Electromagnetic phenomena in a system moving with any velocity smaller that that of light* (*Proceedings* of the Amsterdam Academy, May 27, 1904).

The importance of the question persuaded me to take it up in turn; the results I obtained agree with those of Mr. Lorentz on all the significant points. I was led merely to modify and extend them only in a few details; further on we will see the points of divergence, which are of secondary importance.

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Lorentz's idea may be summed up like this: if we are able to impress a translation upon an entire system without modifying any observable phenomena, it is because the equations of an electromagnetic medium are unaltered by certain transformations, which we will call *Lorentz transformations*. Two systems, one of which is at rest, the other in translation, become thereby exact images of each other.

Langevin *) sought to modify Lorentz's idea; for both authors, the moving electron takes the form of a flattened ellipsoid. For Lorentz, two axes of the ellipsoid remain constant, while for Langevin, ellipsoid volume remains constant. The two scientists also showed that these two hypotheses are corroborated by Kaufmann's experiments to the same extent as the original hypothesis of Abraham (rigid-sphere electron).

The advantage of Langevin's theory is that it requires only electromagnetic forces, and bonds; it is, however, incompatible with the postulate of relativity. This is what Lorentz showed, and this is what I found in turn using a different method, which calls on principles of group theory.

We must return therefore to Lorentz's theory, but if we want to do this and avoid intolerable contradictions, we must posit the existence of a special force that explains both the contraction, and the constancy of two of the axes. I sought to determine this force, and found that *it may be assimilated to a constant external pressure on the deformable and compressible electron, whose work is proportional to the electron's change in volume.*

If the inertia of matter is exclusively of electromagnetic origin, as generally admitted in the wake of Kaufmann's experiment, and all forces are of electromagnetic origin (apart from this constant pressure that I just mentioned), the postulate of relativity may be established with perfect rigor. This is what I show by a very simple calculation based on the principle of least action.

But that is not all. In the article cited above, Lorentz judged it necessary to extend his hypothesis in such a way that the postulate remains valid in case there are forces of non-electromagnetic origin. According to Lorentz, all forces are affected by the Lorentz transformation (and consequently by a translation) in the same way as electromagnetic forces.

It was important to examine this hypothesis closely, and in particular to ascertain the modifications we would have to apply to the laws of gravitation.

We find first of all that it requires us to assume that gravitational propagation is not instantaneous, but occurs with the speed of light. One might think that this is reason enough to reject the hypothesis, since Laplace demonstrated that this cannot be the case. In reality, however, the effect of this propagation is compensated in large part by a different cause, in such a way that no contradiction arises between the proposed law and astronomical observations.

Is it possible to find a law satisfying Lorentz's condition, and reducing to Newton's law whenever the speeds of celestial bodies are small enough to allow us to neglect their squares (as well as the product of acceleration and distance) with respect to the square of the speed of light?

To this question we must respond in the affirmative, as we will see later.

Modified in this way, is the law compatible with astronomical observations?

It seems so on first sight, but the question will be settled only after an extended discussion.

Suppose, then, that this discussion is settled in favor of the new hypothesis, what

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*Langevin was anticipated by Mr. Bucherer of Bonn, who earlier advanced the same idea. (See: Bucherer, *Mathematische Einführung in die Elektronentheorie*, August, 1904. Teubner, Leipzig).

should we conclude? If propagation of attraction occurs with the speed of light, it could not be a fortuitous accident. Rather, it must be because it is a function of the ether, and then we would have to try to penetrate the nature of this function, and to relate it to other fluid functions.

We cannot be content with a simple juxtaposition of formulas that agree with each other by good fortune alone; these formulas must, in a manner of speaking, interpenetrate. The mind will be satisfied only when it believes it has perceived the reason for this agreement, and the belief is strong enough to entertain the illusion that it could have been predicted.

But the question may be viewed from a different perspective, better shown via an analogy. Let us imagine a pre-Copernican astronomer who reflects on Ptolemy's system; he will notice that for all the planets, one of two circles – epicycle or deferent – is traversed in the same time. This fact cannot be due to chance, and consequently between all the planets there is a mysterious link we can only guess at.

Copernicus, however, destroys this apparent link by a simple change in the coordinate axes that were considered fixed. Each planet now describes a single circle, and orbital periods become independent (until Kepler reestablishes the link that was believed to have been destroyed).

It is possible that something analogous is taking place here. If we were to admit the postulate of relativity, we would find the same number in the law of gravitation and the laws of electromagnetism—the speed of light—and we would find it again in all other forces of any origin whatsoever. This state of affairs may be explained in one of two ways: either everything in the universe would be of electromagnetic origin, or this aspect—shared, as it were, by all physical phenomena—would be a mere epiphenomenon, something due to our methods of measurement. How do we go about measuring? The first response will be: we transport objects considered to be invariable solids, one on top of the other. But that is no longer true in the current theory if we admit the Lorentzian contraction. In this theory, two lengths are equal, by definition, if they are traversed by light in equal times.

Perhaps if we were to abandon this definition Lorentz's theory would be as fully overthrown as was Ptolemy's system by Copernicus's intervention. Should that happen some day, it would not prove that Lorentz's efforts were in vain, because regardless of what one may think, Ptolemy was useful to Copernicus.

I, too, have not hesitated to publish these few partial results, even if at this very moment the discovery of magneto-cathode rays seems to threaten the entire theory.

§ 1. — Lorentz Transformation.

Lorentz adopted a certain system of units in order to do away with 4π factors in formulas. I will do the same, and in addition, select units of length and time in such a way that the speed of light equals 1. Under these conditions, and denoting electric displacement f, g, h, magnetic intensity α, β, γ, vector potential F, G, H, scalar potential ψ, charge density ρ, electron velocity ξ, η, ζ, and current u, v, w, the fundamental formulas become:

$$\left.\begin{array}{c} u=\dfrac{df}{dt}+\rho\xi=\dfrac{d\gamma}{dy}-\dfrac{d\beta}{dz}, \quad \alpha=\dfrac{dH}{dy}-\dfrac{dG}{dz}, \quad f=-\dfrac{dF}{dt}-\dfrac{d\psi}{dx}, \\ \dfrac{d\alpha}{dt}=\dfrac{dg}{dz}-\dfrac{dh}{dy}, \quad \dfrac{d\rho}{dt}+\sum\dfrac{d\rho\xi}{dx}=0, \quad \sum\dfrac{df}{dx}=\rho, \quad \dfrac{d\psi}{dt}+\sum\dfrac{dF}{dx}=0, \\ \Box=\Delta-\dfrac{d^2}{dt^2}=\sum\dfrac{d^2}{dx^2}-\dfrac{d^2}{dt^2}, \qquad \Box\psi=-\rho, \qquad \Box F=-\rho\xi. \end{array}\right\} \quad (1)$$

An elementary particle of matter of volume $dxdydz$ is acted upon by a mechanical force, the components of which are derived from the formula:

$$X=\rho f+\rho(\eta\gamma-\zeta\beta). \quad (2)$$

These equations admit a remarkable transformation discovered by Lorentz, which owes its interest to the fact that it explains why no experiment can inform us of the absolute motion of the universe. Let us put:

$$x'=kl(x+\varepsilon t), \qquad t'=kl(t+\varepsilon x), \qquad y'=\ell y, \qquad z'=\ell z, \quad (3)$$

where ℓ and ε are two arbitrary constants, such that

$$k=\frac{1}{\sqrt{1-\varepsilon^2}}.$$

Now if we put:

$$\Box'=\sum\frac{d^2}{dx'^2}-\frac{d^2}{dt'^2},$$

we will have:

$$\Box'=\Box\ell^{-2}.$$

Let a sphere be carried along with the electron in uniform translation, and let the equation of this mobile sphere be:

$$(x-\xi t)^2+(y-\eta t)^2+(z-\zeta t)^2=r^2,$$

and the volume of the sphere be $\frac{4}{3}\pi r^3$.[1]

The transformation will change the sphere into an ellipsoid, the equation of which is easy to find. We thus deduce easily from (3):

$$x=\frac{k}{\ell}(x'-\varepsilon t'), \quad t=\frac{k}{\ell}(t'-\varepsilon x'), \quad y=\frac{y'}{\ell}, \quad z=\frac{z'}{\ell}. \quad (3')$$

The equation of the ellipsoid then becomes:

$$k^2(x'-\varepsilon t'-\xi t'+\varepsilon\xi x')^2+(y'-\eta kt'+\eta k\varepsilon x')^2+(z'-\zeta kt'+\zeta k\varepsilon x')^2=\ell^2r^2.$$

This ellipsoid is in uniform motion; for $t'=0$, it reduces to

$$k^2x'^2(1+\xi\varepsilon)^2+(y'+\eta k\varepsilon x')^2+(z'+\zeta k\varepsilon x')^2=\ell^2r^2,$$

[1]The original reads: "$\frac{4}{3}\pi r^2$".

and has a volume:

$$\frac{4}{3}\pi r^3 \frac{\ell^3}{k(1+\xi\varepsilon)}.$$

If we want electron charge to be unaltered by the transformation, and if we designate the new charge density ρ', we will find:

$$\rho' = \frac{k}{\ell^3}(\rho + \varepsilon\rho\xi). \tag{4}$$

What will be the new velocity components ξ', η' and ζ'? We should have:

$$\xi' = \frac{dx'}{dt'} = \frac{d(x+\varepsilon t)}{d(t+\varepsilon x)} = \frac{\xi+\varepsilon}{1+\varepsilon\xi},$$

$$\eta' = \frac{dy'}{dt'} = \frac{dy}{kd(t+\varepsilon x)} = \frac{\eta}{k(1+\varepsilon\xi)}, \qquad \xi' = \frac{\zeta}{k(1+\varepsilon\xi)},$$

whence:

$$\rho'\xi' = \frac{k}{\ell^3}(\rho\xi + \varepsilon\rho), \qquad \rho'\eta' = \frac{1}{\ell^3}\rho\eta, \qquad \rho'\zeta' = \frac{1}{\ell^3}\rho\zeta. \tag{4'}$$

Here is where I must point out for the first time a difference with Lorentz. In my notation, Lorentz put (l.c., page 813, formulas 7 and 8):

$$\rho' = \frac{1}{k\ell^3}\rho, \qquad \xi' = k^2(\xi+\varepsilon), \qquad \eta' = k\eta, \qquad \zeta' = k\zeta.$$

In this way we recover the formulas:

$$\rho'\xi' = \frac{k}{\ell^3}(\rho\xi + \varepsilon\rho), \qquad \rho'\eta' = \frac{1}{\ell^3}\rho\eta, \qquad \rho'\zeta' = \frac{1}{\ell^3}\rho\zeta;$$

although the value of ρ' differs.

It is important to notice that the formulas (4) and (4′) satisfy the condition of continuity

$$\frac{d\rho'}{dt'} + \sum \frac{d\rho'\xi'}{dx'} = 0.$$

To see this, let λ be an undetermined coefficient and D the Jacobian of

$$t+\lambda\rho, \qquad x+\lambda\rho\xi, \qquad y+\lambda\rho\eta, \qquad z+\lambda\rho\zeta \tag{5}$$

with respect to t, x, y and z. It follows that:

$$D = D_0 + D_1\lambda + D_2\lambda^2 + D_3\lambda^3 + D_4\lambda^4,$$

with $D_0 = 1$, $D_1 = \frac{d\rho}{dt} + \sum \frac{d\rho\xi}{dx} = 0$.

Let $\lambda' = \ell^4\rho'$;[2] then the 4 functions

$$t'+\lambda'\rho', \qquad x'+\lambda'\rho'\xi', \qquad y'+\lambda'\rho'\eta', \qquad z'+\lambda'\rho'\zeta' \tag{5'}$$

are related to the functions (5) by the same linear relationships as the old variables to the new ones. Therefore, if we denote D' the Jacobian of the functions (5′) with respect to the new variables, it follows that:

$$D' = D, \qquad D' = D'_0 + D'_1\lambda' + \cdots + D'_4\lambda'^4,$$

[2]The original reads: "$\lambda' = \ell^2\rho'$".

and thereby:[3]

$$D'_0 = D_0 = 1, \qquad D'_1 = \ell^{-4} D_1 = 0 = \frac{d\rho'}{dt'} + \sum \frac{d\rho'\xi'}{dx'}. \qquad \text{Q.E.D.}$$

Under Lorentz's hypothesis, this condition would not be met since ρ' has a different value.

We will define the new vector and scalar potentials in such a way as to satisfy the conditions

$$\Box'\psi' = -\rho', \qquad \Box' F' = -\rho'\xi'. \tag{6}$$

From this we deduce:

$$\psi' = \frac{k}{\ell}(\psi + \varepsilon F), \quad F' = \frac{k}{\ell}(F + \varepsilon\psi), \quad G' = \frac{1}{\ell}G, \quad H' = \frac{1}{\ell}H. \tag{7}$$

These formulas differ noticeably from those of Lorentz, although the divergence stems ultimately from the definitions employed.

New electric and magnetic fields are now chosen in order to satisfy the equations

$$f' = -\frac{dF'}{dt'} - \frac{d\psi'}{dx'}, \qquad \alpha' = \frac{dH'}{dy'} - \frac{dG'}{dz'}. \tag{8}$$

It is easy to see that:

$$\frac{d}{dt'} = \frac{k}{\ell}\left(\frac{d}{dt} - \varepsilon\frac{d}{dx}\right), \quad \frac{d}{dx'} = \frac{k}{\ell}\left(\frac{d}{dx} - \varepsilon\frac{d}{dt}\right), \quad \frac{d}{dy'} = \frac{1}{\ell}\frac{d}{dy}, \quad \frac{d}{dz'} = \frac{1}{\ell}\frac{d}{dz}$$

and we deduce thereby:

$$\left.\begin{array}{lll} f' = \dfrac{1}{\ell^2} f, & g' = \dfrac{k}{\ell^2}(g + \varepsilon\gamma), & h' = \dfrac{k}{\ell^2}(h - \varepsilon\beta), \\ \alpha' = \dfrac{1}{\ell^2}\alpha, & \beta' = \dfrac{k}{\ell^2}(\beta - \varepsilon h), & \gamma' = \dfrac{k}{\ell^2}(\gamma + \varepsilon g). \end{array}\right\} \tag{9}$$

These formulas are identical to those of Lorentz.

Our transformation does not alter (1). In fact, the condition of continuity, as well as (6) and (8) were already featured in (1) (neglecting the primes).

Combining (6) with the condition of continuity, we obtain:

$$\frac{d\psi'}{dt'} + \sum \frac{dF'}{dx'} = 0. \tag{10}$$

It remains for us to establish:

$$\frac{df'}{dt'} + \rho'\xi' = \frac{d\gamma'}{dy'} - \frac{d\beta'}{dz'}, \qquad \frac{d\alpha'}{dt'} = \frac{dg'}{dz'} - \frac{dh'}{dy'}, \qquad \sum \frac{df'}{dx'} = \rho'$$

and it is easy to see that these are necessary consequences of (6), (8) and (10).

We must now compare forces before and after the transformation.

[3]The original reads: "$D'_1 = \ell^{-2} D_1$".

Let X, Y, Z be the force prior to the transformation, and X', Y', Z' the force after the transformation, both forces being per unit volume. In order for X' to satisfy the same equations as before the transformation, we must have:

$$X' = \rho' f' + \rho'(\eta'\gamma' - \zeta'\beta'),$$
$$Y' = \rho' g' + \rho'(\zeta'\alpha' - \xi'\gamma'),$$
$$Z' = \rho' h' + \rho'(\xi'\beta' - \eta'\alpha'),$$

or, replacing all quantities by their values (4), (4′) and (9), and in light of (2):

$$\left.\begin{aligned} X' &= \frac{k}{\ell^5}(X + \varepsilon \sum X\xi), \\ Y' &= \frac{1}{\ell^5} Y, \\ Z' &= \frac{1}{\ell^5} Z. \end{aligned}\right\} \tag{11}$$

Instead of representing the components of force per unit volume by X_1, Y_1, Z_1, we now let these terms represent the force per unit electron charge, and we let X_1', Y_1', Z_1' represent the latter force after transformation. It follows that:

$$X_1 = f + \eta\gamma - \zeta\beta, \quad X_1' = f' + \eta'\gamma' - \zeta'\beta', \quad X = \rho X_1, \quad X' = \rho X_1',$$

and we obtain the equations:

$$\left.\begin{aligned} X_1' &= \frac{k}{\ell^5}\frac{\rho}{\rho'}(X_1 + \varepsilon \sum X_1\xi), \\ Y_1' &= \frac{1}{\ell^5}\frac{\rho}{\rho'} Y_1, \\ Z_1' &= \frac{1}{\ell^5}\frac{\rho}{\rho'} Z_1. \end{aligned}\right\} \tag{11′}$$

Lorentz found (page 813, equation (10) with different notation):

$$\left.\begin{aligned} X_1 &= \ell^2 X_1' - \ell^2\varepsilon(\eta' g' + \zeta' h'), \\ Y_1 &= \frac{\ell^2}{k} Y_1' + \frac{\ell^2\varepsilon}{k}\xi' g', \\ Z_1 &= \frac{\ell^2}{k} Z_1' + \frac{\ell^2\varepsilon}{k}\xi' h'. \end{aligned}\right\} \tag{11″}$$

Before going any further, it is important to locate the source of this significant divergence. It obviously springs from the fact that the formulas for ξ', η' and ζ' are not the same, while the formulas for the electric and magnetic fields are the same.

If electron inertia is exclusively of electromagnetic origin, and if electrons are subject only to forces of electromagnetic origin, then the conditions of equilibrium require that:

$$X = Y = Z = 0$$

inside the electrons.

According to (11), these relationships are equivalent to

$$X' = Y' = Z' = 0.$$

The electron's equilibrium conditions are therefore unaltered by the transformation.

Unfortunately, such a simple hypothesis is inadmissible. In fact, if we assume $\xi = \eta = \zeta = 0$, the condition $X = Y = Z = 0$ leads necessarily to $f = g = h = 0$, and consequently, to $\sum \frac{df}{dx} = 0$, i.e., $\rho = 0$. Similar results obtain for the most general case. We must then admit that in addition to electromagnetic forces there are either non-electromagnetic forces or bonds. Therefore, we need to identify the conditions that these forces or these bonds must satisfy for electron equilibrium to be undisturbed by the transformation. This will be the object of an upcoming section.

§ 9. — Hypotheses Concerning Gravitation.

In this way Lorentz's theory would fully explain the impossibility of detecting absolute motion, if all forces were of electromagnetic origin.

But there exist other forces to which an electromagnetic origin cannot be attributed, such as gravitation, for example. It may in fact happen, that two systems of bodies produce equivalent electromagnetic fields, i.e., exert the same action on electrified bodies and on currents, and at the same time, these two systems do not exert the same gravitational action on Newtonian masses. The gravitational field is therefore distinct from the electromagnetic field. Lorentz was obliged thereby to extend his hypothesis with the assumption that *forces of any origin whatsoever, and gravitation in particular, are affected by a translation* (or, if one prefers, by the Lorentz transformation) *in the same manner as electromagnetic forces*.

It is now appropriate to enter into the details of this hypothesis, and to examine it more closely. If we want the Newtonian force to be affected by the Lorentz transformation in this fashion, we can no longer suppose that it depends only on the relative position of the attracting and attracted bodies at the instant considered. The force should also depend on the velocities of the two bodies. And that is not all: it will be natural to suppose that the force acting on the attracted body at the instant t depends on the position and velocity of this body at this same instant t, but it will also depend on the position and velocity of the *attracting* body, not at the instant t, but at *an earlier instant*, as if gravitation had taken a certain time to propagate.

Let us now consider the position of the attracted body at the instant t_0, and let x_0, y_0, z_0 be its coordinates, and ξ, η, ζ its velocity components at this instant; let us consider also the attracting body at the corresponding instant $t_0 + t$, and let its coordinates be $x_0 + x$, $y_0 + y$, $z_0 + z$, and its velocity components be ξ_1, η_1, ζ_1 at this instant.

First we should have a relationship

$$\varphi\left(t,\ x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ \xi_1,\ \eta_1,\ \zeta_1\right) = 0 \tag{1}$$

in order to define the time t. This relationship will define the law of propagation of gravitational action (I do not constrain myself by any means to a propagation velocity equal in all directions).

Now let X_1, Y_1, Z_1 be the three components of the action exerted on the attracted body at the instant t_0;[4] we want to express X_1, Y_1, Z_1 as functions of

$$t,\ x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ \xi_1,\ \eta_1,\ \zeta_1. \tag{2}$$

What conditions must be satisfied?

[4]The original reads: "à l'instant t".

1° The condition (1) should not be altered by transformations of the Lorentz group.

2° The components X_1, Y_1, Z_1 should be affected by transformations of the Lorentz group in the same manner as the electromagnetic forces designated by the same letters, i.e., in accordance with (11′) of section 1.

3° When the two bodies are at rest, the ordinary law of attraction will be recovered.

It is important to note that in the latter case, the relationship (1) vanishes, because if the two bodies are at rest the time t plays no role.

Posed in this fashion the problem is obviously indeterminate. We will therefore seek to satisfy to the utmost other, complementary conditions.

4° Since astronomical observations do not seem to show a sensible deviation from Newton's law, we will choose the solution that differs the least with this law for small velocities of the two bodies.

5° We will make an effort to arrange matters in such a way that t is always negative. Although we can imagine that the effect of gravitation requires a certain time in order to propagate, it would be difficult to understand how this effect could depend on the position *not yet attained* by the attracting body.

There is one case where the indeterminacy of the problem vanishes; it is the one where the two bodies are in mutual *relative* rest, i.e., where

$$\xi = \xi_1, \qquad \eta = \eta_1, \qquad \zeta = \zeta_1;$$

this is then the case we will examine first, by supposing that these velocities are constant, such that the two bodies are engaged in a common uniform rectilinear translation.

We may suppose that the $x-$axis is parallel to this translation, such that $\eta = \zeta = 0$, and we will let $\varepsilon = -\xi$.

If we apply the Lorentz transformation under these conditions, after the transformation the two bodies will be at rest, and it follows that:

$$\xi' = \eta' = \zeta' = 0.$$

The components X_1, Y_1, Z_1 should then agree with Newton's law and we will have, apart from a constant factor:

$$X_1' = -\frac{x}{r'^3}, \quad Y_1' = -\frac{y}{r'^3}, \quad Z_1' = -\frac{z}{r'^3}, \quad r'^2 = x'^2 + y'^2 + z'^2. \tag{3}$$

But according to section 1 we have:

$$\begin{aligned}
&x' = k(x + \varepsilon t), \quad y' = y, \quad z' = z, \quad t' = k(t + \varepsilon x),\\
&\frac{\rho'}{\rho} = k(1 + \xi\varepsilon) = k(1 - \varepsilon^2) = \frac{1}{k}, \quad \sum X_1\xi = -X_1\varepsilon,\\
&X_1' = k\frac{\rho}{\rho'}(X_1 + \varepsilon \sum X_1\xi) = k^2 X_1(1 - \varepsilon^2) = X_1,\\
&Y_1' = k\frac{\rho}{\rho'}Y_1 = kY_1\\
&Z_1' = kZ_1.
\end{aligned}$$

We have in addition:

$$x + \varepsilon t = x - \xi t, \qquad r'^2 = k^2(x - \xi t)^2 + y^2 + z^2$$

and

$$X_1 = \frac{-k(x-\xi t)}{r'^3}, \quad Y_1 = \frac{-y}{kr'^3}, \quad Z_1 = \frac{-z}{kr'^3}; \tag{4}$$

which may be written:

$$X_1 = \frac{dV}{dx}, \quad Y_1 = \frac{dV}{dy}, \quad Z_1 = \frac{dV}{dz}; \quad V = \frac{1}{kr'}. \tag{4'}$$

It seems at first that the indeterminacy remains, since we made no hypotheses concerning the value of t, i.e., the transmission speed; and that besides, x is a function of t. It is easy to see, however, that the terms appearing in our formulas, $x - \xi t,\ y,\ z$, do not depend on t.

We see that if the two bodies translate together, the force acting on the attracted body is perpendicular to an ellipsoid, at the center of which lies the attracting body.

To advance further, we need to look for the *invariants of the Lorentz group*.

We know that the substitutions of this group (assuming $\ell = 1$) are linear substitutions that leave unaltered the quadratic form

$$x^2 + y^2 + z^2 - t^2.$$

Let us also put:

$$\xi = \frac{\delta x}{\delta t}, \qquad \eta = \frac{\delta y}{\delta t}, \qquad \zeta = \frac{\delta z}{\delta t},$$
$$\xi_1 = \frac{\delta_1 x}{\delta_1 t}, \qquad \eta_1 = \frac{\delta_1 y}{\delta_1 t}, \qquad \zeta_1 = \frac{\delta_1 z}{\delta_1 t};$$

we see that the Lorentz transformation will make $\delta x,\ \delta y,\ \delta z,\ \delta t$, and $\delta_1 x,\ \delta_1 y,\ \delta_1 z,\ \delta_1 t$ undergo the same linear substitutions as $x,\ y,\ z,\ t$.

Let us regard

$$\begin{array}{cccc} x, & y, & z, & t\sqrt{-1}, \\ \delta x, & \delta y, & \delta z, & \delta t\sqrt{-1}, \\ \delta_1 x, & \delta_1 y, & \delta_1 z, & \delta_1 t\sqrt{-1}, \end{array}$$

as the coordinates of 3 points $P,\ P',\ P''$ in space of 4 dimensions. We see that the Lorentz transformation is merely a rotation in this space about the origin, assumed fixed. Consequently, we will have no distinct invariants apart from the 6 distances between the 3 points $P,\ P',\ P''$, considered separately and with the origin, or, if one prefers, apart from the two expressions

$$x^2 + y^2 + z^2 - t^2, \qquad x\delta x + y\delta y + z\delta z - t\delta t,$$

or the 4 expressions of like form deduced from an arbitrary permutation of the 3 points $P,\ P',\ P''$.

But what we seek are invariants that are functions of the 10 variables (2). Therefore, among the combinations of our 6 invariants we must find those depending only on these 10 variables, i.e., those that are 0th degree homogeneous with respect both to $\delta x,\ \delta y,\ \delta z,\ \delta t$, and to $\delta_1 x,\ \delta_1 y,\ \delta_1 z,\ \delta_1 t$. We will then be left with 4 distinct invariants:

$$\sum x^2 - t^2, \quad \frac{t - \sum x\xi}{\sqrt{1 - \sum \xi^2}}, \quad \frac{t - \sum x\xi_1}{\sqrt{1 - \sum \xi_1^2}}, \quad \frac{1 - \sum \xi\xi_1}{\sqrt{\left(1 - \sum \xi^2\right)\left(1 - \sum \xi_1^2\right)}}. \tag{5}$$

Next let us see how the force components are transformed; we recall the equations (11) of section 1, that refer not to the force X_1, Y_1, Z_1 considered at present, but to the force per unit volume: X, Y, Z.

We designate moreover

$$T = \sum X\xi;$$

we will see that (11) can be written ($\ell = 1$):

$$\left.\begin{aligned} X' &= k(X + \varepsilon T), \qquad & T' &= k(T + \varepsilon X), \\ Y' &= Y, & Z' &= Z; \end{aligned}\right\} \tag{6}$$

in such a way that X, Y, Z, T undergo the same transformation as x, y, z, t. Consquently, the group invariants will be

$$\sum X^2 - T^2, \quad \sum Xx - Tt, \quad \sum X\delta x - T\delta t, \quad \sum X\delta_1 x - T\delta_1 t.$$

However, it is not X, Y, Z that we need, but X_1, Y_1, Z_1, with

$$T_1 = \sum X_1\xi.$$

We see that

$$\frac{X_1}{X} = \frac{Y_1}{Y} = \frac{Z_1}{Z} = \frac{T_1}{T} = \frac{1}{\rho}.$$

Therefore, the Lorentz transformation will act in the same manner on X_1, Y_1, Z_1, T_1, as on X, Y, Z, T, except that these expressions will be multiplied moreover by

$$\frac{\rho}{\rho'} = \frac{1}{k(1 + \xi\varepsilon)} = \frac{\delta t}{\delta t'}.$$

Likewise, the Lorentz transformation will act in the same way on ξ, η, ζ, 1 as on δx, δy, δz, δt, except that these expressions will be multiplied moreover by the *same* factor:

$$\frac{\delta t}{\delta t'} = \frac{1}{k(1 + \xi\varepsilon)}.$$

Next we consider X, Y, Z, $T\sqrt{-1}$ as the coordinates of a fourth point Q; the invariants will then be functions of the mutual distances of the five points

$$0, \quad P, \quad P', \quad P'', \quad Q$$

and among these functions we must retain only those that are 0th degree homogeneous with respect, on one hand, to

$$X, \quad Y, \quad Z, \quad T, \quad \delta x, \quad \delta y, \quad \delta z, \quad \delta t$$

(variables that can be replaced further by X_1, Y_1, Z_1, T_1, ξ, η, ζ, 1), and on the other hand, with respect to[5]

$$\delta_1 x, \qquad \delta_1 y, \qquad \delta_1 z, \qquad \delta_1 t$$

(variables that can be replaced further by ξ_1, η_1, ζ_1, 1).

[5]The original reads "$\delta_1 x$, $\delta_1 y$, $\delta_1 z$, 1."

In this way we find, beyond the four invariants (5), four distinct new invariants:

$$\frac{\sum X_1^2 - T_1^2}{1-\sum \xi^2}, \quad \frac{\sum X_1 x - T_1 t}{\sqrt{1-\sum \xi^2}}, \quad \frac{\sum X_1 \xi_1 - T_1}{\sqrt{1-\sum \xi^2}\sqrt{1-\sum \xi_1^2}}, \quad \frac{\sum X_1 \xi - T_1}{1-\sum \xi^2}. \tag{7}$$

The latter invariant is always null according to the definition of T_1.

These terms being settled, what conditions must be satisfied?

1° The first term of (1), defining the velocity of propagation, has to be a function of the 4 invariants (5).

A wealth of hypotheses can obviously be entertained, of which we will examine only two:

A) We can have

$$\sum x^2 - t^2 = r^2 - t^2 = 0,$$

from whence $t = \pm r$, and, since t has to be negative, $t = -r$. This means that the velocity of propagation is equal to that of light. It seems at first that this hypothesis ought to be rejected outright. Laplace showed in effect that the propagation is either instantaneous or much faster than that of light. However, Laplace examined the hypothesis of finite propagation velocity *ceteris non mutatis*; here, on the contrary, this hypothesis is conjoined with many others, and it may be that between them a more or less perfect compensation takes place. The application of the Lorentz transformation has already provided us with numerous examples of this.

B) We can have

$$\frac{t - \sum x\xi_1}{\sqrt{1-\sum \xi_1^2}} = 0, \qquad t = \sum x\xi_1.$$

The propagation velocity is therefore much faster than that of light, but in certain cases t could be positive, which, as we mentioned, seems hardly admissible.[6] *We will therefore stick with hypothesis* (A).

2° The four invariants (7) ought to be functions of the invariants (5).

3° When the two bodies are at absolute rest, X_1, Y_1, Z_1 ought to have the values given by Newton's law, and when they are at relative rest, the values given by (4).

For the case of absolute rest, the first two invariants (7) ought to reduce to

$$\sum X_1^2, \qquad \sum X_1 x,$$

or, by Newton's law, to

$$\frac{1}{r^4}, \qquad -\frac{1}{r};$$

in addition, according to hypothesis (A), the 2^d^ and 3^rd^ invariants in (5) become:

$$\frac{-r - \sum x\xi}{\sqrt{1-\sum \xi^2}}, \quad \frac{-r - \sum x\xi_1}{\sqrt{1-\sum \xi_1^2}},$$

that is, for absolute rest,

$$-r, \qquad -r.$$

[6]The original reads "t pourrait être négatif."

We may therefore admit, *for example*, that the first two invariants in (7) reduce to[7]

$$\frac{\left(1-\sum\xi_1^2\right)^2}{(r+\sum x\xi_1)^4},\quad -\frac{\sqrt{1-\sum\xi_1^2}}{r+\sum x\xi_1},$$

although other combinations are possible.

A choice must be made among these combinations, and furthermore, we need a 3rd equation in order to define X_1, Y_1, Z_1. In making such a choice, we should try to come as close as possible to Newton's law. Let us see what happens when we neglect the squares of the velocities ξ, η, etc. (still letting $t=-r$). The 4 invariants (5) then become:

$$0,\quad -r-\sum x\xi,\quad -r-\sum x\xi_1,\quad 1$$

and the 4 invariants (7) become:

$$\sum X_1^2,\quad \sum X_1(x+\xi r),\quad \sum X_1(\xi_1-\xi),\quad 0.$$

Before we can make a comparison with Newton's law, another transformation is required. In the case under consideration, x_0+x, y_0+y, z_0+z, represent the coordinates of the attracting body at the instant t_0+t, and $r=\sqrt{\sum x^2}$. With Newton's law we have to consider the coordinates of the attracting body x_0+x_1, y_0+y_1, z_0+z_1 at the instant t_0, and the distance $r_1=\sqrt{\sum x^2}$.

We may neglect the square of the time t required for propagation, and proceed, consequently, as if the motion were uniform; we then have:

$$x=x_1+\xi_1 t,\quad y=y_1+\eta_1 t,\quad z=z_1+\zeta_1 t,\quad r(r-r_1)=\sum x\xi_1 t;$$

or, since $t=-r$,

$$x=x_1-\xi_1 r,\quad y=y_1-\eta_1 r,\quad z=z_1-\zeta_1 r,\quad r=r_1-\sum x\xi_1;$$

such that our 4 invariants (5) become:

$$0,\quad -r_1+\sum x(\xi_1-\xi),\quad -r_1,\quad 1$$

and our 4 invariants (7) become:

$$\sum X_1^2,\quad \sum X_1[x_1+(\xi-\xi_1)r_1],\quad \sum X_1(\xi_1-\xi),\quad 0.$$

In the second of these expressions I wrote r_1 instead of r, because r is multiplied by $\xi-\xi_1$, and because I neglect the square of ξ.

For these 4 invariants (7), Newton's law would yield

$$\frac{1}{r_1^4},\qquad -\frac{1}{r_1}-\frac{\sum x_1(\xi-\xi_1)}{r_1^2},\qquad \frac{\sum x_1(\xi-\xi_1)}{r_1^3},\qquad 0.$$

Therefore, if we designate the 2nd and 3rd of the invariants (5) as A and B, and the first 3 invariants of (7) as M, N, P, we will satisfy Newton's law to first-order terms in the square of velocity by setting:

$$M=\frac{1}{B^4},\quad N=\frac{+A}{B^2},\quad P=\frac{A-B}{B^3}. \tag{8}$$

[7]The original has (4) instead of (7).

This solution is not unique. Let C be the 4th invariant in (5); $C-1$ is of the order of the square of ξ, and it is the same with $(A-B)^2$.

The solution (8) appears at first to be the simplest, nevertheless, it may not be adopted. In fact, since M, N, P are functions of X_1, Y_1, Z_1, and $T_1 = \sum X_1\xi$, the values of X_1, Y_1, Z_1 can be drawn from these three equations (8), but in certain cases these values would become imaginary.

To avoid this difficulty we will proceed in a different manner. Let us put:

$$k_0 = \frac{1}{\sqrt{1-\sum \xi^2}}, \quad k_1 = \frac{1}{\sqrt{1-\sum \xi_1^2}},$$

which is justified by analogy with the notation

$$k = \frac{1}{\sqrt{1-\sum \xi^2}}$$

featured in the Lorentz substitution.

In this case, and in light of the condition $-r = t$, the invariants (5) become:

$$0, \quad A = -k_0(r + \sum x\xi), \quad B = -k_1(r + \sum x\xi_1), \quad C = k_0k_1(1 - \sum \xi\xi_1).$$

Moreover, we notice that the following systems of quantities:

$$\begin{array}{cccc} x, & y, & z, & -r = t, \\ k_0X_1, & k_0Y_1, & k_0Z_1, & k_0T_1, \\ k_0\xi, & k_0\eta, & k_0\zeta, & k_0, \\ k_1\xi_1, & k_1\eta_1, & k_1\zeta_1, & k_1 \end{array}$$

undergo the *same* linear substitutions when the transformations of the Lorentz group are applied to them. We are led thereby to put:

$$\left.\begin{aligned} X_1 &= x\frac{\alpha}{k_0} + \xi\beta + \xi_1\frac{k_1}{k_0}\gamma, \\ Y_1 &= y\frac{\alpha}{k_0} + \eta\beta + \eta_1\frac{k_1}{k_0}\gamma, \\ Z_1 &= z\frac{\alpha}{k_0} + \zeta\beta + \zeta_1\frac{k_1}{k_0}\gamma, \\ T_1 &= -r\frac{\alpha}{k_0} + \beta + \frac{k_1}{k_0}\gamma. \end{aligned}\right\} \tag{9}$$

It is clear that if α, β, γ are invariants, X_1, Y_1, Z_1, T_1 will satisfy the fundamental condition, i.e., the Lorentz transformations will make them undergo an appropriate linear substitution.

However, for equations (9) to be compatible we must have

$$X_1\xi - T_1 = 0,$$

which becomes, replacing X_1, T_1, Z_1, T_1 with their values in (9) and multiplying by k_0^2:

$$-A\alpha - \beta - C\gamma = 0. \tag{10}$$

What we would like is that the values of X_1, Y_1, Z_1 remain in line with Newton's law when we neglect (as above) the squares of velocities ξ, etc. with respect to the square of the velocity of light, and the products of acceleration and distance.

We could select

$$\beta = 0, \qquad \gamma = -\frac{A\alpha}{C}.$$

To the adopted order of approximation, we obtain

$$k_0 = k_1 = 1,\ C = 1,\ A = -r_1 + \sum x(\xi_1 - \xi),\ B = -r_1,\ x = x_1 + \xi_1 t = x_1 - \xi_1 r.$$

The 1st equation in (9) then becomes

$$X_1 = \alpha(x - A\xi_1).$$

But if the square of ξ is neglected, $A\xi_1$ can be replaced by $-r_1\xi_1$, or by $-r\xi_1$, which yields:

$$X_1 = \alpha(x + \xi_1 r) = \alpha x_1.$$

Newton's law would yield

$$X_1 = -\frac{x_1}{r_1^3}.$$

Consequently, we must select a value for the invariant α which reduces to $-\frac{1}{r_1^3}$ in the adopted order of approximation, that is, $\frac{1}{B^3}$. Equations (9) will become:

$$\left.\begin{aligned} X_1 &= \frac{x}{k_0 B^3} - \xi_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ Y_1 &= \frac{y}{k_0 B^3} - \eta_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ Z_1 &= \frac{z}{k_0 B^3} - \zeta_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ T_1 &= -\frac{r}{k_0 B^3} - \frac{k_1}{k_0}\frac{A}{B^3 C}. \end{aligned}\right\} \tag{11}$$

We notice first that the corrected attraction is composed of two components: one parallel to the vector joining the positions of the two bodies, the other parallel to the velocity of the attracting body.

Remember that when we speak of the position or velocity of the attracting body, this refers to its position or velocity at the instant the gravitational wave takes off; for the attracted body, on the contrary, this refers to the position or velocity at the instant the gravitational wave arrives, assuming that this wave propagates with the velocity of light.

I believe it would be premature to seek to push the discussion of these formulas further; I will therefore confine myself to a few remarks.

1° The solutions (11) are not unique; we may, in fact, replace the the global factor $\frac{1}{B^3}$ by

$$\frac{1}{B^3} + (C-1)f_1(A,\ B,\ C) + (A-B)^2 f_2(A,\ B,\ C),$$

where f_1 and f_2 are arbitrary functions of A, B, C. Alternatively, we may forgo setting β to zero, but add any complementary terms to α, β, γ that satisfy condition (10) and are of second order with respect to the ξ for α, and of first order for β and γ.

2° The first equation in (11) may be written:

$$X_1 = \frac{k_1}{B^3 C}\left[x\left(1 - \sum \xi\xi_1\right) + \xi_1\left(r + \sum x\xi\right)\right] \tag{11'}$$

and the quantity in brackets itself may be written:

$$(x + r\xi_1) + \eta(\xi_1 y - x\eta_1) + \zeta(\xi_1 z - x\zeta_1), \tag{12}$$

such that the total force may be separated into three components corresponding to the three parentheses of expression (12); the first component is vaguely analogous to the mechanical force due to the electric field, the two others to the mechanical force due to the magnetic field; to extend the analogy I may, in light of the first remark, replace $\frac{1}{B^3}$ in (11) by $\frac{C}{B^3}$, in such a way that X_1, Y_1, Z_1 are linear functions of the attracted body's velocity ξ, η, ζ, since C has vanished from the denominator of (11′).

Next we put:

$$\left.\begin{array}{lll} k_1(x + r\xi_1) = \lambda, & k_1(y + r\eta_1) = \mu, & k_1(z + r\zeta_1) = \nu, \\ k_1(\eta_1 z - \zeta_1 y) = \lambda', & k_1(\zeta_1 x - \xi_1 z) = \mu', & k_1(\xi_1 y - x\eta_1) = \nu'; \end{array}\right\} \tag{13}$$

and since C has vanished from the denominator of (11′), it will follow that:

$$\left.\begin{aligned} X_1 &= \frac{\lambda}{B^3} - \frac{\eta\nu' - \zeta\mu'}{B^3}, \\ Y_1 &= \frac{\mu}{B^3} - \frac{\zeta\lambda' - \xi\nu'}{B^3}, \\ Z_1 &= \frac{\nu}{B^3} - \frac{\xi\mu' - \eta\lambda'}{B^3}; \end{aligned}\right\} \tag{14}$$

and we will have moreover:

$$B^2 = \sum \lambda^2 - \sum \lambda'^2. \tag{15}$$

Now λ, μ, ν, or $\frac{\lambda}{B^3}$, $\frac{\mu}{B^3}$, $\frac{\nu}{B^3}$, is an electric field of sorts, while λ', μ', ν', or rather $\frac{\lambda'}{B^3}$, $\frac{\mu'}{B^3}$, $\frac{\nu'}{B^3}$ is a magnetic field of sorts.

3° The postulate of relativity would compel us to adopt solution (11), or solution (14), or any solution at all among those derived on the basis of the first remark. However, the first question to ask is whether or not these solutions are compatible with astronomical observations. The deviation from Newton's law is of the order of ξ^2, i.e., 10000 times smaller than if it were of the order of ξ, i.e., if the propagation were to take place with the velocity of light, *ceteris non mutatis*; consequently, it is legitimate to hope that it will not be too large. To settle this question, however, would require an extended discussion.

Paris, July, 1905.

H. Poincaré