SCIENCE

FRIDAY, MAY 19, 1911

CONTENTS

MSS. intended for publication and books, etc., intended for review should be sent to the Editor of SCIENCE, Garrison-on-Hudson, N. Y.

THE BOLYAI PRIZE[1]

THE problems treated by Hilbert are so varied and their importance is so evident that a long preamble seems unnecessary. It is preferable to enter immediately upon the detailed exposition of his principal memoirs. The reader in the presence of results so important will himself draw conclusions.

INVARIANTS

The first works of Hilbert relate to invariants. We know with what passion this part of mathematics was cultivated about the middle of last century and how it has since been neglected. It seemed in fact that Clebsch, Gordan, Cayley and Sylvester had used up all that it was possible to deduce from the old methods and that after them there remained only slight gleanings. But the progress of algebra and arithmetic, and in particular the theory of whole algebraic numbers, the extension soon made of it to integral polynomials, and Kronecker's theory of moduli, made possible the approach of the question from a side still unexplored.

This Hilbert did in attacking at first the celebrated theorem of Gordan, according to which all the invariants of a system of forms can be expressed in a rational and integral way as functions of a finite number of them.

We could not better measure the advance made than by comparing the volume Gordan had to devote to his demonstration with the few lines with which Hilbert has been satisfied. The method gained in gen-

[1] Report on the works of Hilbert by Poincaré. Translated by G. B. Halsted.

erality as much as in simplicity and one could make out a whole series of possible generalizations. A very simple lemma inspired by Kronecker's ideas had made this result possible.

Consider an indefinite series of forms F depending upon n variables; we can find among these a finite number of forms $F_1, \cdots, F_p$, such that any form F of the series can be equated to

(1) $$F \equiv A_1F_1 + \cdots A_pF_p,$$

the A's being forms depending upon the same variables. This is a consequence of the fundamental notion of the modulus introduced by Kronecker. This means, in Kronecker's language, that the divisors common to many moduli, even were they infinite in number, are submultiples of one of them which is their greatest common divisor, and in geometric language (supposing four variables and regarding them as homogeneous coordinates of a point in space) that the aggregate of points common to an infinite number of algebraic surfaces is composed of a finite number of isolated points and a finite number of skew algebraic curves.

But this is not all; suppose the F's are the invariants of a system of forms and the A's functions of the coefficients of these forms.

We may always suppose that the A's are also invariants, otherwise we could perform an arbitrary linear transformation upon the forms. Then in the relation (1) thus transformed would appear the coefficients of this transformation. In applying to the relation (1) transformed a certain train of successive differentiations (the differentiations are performed with respect to the coefficients of the linear transformation) we reach a relation of the same form as (1) but where the A's are invariants. From this the proof of Gordan's theorem follows immediately.

But this is not all; among these fundamental invariants there is a certain number of relations called syzygies. All the syzygies can be deduced from a finite number of them by addition and multiplication. Among these fundamental syzygies of the first order there are syzygies of the second order, which can also be obtained from a finite number of them by addition and multiplication, and so on.

Hilbert gets this result from a general theorem of algebra. Consider a system of linear equations of the form

$$\Sigma F_{ik} X_i = 0,$$

where the F's are given forms and the X's unknown forms homogeneous in regard to certain variables; the study of the solutions of this system and of the relations which connect them leads to the consideration of a series of derived systems continued until we reach a derived system which no longer admits of any solution. Thus it was that Hilbert was led to determine and to study the number $X(R)$ of distinct conditions which a form of degree R should satisfy to be congruent to zero with regard to a given modulus.

But to complete the theory it was not enough to establish the existence of a system of fundamental invariants; it was necessary to give the means of actually forming this, and this problem was made by our author to depend upon a question which connects it with the theory of whole algebraic numbers extended to integral polynomials.

The problem is thus broken up into three others.

1. To find invariants J_m as functions of which all the others can be expressed in *algebraic and integral* form, that is to say, such that any invariant J satisfies an algebraic equation

$$J^k + G_1J^{k-1} + G_2J^{k-2} + \cdots + G_{k-1}J + G_k = 0,$$

the *G's* being polynomials integral with regard to J_m.

2. To find invariants as functions of which all the others can be expressed rationally.

3. To find invariants as functions of which all the others can be expressed in rational and integral form.

Of these three problems the first is the most difficult. If it be supposed solved, the aggregate of invariants presents itself as an *algebraic corpus,* and the first step to make is to determine the degree of this corpus; this it is at which Hilbert arrives at least for binary forms by evaluating in two different ways the number $\phi(\sigma)$ of invariants linearly independent of degree σ, or rather the asymptotic value of this numeric function $\phi(\sigma)$ for σ very great.

The first problem once solved, the solution of the other two goes back to a classic question of the arithmetic of polynomials and of the theory of algebraic corpora. The question is then to find the fundamental invariants by whose aid all the others can be expressed in algebraic and integral form.

With this purpose Hilbert remarks that these are those which can not be annulled without annulling all the others. So we see that the search for these fundamental invariants will be singularly facilitated by the study of *null forms,* that is to say, of those whose numeric coefficients are chosen in such a way that the numeric values of all the invariants may be null.

In the case of binary forms, the null forms are those which are divisible by a sufficiently high power of a linear factor; but in the other cases the problem is more delicate. Our author first establishes a certain number of theorems.

Consider a form with numeric coefficients and its transform by any linear substitution; the coefficients of this transform will be integral polynomials with regard to the coefficients of the substitution. If the determinant of the substitution is an *algebraic and integral* function of these integral polynomials, the proposed form is not a null form. In the contrary case, it is a null form.

Consider, on the other hand, the transforms of a form by a linear substitution depending upon an arbitrary parameter t and in such a way that the coefficients of this substitution are series developable in positive or negative integral powers but increasing with this parameter. If it be a question of a null form, we can choose a substitution of this kind of such a sort that its determinant becomes infinite for $t=0$, while the coefficients of the form transformed remain finite. Hilbert shows that this condition is necessary in order that the proposed form may be null, and it is evident, moreover, that it is sufficient. To each null form corresponds therefore one and perhaps several linear substitutions having the enunciated property. This settled, our author proves that, starting from any null form, we can by a linear transformation, transform it into a *canonic* null form. A form is called canonic when the linear substitution which corresponds to it and which possesses in relation to it the property we have just stated is of the simple form

$$\begin{vmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{vmatrix}.$$

The investigation of null forms is thus made to depend on that of canonic null forms which is much more simple. We find that the canonic null forms are those in which certain terms are lacking; and the determination of the terms which should be lacking can easily be made, thanks to a simple geometric scheme.

We see under what a new and elegant aspect present themselves to-day, thanks to Hilbert, problems so many geometers had for fifty years attempted.

THE NUMBER e

Hermite was the first to prove that the number e is transcendant, and shortly afterward Lindemann extended this result to the number π.

This was a victory important for science, but Hermite's methods were still susceptible of betterment; however ingenious and however original they were, one felt they did not lead to the goal by the shortest way. This shortest way Hilbert has found, and it seems that henceforth no one can hope to give new simplification to the proof.

This was the second time that Hilbert had given, of a theorem known but only established by means most arduous, a proof of astonishing simplicity. This faculty of simplifying what had seemed at first complex thus presents itself as one of the characteristics of his genius.

ARITHMETIC

The arithmetical works of Hilbert pertain principally to algebraic corpora. The aggregate of numbers which can be expressed rationally as functions of one or several algebraic numbers constitutes a domain of rationality, and the aggregate of the numbers of this domain which are algebraic integers constitutes a corpus. If we consider then all the algebraic numbers of a corpus which can be put under the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p$$

where the α's are *given* numbers of the corpus, and the x's indeterminate numbers of the same corpus, the aggregate of these numbers is what is called an *ideal*. That which gives interest to this consideration is that the ideals obey in what concerns their divisibility the ordinary laws of arithmetic and that in particular every ideal is decomposable in one way and only one into ideal primes. This is the *fundamental theorem of Dedekind.*

On the other hand, we may consider numbers which satisfy an algebraic equation of which the coefficients belong to a domain D of rationality. These numbers and those rationally expressible by means of them define a new domain of rationality D' more extended than D; and an algebraic corpus K' more extended than the corpus K which corresponds to D. We then may relate the corpus K', not to the ordinary rational numbers and to the corpus of the integers of ordinary arithmetic, but to the domain D and to the algebraic corpus K. We then may speak of the *relative* degree of K' with reference to K, of the *relative norm* of an algebraic number of K' with reference to K, etc. There will be corpora relatively quadratic obtained by the adjunction to the domain D' of a radical $\sqrt{\mu}$, μ being a number of the domain D, and corpora relatively abelian, obtained by the adjunction to D of the roots of an abelian equation. This is a sort of generalization of the ideas of Dedekind, that Hilbert is doubtless not the first to have seen, but from which he has drawn an unexpected advantage.

We should also speak of galois corpora, whose generating equation is a galois equation. Any corpus is contained in a galois corpus, in the same way the corpus K of which we have just spoken is contained in the corpus K'; and this galois corpus is easily obtained by adjoining to the domain of rationality, not only one of the roots of the generating algebraic equation of K, but *all* its roots.

Questions relative to any corpus are

thus made to depend upon the analogous problems for the galois corpora.

After having shown how we may, by the discussion of a congruence, form all the ideals of given norm, Hilbert has sought a new proof of the fundamental theorem of Dedekind; he established it first for the galois corpora and then easily extended it to any corpus.

Thus Hilbert was led to study the general theory of galois corpora, and he introduced a host of new notions, defining a series of subcorpora, corresponding to different subgroups of the galois group of the generating equation; these subgroups are defined by certain relations they have with any ideal prime of the corpus, and the study of these subgroups opens for us glimpses new and interesting of the structure of the corpus.

Our author gave in 1896 a new proof of Kronecker's theorem according to which the roots of abelian equations can be expressed by the roots of unity. This demonstration purely arithmetical puts in evidence the way of constructing all the abelian corpora of a given group and discriminant.

But the works of Hilbert have had as their principal object the study of corpora relatively quadratic and relatively abelian.

One of the essential points of the theory of numbers is Gauss's law of reciprocity in the subject of quadratic residues; we know with what predilection the great geometer returned to this question and how he multiplied demonstrations.

This law of reciprocity is capable of interesting generalizations when we pass from the domain of ordinary rational numbers to a domain of any rationality. Hilbert has succeeded in realizing this generalization in the case where the corpus is imaginary and has an odd number of classes. He has introduced a symbol analogous to that of Legendre, and the law of reciprocity reached by him presents itself in a simple form; the product of a certain number of such symbols must equal 1.

This generalization presents all the more interest since our author has succeeded in showing that there are genera corresponding to half of all the imaginable systems of characters, a result which should be likened to that of Gauss and which makes possible the extension to a domain of any rationality of this notion of the genus of quadratic forms which is the subject of one of the most attractive chapters of the "Disquisitiones Arithmeticæ."

To go farther, Hilbert is obliged to introduce a new notion and modify the definition of class.

Two ideals belong to the same class in the old or broad sense if their ratio is any existing algebraic number; they belong to the same class in the new or narrow sense if their ratio is an existing algebraic number *which is positive as well as all its conjugates*. The numbers of classes, whether understood in the broad sense or in the narrow sense, are evidently in intimate relation and our author explains what the nature of this relation is. But this new definition allows Hilbert to express in simpler language the theorems he had in view. These theorems stated in their most general form are, as Hilbert says, remarkably simple and of crystalline beauty; their complete proof appeared to our author as the final aim of his studies on algebraic corpora. It is in this general form we shall state them.

If k is any corpus, there is a group Kk which may be called its *class corpus*. Its relative degree is equal to the number of classes in the narrow sense. It is non-ramified, that is to say, no ideal prime of k is divisible by the square of an ideal prime of Kk, and it contains all the non-ramified corpora relatively abelian with regard to k.

Its relative group is isomorphic to the abelian group which defines the composition of the classes of ideals of k.

The ideal primes of k, although prime in relation to k, are not in general prime in relation to Kk; they may, therefore, be broken into factors ideal primes with regard to Kk, the number of these factors and the power to which they are raised, in a word the mode of partition, depending solely upon the class to which the ideal considered belongs in the corpus k.

Call "*ambige*" a number of Kk which is positive as well as all its conjugates, and which differs from these conjugates only by a factor which is a complex unity.

Each ambige of Kk corresponds to an ideal of k and reciprocally. This property is characteristic of the corpus Kk among all the corpora relatively abelian with regard to k.

We see the bearing of these theorems and the light thrown on the notion of class, since the mutual relations of classes of ideals are reproduced as in a faithful picture by those of the algebraic integers of a corpus.

In reality Hilbert has completely proved these theorems only in particular cases, but these particular cases are very numerous, exceedingly varied and broadly extended. He is, besides, he says, convinced that his methods are applicable to the general case. While sharing his conviction, we must make reservation, so long as this hope, legitimate as it may be, has not been actually realized.

We have spoken above of the law of reciprocity relative to quadratic residues; we must add that Hilbert has given an analogous law for his residues of any power, at least for certain particular corpora.

Summarizing, the introduction of ideals by Kummer and Dedekind was an important advance; it generalized and at the same time cleared up the classic results of Gauss on quadratic forms and their composition. The works of Hilbert we have just analyzed constitute a new step in advance, not less important than the first.

THEOREM OF WARING

Let us speak now of another entirely different arithmetical work. It pertains to proving Waring's theorem according to which every integer can be broken into a sum of N nth powers, N depending only upon n, just as, for example, it can always be broken into a sum of four squares. Needless to recall that this theorem up to the present had simply been stated.

What above all deserves to fix the attention in Hilbert's proof is that it rests on a new way of introducing continuous variables into the theory of numbers.

We start from an identity where a 25uple integral is equated to the mth power of the sum of five squares. Breaking up the domain of integration into smaller domains so as to have a series of approximate values of the integral, as if it were a question of evaluating it by mechanical quadratures, and by the methods of passing to the limit familiar to our author, we reach another identity

$$(x_1^2 + \cdots + x_5^2)^m = \Sigma r_h Y_h^{2m},$$

where the r_h's are rational positive numbers and the Y's linear functions of the x's with integral coefficients. The coefficients r and those of the Y's, as also the number of these linear functions, depend only upon m.

Up to this point we have not gone out of algebra, if not in showing that the coefficients r and those of Y are rational. To get further our author establishes a series of lemmas whose statement is too complicated to be here reproduced and which lead finally to the complete proof of the theorem. We can not doubt that these considerations, which allow also the obtaining of arithmetical relations in making them come from identities where definite integrals figure, can some day, when we shall have grasped their meaning, be applied to problems much more extended than that of Waring.

GEOMETRY

I come to Hilbert's works so very original on the foundations of geometry.

There are in the history of this geometric philosophy three principal epochs; the first is that where thinkers at whose head we should cite Bolyai founded the non-euclidean geometry; the second is that in which Helmholtz and Lie showed the rôle in geometry of the notion of motion and of group; the third was inaugurated by Hilbert.

The German author takes the logical point of view. What are the axioms enunciated and those unconsciously assumed; what is their real logical content and what may be deduced from them by the simple application of the rules of logic and without new appeal to intuition? Finally, are they independent, or can we on the contrary, deduce them from one another? These are the questions to face.

Hilbert commences, therefore, by establishing the complete list of assumptions, striving not to forget a single one. That is not as easy as one might think, and Euclid himself uses some he does not state. Geometric intuition is so familiar to us that we use intuitive verities, so to speak, without our perceiving them; hence to attain the aim Hilbert proposed to himself, the necessity of not according to intuition the least place.

The savant professor divides the assumptions into five groups:

I. Assumptions of Association (I shall translate by *projective assumptions* in place of seeking a literal translation, as, for example, *assumptions of connection,* which would not be satisfactory).

II. Betweenness assumptions (assumptions of order).

III. Congruence assumptions or metric assumptions.

IV. Euclid's postulate.

V. The Archimedes assumption.

Among the projective assumptions we distinguished those of the plane and those of space; the first come from the well-known proposition: *through two points passes one straight, and only one.*

Going on to the second group, the order assumptions, here is the statement of the first two:

"If three points are on the same straight, they have a certain relation which we express by saying that one of the points, and only one, is between the other two. If C is between A and B, if D is between A and C, D will also be between A and B, etc."

Here still we note that intuition is not brought in; we seek not to fathom the meaning of the word *between,* every relation satisfying the assumptions may be designated by this same word.

The third group comprises the metric assumptions where we distinguish three subgroups, relative respectively to sects, to angles and to triangles.

An important point here was not stressed (in the first German edition, though it appears in the French translation). To complete the list of assumptions it needs to be said that the sect AB is congruent to the inverse sect BA. This assumption implies the symmetry of space and the equality of the angles at the base in an isosceles triangle. Hilbert does not here treat this question, but he has made it the subject of a memoir to which we shall return later.

The fourth group contains only Euclid's postulate.

The fifth group comprises two assumptions; the first and most important is that of Archimedes.

Let there be any two points A and B on a straight d; let a be any sect; construction d, starting from the point A, and in the sense AB, a series of sects all equal to one another and equal to a:

AA_1, A_1A_2, $\cdots$ $A_{n-1}A_n$; we can always take n so great that the point B is on one of these sects.

This is to say that, if we take any two sects l and L, we can always find a whole number n so great that by adding the sect l to itself n times, we obtain a total sect greater than L.

The second is the assumption of completeness of which I shall explain the meaning further on.

INDEPENDENCE OF THE ASSUMPTIONS

The list of assumptions once drawn up, it is necessary to see if it is free from contradictions. We well know that it is, since geometry exists; and Hilbert first replied yes by constructing a geometry. But strange to say, this geometry is not exactly ours, his space is not ours, or at least is only a part of it. In Hilbert's space are not all the points which are in ours, but only those that, starting from two given points, we can construct with ruler and compasses. In this space, for example, there is no angle of 10°.

In his second edition Hilbert tried to fill out his list so as to obtain our geometry and no other, and so he introduced the assumption of completeness which he states as follows:

To the system of points, straights and planes it is impossible to adjoin another system of objects such that the complete system satisfies all the other assumptions.

It is evident then that this space of which I spoke, which does not contain all the points of our space, does not satisfy this new axiom, because we can adjoin to it those points of our space which it does not contain, without its ceasing to satisfy all the assumptions.

There is, therefore, an infinity of geometries which satisfy all the assumptions except the assumption of completeness, but only one, ours, which satisfies also this latter assumption.

We then must ask if the assumptions are independent, that is to say, if we could sacrifice one of the five groups, retaining the other four, and nevertheless obtain a coherent geometry. Thus it is, suppressing group IV. (Euclid's postulate), we obtain Bolyai's non-euclidean geometry.

We can equally suppress group III. Hilbert has succeeded in retaining groups I., II., IV. and V., as also the two subgroups of the metric assumptions of sects and angles, while rejecting the metric assumption of triangles, that is to say, the proposition III., 6.

Non-archimedean Geometry.—But Hilbert's most original conception is that of non-archimedean geometry, where all the assumptions remain true save that of Archimedes. For this it is needful first to make a *system of non-archimedean numbers*, that is to say, a system of elements between which we can conceive relations of equality and inequality, and to which we can apply operations corresponding to arithmetical addition and multiplication, and this in a way to satisfy the following conditions:

1. The arithmetical rules of addition and of multiplication (commutativity, associativity, distributivity, etc., *arithmetical*

assumptions of combination) remain without change.

2. The rules of the calculus and transformation of inequalities (arithmetical assumptions of ordering) likewise remain.

3. The Archimedes assumption is not true.

We may attain this result by choosing for elements series of the following form:

$$A_0t^m + A_1t^{m-1} + A_2t^{m-2} + \cdots,$$

where m is an integer positive or negative and where the coefficients A are real, and convening to apply to these series the ordinary rules of addition and of multiplication. It is necessary then to define the conditions of inequality of these series so as to *arrange* our elements in a determined order. We shall attain this by the following convention: we will attribute to our series the sign of A_0 and we will say that a series is smaller than another when, if taken away from this, it gives a positive difference.

It is clear that with this convention the rules of the calculus of inequalities hold good, but the Archimedes assumption is no longer true.

Our common numbers come in as particular cases among these *non-archimedean numbers*. The new numbers intercalate themselves, so to speak, in the series of our common numbers, in such a way that there is for example an infinity of new numbers less than a given common number A and greater than all the common numbers less than A.

That settled, imagine a tri-dimensional space wherein the coordinates of a point are measured not by common numbers, but by non-archimedean numbers, but where the usual equations of the straight and of the plane hold good, as also the analytic expressions for angles and sects. It is clear that in this space all the assumptions remain true, save that of Archimedes.

On any straight between our common points would intercalate themselves new points. Likewise there will be on this straight an infinity of new points to the right of all the common points. In a word, our common space is only a part of non-archimedean space.

We see what is the bearing of this invention and wherein it constitutes in the progress of our ideas a step almost as bold as that which Bolyai made us take; the geometry non-euclidean respected, so to speak, our qualitative conception of the geometric continuum while it overturned our ideas on the measure of this continuum. The non-archimedean geometry destroys this conception; it dissects the continuum to introduce into it new elements.

In this conception so audacious Hilbert had had a precursor. In his foundations of geometry Veronese had had an analogous idea. Chapter VI. of his introduction is the development of a veritable non-archimedean arithmetic and geometry where the transfinite numbers of Cantor play a preponderant rôle. Nevertheless, by the elegance and simplicity of his exposition, by the depth of his philosophic views, by the advantage he has derived from the fundamental idea, Hilbert has made the new geometry his own.

Non-arguesian Geometry.—The fundamental theorem of projective geometry is the theorem of Desargues.

Two triangles are called *homologs* when, the straights joining each to each, the corresponding vertices are copunctal. Desargues proved that the intersection points of the corresponding sides of two homologous triangles are costraight; the dual is equally true.

The theorem of Desargues can be established in two ways:

1. By using the projective assumptions of the plane and the metric assumptions of the plane.

2. By using the projective assumptions of the plane and those of space.

Therefore, the theorem could be discovered by a two-dimensional animal, to whom a third dimension would seem as inconceivable as to us a fourth; who consequently would not know the projective assumptions of space, but who would have seen displaced in the plane he inhabited rigid figures analogous to our solid bodies, and who consequently would know the metric assumptions. Equally well the theorem could be discovered by a tri-dimensional animal who should know the projective assumptions of space, but who, having never seen solid bodies displaced, would not know the metric assumptions.

But would it be possible to establish the theorem of Desargues without using either the projective assumptions of space or the metric assumptions, but only the projective assumptions of the plane? It was thought not, but we were not sure. Hilbert has settled the question by constructing a *non-arguesian geometry,* which is of course a plane geometry.

Non-pascalean Geometry.—Hilbert does not stop there; he introduces still a new conception. To understand it, we must return a moment to the domain of arithmetic. We have above seen the notion of number enlarged by the introduction of *non-archimedean numbers.* We need a classification of these new numbers, and to get it we first classify the assumptions of arithmetic into four groups:

1. The laws of associativity and of commutativity of addition, the associative law for multiplication, the two laws of distributivity of multiplication; or, to summarize, all the rules of addition and of multiplication, save the law of the commutativity of multiplication.

2. The assumptions of order, that is to say, the rules of the calculus of inequalities.

3. The law of commutativity of multiplication according to which we can invert the order of the factors without changing the product.

4. The Archimedes assumption.

The numbers which admit the first two groups are called *arguesian;* they may be *pascalean* or *non-pascalean,* according as they satisfy or do not satisfy the assumption of the third group; they will be *archimedean* or *non-archimedean,* according as they satisfy or do not the assumption of the fourth group. We soon shall see the reason for these names.

The ordinary numbers are at once arguesian, pascalean and archimedean. We can prove the law of commutativity from the assumptions of the first two groups and the Archimedes assumption; so there are no numbers arguesian, archimedean and not pascalean.

On the other hand, it is easy to make a system of numbers arguesian, non-pascalean and non-archimedean. The elements of this system will be series of the form

$$S = T_0 s^n + T_1 s^{n-1} + \cdots,$$

where s is a symbol analogous to t, n an integer positive or negative, and T_0, T_1, $\cdots$ numbers of the system T. If therefore we replace the coefficients T_0, T_1, $\cdots$ by the corresponding series in t, we shall have a series depending at the same time upon t and upon s. We add these series S according to the ordinary rules, and likewise for the multiplication of these series, we shall admit the rules of distributivity and of associativity, but we shall hold that the law of commutativity is not true and that, on the contrary, $st = -ts$.

It remains to *arrange* these series in an order so determined as to satisfy the assumptions of order. For that, we give to the series S the sign of the first coefficient

T_0; we shall say that a series is less than another, when if taken away from this, it gives a positive difference. This, therefore, is always the same rule: t is looked upon as very great with regard to any ordinary real number, and s is looked upon as very great with regard to any number of the system T.

The law of commutativity not being true, these now are non-pascalean numbers.

Before going farther I recall that Hamilton long ago introduced a system of complex numbers where the multiplication is not commutative; these are the *quaternions,* which the English so often use in mathematical physics. But, for quaternions the assumptions of order are not true; what therefore is original in Hilbert's conception is that his new numbers satisfy the assumptions of order without satisfying the rule of commutativity.

To return to geometry. Admit the assumptions of [the first] three groups, that is to say, the projective assumptions of the plane and of space, the assumptions of order, and Euclid's postulate; the theorem of Desargues will follow from them since it is a consequence of the projective assumptions of space.

We wish to establish our geometry *without using metric assumptions;* the word *length* therefore has now for us no meaning; we have no right to use the compasses; on the other hand, we may use the ruler, since we admit that we may pass a straight through two points, in virtue of one of the projective assumptions; equally we know how through a point to draw a parallel to a given straight, since we admit Euclid's postulate. Let us see what we can do with these resources.

We can define the homothety (perspective similarity) of two figures; and through it proportion. We can also define equality in a certain measure.

The two opposite sides of a parallelogram shall be equal *by definition;* thus we know how to recognize whether two sects are equal to one another, *provided they be parallel.*

Thanks to these conventions, we now are prepared to compare the lengths of two sects, but *with the proviso that these sects be parallel.*

The comparison as to length of two sects differing in direction has no meaning, and there would be needed, so to speak, a different unit of length for each direction. It is unnecessary to add that the word *angle* has no meaning.

Sects will thus be expressed by numbers; but necessarily these will not be ordinary numbers. All we can say is that if the theorem of Desargues is true, as we suppose, these numbers will belong to an *arguesian system.*

Inversely, having given any system S of arguesian numbers, we can make a geometry such that the lengths of the sects of a straight may be exactly expressed by these numbers.

The equation of the plane will be a linear equation as in the ordinary analytic geometry; but since in the system S multiplication will not be commutative, in general it is needful to make a distinction and to say that in each of the terms of this linear equation the coordinate will play the rôle of multiplicand, and the constant coefficient the rôle of multiplier.

Thus to each system of arguesian numbers will correspond a new geometry satisfying the projective assumptions and those of order, the theorem of Desargues and Euclid's postulate. What now is the geometric meaning of the arithmetical assumption of the third group, that is to say, of the rule of commutativity of multiplication?

The geometric translation of this rule is

Pascal's theorem; I mean the theorem about the hexagon inscribed in a conic, supposing that this conic reduces to two straights. So Pascal's theorem will be true or false according as the system S is pascalean or non-pascalean; and since there are non-pascalean systems, *there likewise are non-pascalean geometries.* The theorem of Pascal can be deduced from the metric axioms; it therefore will be true if we suppose figures may be transformed not only by homothety and translation, as we have done, but also by rotation. Pascal's theorem can likewise be deduced from the Archimedes axiom, since we have seen that every system of numbers arguesian and archimedean is at the same time pascalean; *every non-pascalean geometry is therefore at the same time non-archimedean.*

The Sect-carrier.—We cite still another conception of Hilbert's. He studies the constructions we can make, not with ruler and compasses, but with ruler and a special instrument which he calls the *sect-carrier,* and which enables us to set off on a straight a sect equal to another sect taken on another straight. The *sect-carrier* is not the equivalent of the compasses; this latter instrument enables us to construct the intersection of two circles, or of a circle and any straight; the *sect-carrier* will only give us the intersection of a circle and a straight *passing through the center of this circle.* Hilbert seeks therefore what are the constructions which are possible with these two instruments, and he reaches a very remarkable conclusion.

The constructions which can be achieved with ruler and compasses can likewise be made with the ruler and the *sect-carrier, provided these constructions are such that their result is always real.*

It is evident in fact that this condition is necessary, because a circle is always cut *in two real points* by a straight drawn through its center. But it was hard to foresee that this condition would likewise be sufficient.

But this is not all; in all these constructions, as Kürschák first noticed, it is possible to replace the *sect-carrier* by the *unit-sect carrier,* an instrument which enables us to set off on any straight from any point of it, no longer any sect, but a sect equal to unity.

An analogous question is treated in another article of Hilbert's: *On the equality of the angles at the base of an isosceles triangle.*

In the ordinary plane geometry, the plane is symmetric, which expresses itself in the equality of the angles at the base of the isosceles triangle.

We should make this *symmetry of the plane* appear in the list of metric assumptions. In all the geometries more or less strange of which we have spoken hitherto, in those at least where we admit the metric assumptions, in the non-archimedean metric geometry, in the new geometries of Dehn, in those which are the subject of the memoir "On a New Foundation, etc.," this symmetry of the plane is always supposed. Is it a consequence of the other metric assumptions? Yes, as Hilbert shows, if we admit the Archimedes assumption. No, in the contrary case. There are non-archimedean geometries where all the metric assumptions are true with the exception of this of the symmetry of the plane.

In this geometry it is not true that the angles at the base of an isosceles triangle are equal; it is not true that in a triangle one side is less than the sum of the other two; the theorem of Pythagoras about the square on the hypothenuse is not true. That is why this geometry is called *non-pythagorean.*

I come to an important memoir of Hilbert's which is entitled "Foundations of Geometry," which bears then the same title as his "Festschrift," but where he takes, however, a wholly different point of view. In his "Festschrift," in fact, as we have seen by the preceding analysis, the relations of the notion of space and the notion of group resulting from the works of Lie are laid aside or relegated to an inferior place. The general properties of groups do not appear in the list of fundamental assumptions. Not so in the memoir of which we are to speak.

As regards the ideas of Lie, the progress made is considerable. Lie supposed his groups defined by analytic equations. Hilbert's hypotheses are far more general. Without doubt this is still not entirely satisfactory, since though the *form* of the group is supposed any whatever, its *matter*, that is to say, the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, and besides Hilbert analyzes better than any one before him the idea of *number-manifold* and gives outlines which may become the germ of an assumptional theory of analysis situs.

It is impossible not to be struck by the contrast between the point of view here taken by Hilbert and that adopted in his "Festschrift." In this "Festschrift" the continuity assumptions took lowest rank and the great thing was to know what geometry became when they were put aside. Here, on the contrary, continuity is the point of departure and Hilbert is above all anxious to see what we get from continuity alone, joined to the notion of group.

It remains for us to speak of a memoir entitled "Surfaces of Constant Curvature."

We know that Beltrami has shown that there are in ordinary space surfaces which image the non-euclidean plane; these are the surfaces of constant negative curvature; we know what an impulse this discovery gave to the non-euclidean geometry. But is it possible to represent the non-euclidean plane entire on a Beltrami surface without singular point? Hilbert has proved that it is not.

As to the surfaces of constant positive curvature, to which Riemann's geometry corresponds, Hilbert proves that besides the sphere there is no other closed surface of this sort.

(*To be concluded*)

SCIENTIFIC NOTES AND NEWS

DR. DAVID STARR JORDAN has tendered to President Taft his resignation as international commissioner of fisheries, this position having been created three years ago under the treaty of April 11, 1908, with Great Britain. Under the terms of the appointment, the work of the commissioner ceases on the completion of the series of fishery regulations of the boundary waters, and the technical investigations necessary for their completion. This work being finished, the administration of the treaty passes to the Bureau of Fisheries.

DR. WILLY KUKENTHAL, professor of zoology at Breslau, has been appointed exchange professor at Harvard University during the academic year of 1911–12.

DR. EDWARD MINER GALLAUDET has retired from the presidency of Gallaudet College, which he has held for fifty-four years.

DR. OSCAR RIDDLE, of the University of Chicago, has returned from a year of study and travel in Europe. He spent the past six months at the Zoological Station at Naples, whence he now returns to Chicago to take charge of the preparation for publication of the manuscripts left by the late Professor C. O. Whitman. He will also continue certain features of Professor Whitman's investigations.

PROFESSOR GEORGE E. SEVER has been elected president of the Columbia Chapter of