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# NEW METHODS OF CELESTIAL MECHANICS

## VOLUME I. PERIODIC SOLUTIONS, THE NON-EXISTENCE OF INTEGRAL INVARIANTS, ASYMPTOTIC SOLUTIONS

*by H. Poincaré*

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By H. Poincaré

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NEW METHODS OF CELESTIAL MECHANICS

## INTRODUCTION

The Three - Body Problem is of such importance in astronomy, and is at the same time so difficult, that all efforts of geometers have long been directed toward it. A complete and rigorous integration being manifestly impossible, we must turn to the processes of approximation. The methods first employed consisted of seeking developments in terms of powers of the masses. At the beginning of this century the achievements of Lagrange and Laplace and, more recently, Le Verrier's calculations, have added such a degree of perfection to these methods that until now they have been sufficient for practical use. I may add that they will suffice for some time to come in spite of some divergences in details. It is certain, nevertheless, that they will not always be adequate, which a little reflection makes easily understandable. /1*

The final goal of Celestial Mechanics is to resolve the great problem of determining if Newton's law alone explains all astronomical phenomena. The only means of deciding is to make the most precise observations, and then compare them to calculated results. This calculation can only be approximate, and it would be pointless to calculate to more decimals than observation can give us. It is therefore useless to ask more precision from calculation than from observation, but neither should we ask less. Furthermore, the approximation with which we can content ourselves today will be insufficient in several centuries. /2 And, in fact, even admitting the improbability of perfecting measurement instruments, the very accumulation of observations over several centuries will permit us to know the coefficients of the various inequalities with greater precision.

This era, when we will be obliged to relinquish old methods, is without doubt still quite distant. However, the theorist must anticipate it, because his work must precede, and often by a great number of years, that of the numerical calculator.

We must not believe that in order to obtain precise ephemerides over a great number of years it will suffice to calculate a greater number of terms in the developments implicit in the old methods.

These methods, which consist of developing the coordinates of the heavenly bodies in terms of the powers of the masses, have, in fact, a common character which is opposed to their use for long-term calculation of the ephemerides. The series obtained contain terms called secular, where time occurs outside the sine and cosine terms, with the result that their convergence would become doubtful if we were to give this time t a large value.

The presence of these secular terms is not basic to the nature of the problem, but only to the method used. It is easy to realize, in fact, that if the true expression of a coordinate contains a term in

$$\sin \alpha mt,$$

*Numbers given in the margin indicate pagination in original foreign text.

$\alpha$ being a constant and m being one of the masses, we will find, when we wish to develop in powers of m, the secular terms

$$\alpha m t - \frac{\alpha^3 m^3 t^3}{6} + \dots,$$

and the presence of these terms would give a very false idea of the true form of the particular function.

This is a point on which astronomers (including the founders of Celestial Mechanics themselves) have long agreed: under all circumstances when they wanted to obtain formulas applicable to long time periods, as for example in the calculation of secular inequalities, they have had to operate differently /3
and give up the method of development only in powers of the masses. The study of secular inequalities by means of a system of linear differential equations with constant coefficients can, therefore, be regarded as related to new methods rather than old.

All efforts of geometers in the second half of this century have had as main objective the elimination of secular terms. The first serious attempt made in this direction was that of Delaunay, whose method is still used, without doubt, to great advantage.

We will also cite Hill's researches on Lunar Theory (American Journal of Mathematics, Vol. 1; Acta mathematica, Vol. VIII). In this work, unfortunately incomplete, we are permitted to perceive the germ of the major part of the progress which science has since made.

However, the scientist who has given this branch of astronomy the most eminent service is without question Gyldén. His work touches all parts of Celestial Mechanics, and it uses with ease all resources of modern analysis. Gyldén has succeeded in eliminating entirely from his development all secular terms, which so troubled his predecessors.

On the other hand, Lindstedt has proposed a method much simpler than that of Gyldén, but of less power, because it is no longer applicable when we are confronted with those terms which Gyldén calls critical.

Through the efforts of these scientists, the difficulty arising from the secular terms can be regarded as finally overcome, and the new processes will probably long satisfy practical requirements.

However, all is not yet resolved. Most of these developments are not convergent in the sense that geometers understand this word. Without doubt, this has little importance for the moment, since we are certain that calculation of the first terms provides a very satisfactory approximation. However, it is no less true that these series are not capable of giving an arbitrarily close approximation. Thus there will come a time when they too will be insufficient. In addition, certain theoretical consequences which we might be tempted to /4
obtain from the form of these series are not legitimate because of their divergence. For this reason they fail to resolve the problem of the stability of the solar system.

The discussion of the convergence of these developments should attract the attention of geometers, first for the reasons I have just given, and in addition for the following: the objective of Celestial Mechanics has not been attained when the ephemerides have been only more or less approximately calculated, unless we can account for the degree of approximation obtained. If we in fact verify a divergence between these ephemerides and observations, we must be able to recognize if Newton's law is at fault or if all can be explained by imperfection in theory. It is therefore important to determine an upper limit for the error committed, a factor with which we have, until now, perhaps not been sufficiently concerned. Thus the methods which permit determination of convergence at the same time give us this upper limit, which therefore increases its importance and usefulness. No one should be astonished at the space I will allow them in this work, even though I may not have given the fullest account.

I have concerned myself with these questions, and I have devoted a memoir to them which appeared in Volume XIII of Acta mathematica. I have especially made every effort to display the rare results relative to the Problem of Three Bodies, which can be established with the absolute rigor demanded by mathematicians. It is this rigor which alone gives some value to my theorems on the periodic, asymptotic and doubly asymptotic solutions. Here, we can find in fact, a solid ground which can be confidently relied on, and which will be a great advantage in all research areas, even in those not restricted to the same rigor.

On the other hand, it has appeared to me that my results permitted me to unite, in a sort of synthesis, the greater part of the new methods recently proposed, and this is what made me decide to undertake the present work.

In this first volume I have had to restrict myself to the study of periodic solutions of the first order, to the demonstration of the non-existence of uniform integrals, and to the exposition and discussion of Lindstedt's methods.

I will devote the subsequent volumes to the discussion of Gyldén's /5
methods, the theory of integral invariants, the question of stability, the study of periodic solutions of the second order, of asymptotic and doubly asymptotic solutions, and finally to the results which I may obtain between now and their publication.

Moreover, in the following volumes I will doubtless be forced to return to the materials presented in Volume I. Logic will suffer somewhat from this, it is true, but it is impossible to do otherwise in a branch of science which is in the process of formation and where there is continuous progress. I therefore excuse myself in advance.

One last remark: it is usual to put the results in the form most convenient for calculation of the ephemerides by expressing the coordinates as explicit functions of time. This procedure obviously has many advantages, and I have conformed to it as often as possible. However, I have not always done so, and I have frequently put my results in the form of integrals, i.e., in the form of implicit relations between the coordinates alone or between the coordinates and time. These relations can first be used to determine the formulas which give the coordinates explicitly. But this is not all; the true aim of Celestial Mechanics is not to calculate the ephemerides, because for this purpose we could be satisfied with a short-term forecast, but to ascertain whether Newton's law is sufficient to explain all the phenomena. From this point of view, the implicit relations which I have just spoken of can serve just as well as explicit formulas. In fact, it is sufficient to substitute in them the observed values of the coordinates and to verify whether they are satisfied.

# CHAPTER 1. GENERALITIES AND THE JACOBI METHOD

## Generalities

1. Before beginning my principal subject, I must go into certain preliminary details and briefly summarize the fundamental principles of Jacobi's "Vorlesungen über Dynamik" and Cauchy's theory relative to the integration of differential equations by series. I will therefore devote this first chapter to the exposition of Jacobi's method, limiting myself for the most part to stating some results whose demonstration is well known. /7

Let us first give some explanations on the subject of notation and definition which will be used throughout this memoir.

The differential equations with which we will deal will have the following form

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n, \tag{1}$$

$X_1, X_2, \dots, X_n$ being analytic and uniform functions of the n variables $x_1, x_2, \dots,$ $x_n$. As for the independent variable t, which we will consider as representing time, we will suppose most often that it does not enter explicitly into the functions X.

System (1) can be considered of the order n since it equals a single differential equation of the order n, but if the functions X are independent of t this order can be decreased by one. To do so, we need only eliminate time and write equations (1) in the form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n}.$$

In order to avoid all confusion, we will fix the meanings of the words solution and integral according to what follows. /8

If equations (1) are satisfied when we set

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \dots, \quad x_n = \varphi_n(t), \tag{2}$$

we will say that equations (2) define a particular solution of equations (1).

If a certain function of $x_1, x_2, \dots, x_n$,

$$F(x_1, x_2, \dots, x_n),$$

remains constant by virtue of equations (1), we will say that this function F is a particular integral of system (1).

It is clear that knowledge of an integral permits reduction of the order of the system by unity.

In problems of Dynamics, equations (1) are presented in a more special form, known under the name of Hamiltonian or the canonical form.

Variables are separated into two sets; we will regularly designate by

$$x_1, x_2, \ldots, x_p$$

variables of the first set and by

$$y_1, y_2, \ldots, y_p$$

those of the second set, and the differential equations will be written

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots, p), \tag{3}$$

F being a uniform function of the 2p variables x and y.

These equations admit a particular integral which is the function F itself and which is known under the name of vis viva integral.

We say that $x_1, y_1, x_2, y_2, \ldots, x_p, y_p$ form p pairs of conjugate variables.

We will say, from the example set by the English, that system (3) contains p degrees of freedom. This system is of the order 2p; however, knowledge of the vis viva integral permits decreasing this order by unity. Time not entering explicitly into the second members of equations (3), by eliminating time, as we /9
said above, we can again decrease the order by unity, such that finally a system containing p degrees of freedom can always be reduced to one of the order 2p-2.

We know, for example, that if there is only one degree of freedom, the system can be reduced to the order 0, i.e., completely integrated.

## Examples of Canonical Equations

2. The simplest case of equations of Dynamics is that where we study the motion of q free mass points in space. Let $m_1$ be the mass of the first of these points, $x_1, x_2, x_3$ its Cartesian coordinates; in the same manner let $m_2$ be the mass of the second of these points, $x_4, x_5, x_6$, its coordinates, and so forth; finally, let $m_q$ be the mass of the q-th point, $x_{3q-2}, x_{3q-1}$ and $x_{3q}$, its coordinates.

Let us project the quantity of motion of point $m_1$ on the three axes: let $y_1$, $y_2$, $y_3$ be the three projections; similarly let $y_4$, $y_5$, $y_6$ be the projections of the quantity of motion of point $m_2$, etc.; finally, let $y_{3q-2}$, $y_{3q-1}$, $y_{3q}$ be the projections of the quantity of motion of point $m_q$.

Let $F_1$, $F_2$, $F_3$ be the components of the force which acts on $m_1$; let $F_4$, $F_5$, $F_6$ be the components of the force which acts on $m_2$, etc.; finally, let $F_{3q-2}$, $F_{3q-1}$, $F_{3q}$ be the components of the force which acts on $m_q$.

We will suppose that the components F depend only on the 3q coordinates x. If there is conservation of energy, there will exist a function V of the coordinates x, called the force function, and such that

$$F_i = \frac{dV}{dx_i}.$$

The semi-vis viva T will be expressed as

$$T = \frac{y_1^2 + y_2^2 + y_3^2}{2m_1} + \frac{y_4^2 + y_5^2 + y_6^2}{2m_2} + \dots + \frac{y_{3q-2}^2 + y_{3q-1}^2 + y_{3q}^2}{2m_q},$$

and we will be able to write the vis viva equation as

$$T - V = \text{const.}$$

If I set

$$T - V = F(x_1, x_2, \dots, x_{3q}; y_1, y_2, \dots, y_{3q}),$$ /10

the equations of motion will be written

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \dots, 3q). \tag{1}$$

Thus the equations of the motion of q free mass points allow 3q degrees of freedom whenever the forces depend only on the positions of these points in space and there is conservation of energy. In particular the Problem of Three Bodies allows 9 degrees of freedom. We will see below that this number can be lowered considerably.

If our q mass points all move in the same plane, the position of each of these points will be defined no longer by three coordinates, but only by two. The number of degrees of freedom will consequently be reduced to 2q.

Thus, when the orbits of the three bodies are plane and all three are situated in the same plane, the Problem of Three Bodies (which we will then call the Problem of Three Bodies in a plane) will allow no more than 6 degrees of freedom.

The case where there is only one degree of freedom being immediately integrable, we will consider throughout the case presented immediately after, that is to say, the case where there are only 2 degrees of freedom. Most of the results which follow will only be applied to this relatively simple case.

In many mechanical problems the number of degrees of freedom can, in fact, be reduced to 2. This is what occurs, for example, when we study the motion of a free mass point in a plane or, more generally, the motion of a mass point constrained to a surface, whenever the force depends only on the position of this point. We will cite among others the well-known problem of the moving body attracted by two fixed centers, when the initial velocity of the moving point is in the plane of the three bodies.

But this is a somewhat more complicated case, which will have greater importance later.

Let there be in a plane two rectangular axes $O\xi$ and $O\eta$ moving with uniform rotational motion about the origin O. Let n be the angular velocity of this rotational motion. Let P be a point moving in this same plane, whose coordinates with respect to these two axes are called $\xi$ and $\eta$ and whose mass will be taken as unity. /11

Let V be a function of the forces depending only on $\xi$ and on $\eta$, such that the projections on $O\xi$ and on $O\eta$ of the force which acts on point P are, respectively, $\frac{dV}{d\xi}$ and $\frac{dV}{d\eta}$.

The equations of the relative motion of point P with respect to the moving axes $O\xi$ and $O\eta$ are written

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} &= \frac{dV}{d\xi} + n^2\xi, \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} &= \frac{dV}{d\eta} + n^2\eta, \end{aligned}\right\} \qquad (2)$$

whence we deduce the following integral, called the Jacobi integral,

$$\frac{1}{2}\left[\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right] - V - \frac{n^2}{2}(\xi^2 + \eta^2) = \text{const.},$$

which is nothing other than the vis viva integral of the relative motion.

I state that these equations can be reduced to canonical form, the number of the degrees of freedom being equal to 2.

Let us set, in fact,

$$\xi = x_1, \qquad \eta = x_2,$$

$$\frac{d\xi}{dt} - n\eta = y_1, \qquad \frac{d\eta}{dt} + n\xi = y_2,$$

$$F = \frac{1}{2}(y_1 + nx_2)^2 + \frac{1}{2}(y_2 - nx_1)^2 - V - \frac{n^2}{2}(x_1^2 + x_2^2);$$

equations (2) will become

$$\frac{dx_1}{dt} = \frac{dF}{dy_1}, \quad \frac{dx_2}{dt} = \frac{dF}{dy_2}, \quad \frac{dy_1}{dt} = -\frac{dF}{dx_1}, \quad \frac{dy_2}{dt} = -\frac{dF}{dx_2}. \qquad \text{Q. E. D.}$$

One of the special cases of the Problem of Three Bodies goes back to the problem which we have just treated.

Let us suppose that one of the three masses is infinitesimal such that the motion of the two other masses, being unperturbed, remains Keplerian. Such, for example, would be the case of the motion of a small planet in the presence of Jupiter and the Sun. /12

Let us imagine the eccentricity of the orbits of the two large masses to be 0, such that these two masses describe with uniform motion two concentric circumferences about the common center of gravity, assumed fixed.

Let us suppose finally, the inclination of the orbits being zero, that the small mass moves constantly in the plane of these two circumferences.

The center of gravity of the system, which is the common center of the two circumferences, can always be assumed fixed: we will take it as the origin; we make two moving axes $O\xi$ and $O\eta$ pass around this origin: the axis $O\xi$ will be the straight line which joins the two large masses; the axis $O\eta$ will be perpendicular to $O\xi$.

We see that:

(1) These two axes are moving with a uniform rotational motion;
(2) The two large masses are fixed with respect to the moving axes. We must therefore study the relative motion of a moving point, in relation to two moving axes, under the attraction of two centers fixed with respect to these axes. We therefore return to the question we have just considered.

Thus, in this special case, the equations of the Problem of Three Bodies can be reduced to the canonical form with only two degrees of freedom.

We now proceed to an equation often encountered in the theory of perturbations, and one which Gylden uses frequently.

Let

$$\frac{d^2x}{dt^2} = f(x, t). \qquad (3)$$

This equation can also be reduced to the canonical form.

In fact, $f(x,t)$ can always be regarded as the derivative with respect to $x$ of a certain function $\varphi(x,t)$, such that

$$f = \frac{d\varphi}{dx}.$$

If we now set /13

$$x = x_1, \quad \frac{dx}{dt} = y_1, \quad t = y_2,$$

$$F = \frac{y_1^2}{2} - \varphi(x_1, y_2) - x_2,$$

equation (3) can be replaced by the canonical equations (3) from the preceding article with only 2 degrees of freedom. Q. E. D.

I will cite one last example. Let us consider a solid heavy body suspended at a fixed point, and let us study the oscillations of this body. In order to completely define this body's position, we must be given three conditions; we must, in fact, know the three Euler angles formed by a system of axes invariably connected to the body with a system of fixed axes.

The problem will therefore contain 3 degrees of freedom; however, we will later see that this number can be reduced to 2.

I have said enough of this to show how many problems of Mechanics lead to the integration of a canonical system having 2 degrees of freedom and to make the importance of these systems understood; it is therefore unnecessary to multiply the examples still more.

## First Jacobi Theorem

3. Jacobi has shown that the integration of the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{1}$$

reduces to the integration of one partial differential equation

$$F(x_1, x_2, \ldots, x_p; y_1, y_2, \ldots, y_p) = h_1, \tag{2}$$

where $h_1$ is an arbitrary constant and where $y_1, y_2, \ldots, y_p$ are assumed to represent the partial derivatives of the unknown function.

Let, in fact,

$$S(x_1, x_2, \ldots, x_p; h_1, h_2, \ldots, h_p)$$

be a solution of equation (2) containing, in addition to the constant $h_1$, p-1 constants of integration

$$h_2, \quad h_3, \quad \ldots, \quad h_p,$$

such that we have, for any h, /14

$$F\left(x_1, x_2, \ldots, x_p; \frac{dS}{dx_1}, \frac{dS}{dx_2}, \ldots, \frac{dS}{dx_p}\right) = h_1.$$

Jacobi has shown that the general integral of equations (1) can be written

$$\left.\begin{aligned}\frac{dS}{dx_i} &= y_i \quad (i = 1, 2, \ldots, p),\\ \frac{dS}{dh_i} &= h'_i \quad (i = 2, 3, \ldots, p),\\ \frac{dS}{dh_1} &= t + h'_1.\end{aligned}\right\} \tag{3}$$

The 2p constants of integration are then

$$h_1, \ h_2, \ \ldots, \ h_p,$$
$$h'_1, \ h'_2, \ \ldots, \ h'_p.$$

Another theorem which we will make use of is that of Poisson.

Let U and V be two arbitrary functions of x and y. We agree to write

$$[U, V] = \sum_{i=1}^{i=p} \left( \frac{dU}{dx_i}\frac{dV}{dy_i} - \frac{dU}{dy_i}\frac{dV}{dx_i} \right).$$

Now let $F_1$ and $F_2$ be two integrals of equations (1). We see immediately that we will express $F_1$ as an integral of equations (1), while writing

$$[F, F_1] = 0.$$

$F_2$ also being an integral, we will have in the same manner

$$[F, F_2] = 0.$$

Poisson has demonstrated that the expression $[F_1, F_2]$ is similarly an integral of equations (1). It is thus that, in the problem of n bodies, if we assume that $F_1$ and $F_2$ are the first members of the first and second area equations, $[F_1, F_2]$ will be the first member of the third area equation. /15

### Second Jacobi Theorem; Changes of Variables

4. We will not ordinarily retain the rectangular coordinates and the components of the quantities of motion as independent variables. We will select those better suited for our purposes, attempting always to retain the canonical form for the equations.

Let us therefore see how we can change variables without altering the canonical form of equations (1).

Let

$$S(y_1, y_2, \ldots, y_p; h_1, h_2, \ldots, h_p)$$

be an arbitrary function of p variables y and of the new p variables h.

Let us now set

$$x_i = \frac{dS}{dy_i}, \qquad h'_i = \frac{dS}{dh_i}. \tag{4}$$

Equations (4) are regarded as defining the relations which connect the old variables

$$x_1,\ x_2,\ \dots,\ x_q,$$
$$y_1,\ y_2,\ \dots,\ y_q$$

to the new variables

$$h_1,\ h_2,\ \dots,\ h_q,$$
$$h'_1,\ h'_2,\ \dots,\ h'_q.$$

Jacobi has demonstrated that if we make this change of variables, the equations will remain canonical and do so whatever the function S may be.

### Special Changes of Variables

5. Save for an exceptional case, all changes of variables which do not alter the canonical form can be deduced from the process in article 4. /16
However, there are some cases where it is simpler to operate otherwise. We will give two examples.

Let us assume that we have the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{1}$$

and that we make the change of variables according to

$$\left.\begin{aligned} x_i &= \alpha_{1.i}x'_1 + \alpha_{2.i}x'_2 + \dots + \alpha_{n.i}x'_n, \\ y_i &= \beta_{1.i}y'_1 + \beta_{2.i}y'_2 + \dots + \beta_{n.i}y'_n. \end{aligned}\right\} \tag{2}$$

How must we choose the constants $\alpha$ and $\beta$ so that the equations remain canonical when we take $x'_i$ and $y'_i$ as new variables?

If we designate by

$$\delta x_1,\ \delta x_2,\ \dots,\ \delta x_n;\ \delta y_1,\ \delta y_2,\ \dots,\ \delta y_n$$

the real increments in x and y, and multiply equations (1) respectively by $\delta y_i$ and $-\delta x_i$ and then add them, it will follow that

$$\sum\left(\frac{dx_i}{dt}\delta y_i - \frac{dy_i}{dt}\delta x_i\right) = \delta F.$$

In order for the equations to remain canonical after substitution (2), it is thus necessary and sufficient that we have identically

$$\sum\left(\frac{dx_i}{dt}\delta y_i - \frac{dy_i}{dt}\delta x_i\right) = \sum\left(\frac{dx'_i}{dt}\delta y'_i - \frac{dy'_i}{dt}\delta x'_i\right). \tag{3}$$

Since $dx_i$ depend only on $dx'_i$, $\delta y_i$ on $\delta y'_i$, $dy_i$ on $dy'_i$ and $\delta x_i$ on $\delta x'_i$, we must have identically

$$\Sigma\, dx_i\,\delta y_i = \Sigma\, dx'_i\,\delta y'_i, \qquad \Sigma\, dy_i\,\delta x_i = \Sigma\, dy'_i\,\delta x'_i. \tag{4}$$

Relations (2) being linear, $dx_i$ are related to $dx'_i$, and $\delta x_i$ to $\delta x'_i$

by the same relations which subsist between $x_i$ and $x'_i$. The same is true for $dy_i$, $\delta y'_i$, $y_i$, $dy'_i$, $\delta y'_i$, $y'_i$.

Relations (4) will therefore hold when we replace in them $dx_i$ and $\delta x_i$ by $x_i$, and $dy_i$ and $\delta y_i$ by $y_i$, $dx'_i$ and $\delta x'_i$ by $x'_i$, etc. We must therefore have

$$\Sigma x_i y_i = \Sigma x'_i y'_i. \qquad (5)$$

The reciprocal is true and relation (5) implies relations (3) and (4).

Thus, the condition necessary and sufficient for the equations to remain canonical is that we have identically /17

$$\Sigma x_i y_i = \Sigma x'_i y'_i.$$

What is now the condition for these equations to remain canonical and at the same time for us to have

$$\alpha_{k.i} = \beta_{k.i}?$$

I will say that a linear change of variables, such as (2), is orthogonal if we have identically

$$\Sigma x_i^2 = \Sigma x'^2_i,$$

i.e., if we have

$$\sum_{i=1}^{i=n} \alpha_{ki}^2 = 1, \quad \sum_{i=1}^{i=n} \alpha_{ki}\alpha_{h.i} = 0.$$

This definition justifies itself since, in the case where the number of variables is 2 or 3, and where we can regard x or the x' as the coordinates of a point in the plane or in space, a similar substitution is nothing other than a rectangular change of coordinates.

With this stated, if we make a similar orthogonal substitution give way to x and to y, we will have

$$\Sigma x_i^2 = \Sigma x'^2_i, \qquad \Sigma y_i^2 = \Sigma y'^2_i,$$
$$\Sigma (x_i + y_i)^2 = \Sigma (x'_i + y'_i)^2,$$

whence

$$\Sigma x_i y_i = \Sigma x'_i y'_i.$$

The equations will therefore remain canonical.

6. The equations will still remain canonical if we make a change of variables depending only on $x_1$ and on $y_1$, for example, and if we place

$$x_1 = \varphi(x'_1, y'_1), \qquad y_1 = \psi(x'_1, y'_1),$$

and take $x'_1$ and $y'_1$ in place of $x_1$ and $y_1$ as new variables; these equations will remain canonical, I say, provided that the functional, or Jacobian, determinant of $x_1$ and $y_1$ with respect to $x'_1$ and $y'_1$ is equal to 1. /18

Thus, if we set

$$x_1 = \sqrt{2\rho}\cos\omega, \qquad y_1 = \sqrt{2\rho}\sin\omega,$$

the canonical form of the equations will not be altered and the variables $\rho$ and $\varpi$ will be conjugate as were $x_1$ and $y_1$.

7. Above we defined the change of variables

$$\frac{dS}{dy_i} = x_i, \qquad \frac{dS}{dh_i} = h'_i,$$

which does not alter the canonical form of the equations, when S is an arbitrary function of $y_i$ and $h_i$.

Nor is this form altered if we permute $x_i$ with $y_i$ and if we change F into -F at the same time.

If, therefore, S is an arbitrary function of

$$x_1,\ x_2,\ \ldots;\ x_p,\ h_1,\ h_2,\ \ldots,\ h_p$$

and if we set

$$y_i = \frac{dS}{dx_i}, \qquad h'_i = \frac{dS}{dh_i},$$

the canonical form of the equations will not be altered when we take $h_i$ and $h'_i$ as new variables, and when we change F into -F at the same time.

Nor will it be altered if we change

$$y_1,\ y_2,\ \ldots,\ y_n \text{ and } F$$

into

$$\lambda y_1,\ \lambda y_2,\ \ldots,\ \lambda y_n \text{ and } \lambda F,$$

$\lambda$ being an arbitrary constant.

Let us therefore consider still another function S of $x_i$ and of $h_i$, and let us set

$$\lambda y_i = \frac{dS}{dx_i}, \qquad h'_i = \frac{dS}{dh_i}.$$

The canonical form will not be altered if we take $h_i$ and $h'_i$ as new variables, and if we change F into $-\lambda F$ at the same time. /19

## Keplerian Motion

8. Let us apply the preceding principles to Keplerian motion.

In the following portions we always will suppose that the units have been chosen such that the attraction of the two units of mass to the unit of distance is equal to the unit of force or, in other words, that the Gaussian constant is equal to 1.

Let us therefore consider the motion of a moving mass under the influence of a fixed mass situated at the origin of the coordinates and equal to M. Let $x_1$, $x_2$, $x_3$ be the coordinates of the moving mass, and $y_1$, $y_2$, $y_3$ be the components of velocity; if we set

$$F = \frac{y_1^2+y_2^2+y_3^2}{2} = \frac{2M}{\sqrt{x_1^2+x_2^2+x_3^2}},$$

the equations of motion are written

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}. \tag{1}$$

According to article 3, the integration of these equations is reduced to that of the partial differential equation

$$\left(\frac{dS}{dx_1}\right)^2 + \left(\frac{dS}{dx_2}\right)^2 + \left(\frac{dS}{dx_3}\right)^2 - \frac{2M}{\sqrt{x_1^2+x_2^2+x_3^2}} = 2h, \tag{2}$$

where h is an arbitrary constant. Let us set

$$x_1 = r\sin\omega\cos\varphi, \qquad x_2 = r\sin\omega\sin\varphi, \qquad x_3 = r\cos\omega;$$

the equation will become

$$\left(\frac{dS}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS}{d\omega}\right)^2 + \frac{1}{r^2\sin^2\omega}\left(\frac{dS}{d\varphi}\right)^2 = \frac{2M}{r} + 2h.$$

We can satisfy this equation by introducing two arbitrary constants G and $\Theta$ and making /20

$$\left.\begin{aligned} &\frac{dS}{d\varphi} = \Theta\sqrt{M}, \qquad \left(\frac{dS}{d\omega}\right)^2 + \frac{\Theta^2 M}{\sin^2\omega} = G^2 M, \\ &\left(\frac{dS}{dr}\right)^2 + \frac{G^2 M}{r^2} = \frac{2M}{r} + 2h. \end{aligned}\right\} \tag{3}$$

The function S thus defined will depend on r, $\omega$, $\varphi$, G, $\Theta$, h or, what is the same, on $x_1$, $x_2$, $x_3$, G, $\Theta$, h, and the general solution of equations (1) will be written

$$y_i = \frac{dS}{dx_i}, \qquad h' + t = \frac{dS}{dh}, \qquad g = \frac{dS}{dG}, \qquad \theta = \frac{dS}{d\Theta},$$

h', g and $\theta$ being three new arbitrary constants. If we set

$$L = \sqrt{\frac{-M}{2h}}, \qquad h = -\frac{M}{2L^2}, \qquad n = \frac{M}{L^3}, \qquad l = n(t + h'),$$

we will be able to write

$$\frac{dS}{dL} = \frac{dS}{dh}\frac{dh}{dL} = (h' + t)\frac{M}{L^3} = n(h' + t) = l.$$

The integration constants then number six, namely

$$L, \quad G, \quad \Theta, \quad h', \quad g, \quad \theta.$$

It is easy to see the significance of these constants and to express them as functions of those which are usually used. If a, e and i designate the major axis, the eccentricity and the inclination, we have

$$L = \sqrt{a}, \qquad G = \sqrt{a(1 - e^2)}, \qquad \Theta = G \cos i.$$

On the other hand, $\theta$ is the longitude of the node, $g + \theta$ is that of the perihelion, n is the mean motion and $l$ is nothing other that the mean anomaly.

If the moving mass, instead of being subject to the attraction of the mass M, were subject to other forces, we could, nevertheless, construct the function S and then define six new variables

$$L, G, \Theta, l, g, \theta \tag{4}$$

as a function of $x_i$ and $y_i$ by equations /21

$$y_i = \frac{dS}{dx_i}, \qquad \frac{dS}{dL} = l, \qquad \frac{dS}{dG} = g, \qquad \frac{dS}{d\Theta} = \theta; \tag{5}$$

L, G, $\Theta$, g and $\theta$ would no longer be constants.

We can then use the six variables (4) to define the position and velocity of the moving mass. We will give these variables (4) the name Keplerian variables. It is important to remark that the definition of these Keplerian variables depends on the origin to which the moving mass is related and on the value chosen for M.

If the moving mass is a planet which is subjected to the preponderant action of the mass M and to various perturbing forces, we see that these Keplerian variables are nothing more than what astronomers call the osculating elements of this planet.

In the particular case when the orbit of the body $m_1$ is a plane, we can take

$$L = \sqrt{a}, \qquad G = \sqrt{a(1 - e^2)}$$

as new variables with the mean anomaly $l$ and perihelion longitude g. The Keplerian variables then number no more than 4.

It is important to make some remarks on the subject of the use of these Keplerian variables: we remark first that the old variables

$$x_1, \quad x_2, \quad x_3; \quad y_1, \quad y_2, \quad y_3$$

and the position of the body $m_1$ do not change when there is an increase in $l$, g or $\theta$ of $2\pi$, without touching the other variables. These old variables are therefore periodic functions of $l$, g and $\theta$.

In the second place, we must always have

$$L^2 \geq G^2 \geq \Theta^2.$$

Finally, if $G = \pm \Theta$, the old variables and the position of the body $m_1$ no longer depend on $\Theta$; and if $L = \pm G$, they no longer depend on g. /22

## Special Case of the Problem of Three Bodies

9. Let us return to the special case of the Problem of Three Bodies which we considered above.

Two masses, the first equal to $1-\mu$, the second equal to $\mu$, describe two concentric paths about their common center of gravity assumed fixed. The constant distance of these two masses is taken for the unit of length, in such a way that the radii of the two circumferences become respectively $\mu$ and $1-\mu$, the mean motion being equal to unity.

Let us now suppose that in the plane of these two circumferences there is a third moving body, infinitely small, and attracted by the first two.

We will take as origin O the common center of the two circumferences, and we will be able to relate the position of the third mass, either to the two fixed rectangular axes $Ox_1$ and $Ox_2$, or to the two moving axes $O\xi$ and $O\eta$ defined as in article 2. The mean motion of the first two masses being equal to 1, we can suppose that the angle of $O\xi$ and $Ox_1$ (i.e., the longitude of the mass $\mu$) is equal to t.

Since the Gaussian constant is assumed equal to 1, the force function reduces to

$$V = \frac{m_1 \mu}{r_1} + \frac{m_1(1-\mu)}{r_2},$$

calling $m_1$ the infinitely small mass of the third body, $r_1$ the distance between the two bodies $m_1$, $\mu$ and $r_2$ the distance from the body $m_1$ to the body of mass $1-\mu$, such that

$$r_1^2 = \eta^2 + (\xi + \mu - 1)^2 = [x_2 - (1-\mu)\sin t]^2 + [x_1 - (1-\mu)\cos t]^2,$$
$$r_2^2 = \eta^2 + (\xi + \mu)^2 = [x_2 + \mu \sin t]^2 + [x_1 + \mu \cos t]^2.$$

The vis viva equation is then written $\frac{y_1^2}{2m_1} + \frac{y_2^2}{2m_1} - V = \text{const.}$

We agree to call $-m_1R$ the first member of this equation. R will be a function of $x_1$, $x_2$, of $y_1$, $y_2$ and of t, and the equations of motion will be written /23

$$\frac{dx_1}{dt} = -\frac{d(m_1 R)}{dy_1}, \qquad \frac{dx_2}{dt} = -\frac{d(m_1 R)}{dy_2}.$$
$$\frac{dy_1}{dt} = \frac{d(m_1 R)}{dx_1}, \qquad \frac{dy_2}{dt} = \frac{d(m_1 R)}{dx_2}.$$

Let us replace the variables $x_1$, $y_1$, $x_2$, $y_2$ by their values as functions of the Keplerian variables L, G, $l$, g, as has been said in the preceding article. R will become a function of L, G, $l$, g and t, and the equations of motion will be written

$$\frac{dL}{dt} = \frac{dR}{dl}, \quad \frac{dl}{dt} = -\frac{dR}{dL}, \quad \frac{dG}{dt} = \frac{dR}{dg}, \quad \frac{dg}{dt} = -\frac{dR}{dG}.$$

These equations would already be in the canonical form, if R only depended on the four Keplerian variables, but R is also a function of t; it is therefore necessary to transform these equations, so that time does not enter explicitly. To do so, let us see how R depends on t.

It is easily seen that R can be regarded as a function of L, G, $l$, and g - t. If, in fact, we increase g and t by the same quantity, without touching the other variables, we change neither $\xi$ nor $\eta$, $r_1$, $r_2$, $y_1^2+y_2^2$, nor consequently R.

This results in

$$\frac{dR}{dt} + \frac{dR}{dg} = 0.$$

If we then set

$$x'_1 = L, \quad x'_2 = G,$$
$$y'_1 = l, \quad y'_2 = g - t,$$
$$F' = R + G,$$

F' will depend only on $x'_1$, $x'_2$, $y'_1$ and $y'_2$ and the equations of motion, which will be written

$$\frac{dx'_i}{dt} = \frac{dF'}{dy'_i}, \quad \frac{dy'_i}{dt} = -\frac{dF'}{dx'_i}, \tag{1}$$

will be canonical.

It is in this form that we will ordinarily write the equations of this problem. /24

When mass $\mu$ is assumed to be zero, the mass $1-\mu$ becomes equal to 1 and is related to the origin; $r_2$ reduces to $\sqrt{x_1^2 + x_2^2}$, the force function V reduces to $m_1/r_2$, and we find

$$R = \frac{1}{2a} = \frac{1}{2L^2} = \frac{1}{2x_1'^2}$$

and

$$F' = \frac{1}{2x_1'^2} + x'_2.$$

When $\mu$ is not zero, we see immediately that F' can develop in terms of the increasing powers of $\mu$, which allows us to write

$$F' = F_0 + \mu F_1 + \dots .$$

We see that

$$F_0 = \frac{1}{2x_1'^2} + x_2'$$

is independent of $y_1'$ and of $y_2'$.

In addition, $F_1$ will at the same time depend on the four variables; but this function will be periodic with respect to $y_1'$ and $y_2'$, and it will not change when one of these two variables increases by $2\pi$.

We observe finally that if $x_1' = +x_2'$ the eccentricity is zero and the motion is direct, and that $F_1$ then depends only on $x_1'$, $x_2'$ and $y_1'+y_2'$.

On the contrary, if $x_1'=-x_2'$, the eccentricity is zero, but the motion is retrograde, and $F_1$ then depends only on $x_1'$, $x_2'$ and $y_1'-y_2'$.

### Use of Keplerian Variables

10. Let $x_1$, $x_2$, $x_3$ be the rectangular coordinates of a point; $y_1$, $y_2$, $y_3$ its velocity components; m its mass. Let $V_m$ be the force function, so that the components of the force applied to the point are

$$m\frac{dV}{dx_1}, \quad m\frac{dV}{dx_2}, \quad m\frac{dV}{dx_3}.$$

If we set /25

$$F = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + V,$$

the equations of motion for the point will take the canonical form

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}.$$

In article 8 we defined a certain function

$$S(x_1, x_2, x_3, G, \theta, L).$$

We have seen that if we make the change of variables defined by the equations

$$\frac{dS}{dx_i} = y_i, \quad \frac{dS}{dG} = g, \quad \frac{dS}{d\theta} = \theta, \quad \frac{dS}{dL} = l,$$

the new variables are nothing other than the Keplerian variables we have just defined.

By virtue of the theorem of article 7, the equations will retain the canonical form and will be written

$$\frac{dL}{dt} = -\frac{dF}{dl}, \quad \frac{dG}{dt} = -\frac{dF}{dg}, \quad \frac{d\Theta}{dt} = -\frac{dF}{d\theta},$$
$$\frac{dl}{dt} = \frac{dF}{dL}, \quad \frac{dg}{dt} = \frac{dF}{dG}, \quad \frac{d\theta}{dt} = \frac{dF}{d\Theta}.$$

The force remaining constantly in the $x_1\ x_2$ plane, the same may be true of the moving point.

In this case we will constantly have

$$G = \Theta,$$

and the function F will depend only on G, L, $l$ and on the perihelion longitude $g + \theta = \varpi$; we will have

$$\frac{dF}{dg} = \frac{dF}{d\theta} = \frac{dF}{d\varpi}.$$

In order to maintain symmetry, we will set

$$G = \Theta = \Pi.$$

The number of Keplerian variables will be reduced from six to four, namely /26
$\Pi$, L, $\varpi$, and $l$, and the equations become

$$\frac{dL}{dt} = -\frac{dF}{dl}, \quad \frac{d\Pi}{dt} = -\frac{dF}{d\varpi}, \quad \frac{dl}{dt} = \frac{dF}{dL}, \quad \frac{d\varpi}{dt} = \frac{dF}{d\Pi}.$$

## General Case of the Problem of Three Bodies

11. We come to the general case of the Problem of Three Bodies: let ABC be the triangle formed by the three bodies; a, b, c the sides of this triangle; $m_1$, $m_2$, $m_3$ the masses of the three bodies.

The force function is then written

$$\frac{m_2 m_3}{a} + \frac{m_3 m_1}{b} + \frac{m_1 m_2}{c}.$$

We will call the force function $V\mu$, $\mu$ designating an arbitrary constant which we will determine more completely later.

I will suppose that the center of gravity of the three-body system is fixed and I will call D the center of gravity of the system of the two bodies A and B.

I will consider two systems with moving axes:

The first system, always parallel to the fixed axes, will have its origin in A.

The second system, also parallel to the fixed axes, will have its origin in D. I will call $x_1$, $x_2$, $x_3$ the coordinates of point B with respect to the first moving axes; $x_4$, $x_5$ and $x_6$ the coordinates of point C with respect to the second system of moving axes.

The total vis viva will then be expressed by[1]

$$\frac{m_1 m_2}{m_1+m_2}\left(\frac{dx_1^2}{dt^2}+\frac{dx_2^2}{dt^2}+\frac{dx_3^2}{dt^2}\right)+\frac{(m_1+m_2)m_3}{m_1+m_2+m_3}\left(\frac{dx_4^2}{dt^2}+\frac{dx_5^2}{dt^2}+\frac{dx_6^2}{dt^2}\right).$$

If we then set /27

$$\beta\mu=\frac{m_1 m_2}{m_1+m_2},\quad \beta'\mu=\frac{(m_1+m_2)m_3}{m_1+m_2+m_3},\qquad y_1=\beta\frac{dx_1}{dt},\quad y_2=\beta\frac{dx_2}{dt},\quad y_3=\beta\frac{dx_3}{dt},$$

$$F=\frac{T}{\mu}-V=\frac{y_1^2+y_2^2+y_3^2}{2\beta}+\frac{y_4^2+y_5^2+y_6^2}{2\beta'}-V,\quad y_4=\beta'\frac{dx_4}{dt},\quad y_5=\beta'\frac{dx_5}{dt},\quad y_6=\beta'\frac{dx_6}{dt},$$

the equations will take the canonical form

$$\frac{dx_i}{dt}=\frac{dF}{dy_i},\qquad \frac{dy_i}{dt}=-\frac{dF}{dx_i}.$$

Let us again take the function

$$S(x_1, x_2, x_3; L, G, \theta)$$

defined by equations (4) of article 8.

Let us construct it first by setting

$$M=m_1+m_2.$$

Let us then set

$$\frac{dS}{dL}=l,\qquad \frac{dS}{dG}=g,\qquad \frac{dS}{d\theta}=\theta. \tag{1}$$

Then we construct this same function S by having

$$M=m_1+m_2+m_3;$$

we call

$$S'(x_4, x_5, x_6; L', G', \theta')$$

the function thus constructed, and we set

$$\frac{dS'}{dL'}=l',\qquad \frac{dS'}{dG'}=g',\qquad \frac{dS'}{d\theta'}=\theta'. \tag{2}$$

Then let

$$\Sigma=\beta S+\beta' S'.$$

The derivatives of $\Sigma$ with respect to L, G, $\Theta$, L', G', $\Theta'$ will be, respectively, $\beta l$, $\beta g$, $\beta\theta$, $\beta' l'$, $\beta' g'$, $\beta'\Theta'$.

If in addition we set /28

$$y_i=\frac{d\Sigma}{dx_i}, \tag{3}$$

[1]Tisserand, Mécanique céleste, Chapter IV.

equations (1), (2) and (3) will define the 12 old variables x and y as functions of the 12 new variables, which I will divide into two series in the following manner:

$$\left.\begin{matrix} \beta L, & \beta G, & \beta\Theta, & \beta' L', & \beta' G', & \beta'\Theta', \\ l, & g, & \theta, & l', & g', & \theta'. \end{matrix}\right\} \qquad (4)$$

The theorem for articles 4 and 7 then shows that the canonical form of the equations is not altered.

It is easy to realize the significance of the new variables.

Everything occurs as if two masses, equal respectively to $\beta\mu$ and $\beta'\mu$, had for coordinates with respect to fixed axes, the first $x_1$, $x_2$, $x_3$, the second $x_4$, $x_5$, $x_6$ and as if these two fictitious masses were subjected to forces admitting the force function $V\mu$.

Then, if at some instant the forces applied to the first fictitious mass chance to disappear, and if they are replaced by the attraction of a mass $m_1 + m_2$ placed at the origin, this mass would move according to Kepler's laws and the elements of this Keplerian motion would be L, G, $\Theta$, $l$, g and $\theta$.

Furthermore, if the second fictitious mass were subjected only to the attraction of a fixed mass $m_1 + m_2 + m_3$ placed at the origin, the elements of the Keplerian motion which it would take would then be L', G', $\Theta'$, $l'$, g' and $\theta'$.

Let us observe that F does not depend only on the variables (4), but on $m_1$, $m_2$, $m_3$ and on $\mu$.

In general, $m_2$ and $m_3$ are very small, such that we may set

$$m_2 = \alpha_2\mu, \quad m_3 = \alpha_3\mu,$$

considering $\mu$ small, and retain finite values for $\alpha_2$, $\alpha_3$, $\beta$ and $\beta'$; F, which can then be regarded as a function of the variables (4) of $m_1$, $\alpha_2$, $\alpha_3$ and of $\mu$, could then be advantageously developed in terms of the increasing powers of $\mu$

$$F = F_0 + F_1\mu + \dots.$$

If we set $\mu = 0$, it follows that /29

$$V = \frac{\alpha_3 m_1}{b} + \frac{\alpha_2 m_1}{c}, \qquad \beta = \alpha_2, \qquad \beta' = \alpha_3,$$

and

$$F = F_0 = \frac{\beta^3}{2(\beta L)^2} + \frac{\beta'^3}{2(\beta' L')^2} = \frac{\alpha_2^3}{2(\beta L)^2} + \frac{\alpha_3^3}{2(\beta' L')^2};$$

F then no longer depends on any of the variables of the second series $l$, $g$, $\theta$, $l'$, $g'$, $\theta'$; I will add that, whatever $\mu$ may be, F is a periodic function of period $2\pi$ with respect to those variables of the second series.

Let us say a few words about certain special cases. If the three bodies remain constantly in the $x_1$, $x_2$ plane, we will have $G = \Theta$, $G' = \Theta'$ and F will depend only on $g + \theta$ and $g' + \theta'$, such that we will have only four pairs of conjugate variables

$$\begin{matrix} \beta L, & \beta G = \beta\theta = \beta\Pi, & \beta' L', & \beta' G' = \beta'\theta' = \beta'\Pi', \\ l, & g + \theta = \varpi, & l', & g' + \theta' = \varpi', \end{matrix}$$

as has been said in article 10.

12. Let us again take the notation of article 11 and the equations from this article. I am going to put these equations in a new form which will be useful to me in the following.

Let us first consider the special case where the inclinations are zero and where the three bodies move in the same plane.

Let us set

$$\left.\begin{matrix} \beta L = \Lambda, & \beta\Pi = \Lambda - H, & l + \varpi = \lambda, & \varpi = -h, \\ \beta' L' = \Lambda', & \beta'\Pi' = \Lambda' - H', & l' + \varpi' = \lambda', & \varpi' = -h'. \end{matrix}\right\} \qquad (1)$$

We have

$$\frac{d\lambda}{dt} = -\frac{dF}{d(\beta L)} - \frac{dF}{d(\beta\Pi)} = -\frac{dF}{d\Lambda}, \quad \frac{dh}{dt} = \frac{dF}{d(\beta\Pi)} = -\frac{dF}{dH},$$
$$\frac{d\Lambda}{dt} = \frac{dF}{dl} = \frac{dF}{d\lambda}, \qquad \frac{dH}{dt} = \frac{dF}{dl} - \frac{dF}{d\varpi} = \frac{dF}{dh}.$$

We thus see that the new variables $\Lambda$, $H$, $\Lambda'$, $H'$, $\lambda$, $h$, $\lambda'$, $h'$ are still conjugate, and consequently the change of variables (1) does not alter the canonical form of the equations.

We now come to the general case and resume the notations from article 11.

/30

Let us set

$$\left.\begin{matrix} \beta L = \Lambda, & \beta G = \Lambda - \Pi, & \beta\theta = \Lambda - H - Z, \\ \beta' L' = \Lambda', & \beta' G' = \Lambda' - H', & \beta'\theta' = \Lambda' - H' - Z', \\ \lambda = l + g + \theta, & h = -g - \theta, & \zeta = -\theta, \\ \lambda' = l' + g' + \theta', & h' = -g' - \theta', & \zeta' = -\theta'. \end{matrix}\right\} \qquad (2)$$

We verify, as above, that this change of variables (2) does not alter the canonical form of the equations.

This canonical form will not be altered either, according to the note in article 6, if we make

$$\left.\begin{array}{ll}\sqrt{2H}\cos h = \xi, & \sqrt{2H}\sin h = \eta,\\ \sqrt{2H'}\cos h' = \xi', & \sqrt{2H'}\sin h' = \eta',\\ \sqrt{2Z}\cos\zeta = p, & \sqrt{2Z}\sin\zeta = q,\\ \sqrt{2Z'}\cos\zeta' = p', & \sqrt{2Z'}\sin\zeta' = q'.\end{array}\right\} \quad (3)$$

The equations remain canonical and the two series of conjugate variables are the following

$$\left.\begin{array}{llllll}\Lambda, & \Lambda', & \xi, & \xi', & p, & p',\\ \lambda, & \lambda', & \eta, & \eta', & q, & q'.\end{array}\right\} \quad (4)$$

This is the advantage which the choice of variables (4) can have.

The function F, expressed with the help of these variables, is developable in powers of $\xi$, $\xi'$, $\eta$, $\eta'$, p, p', q, q' as well as the sine and cosine of the multiples of $\lambda$ and of $\lambda'$, the coefficients depending in any event upon $\Lambda$ and $\Lambda'$.

In fact, according to the definitions of the preceding variables, we have

$$H = \Lambda(1-\sqrt{1-e^2}), \qquad Z = \beta G(1-\cos i);$$

we deduce from this that:

(1) H is developable in terms of the powers of $e^2$, the first term of the development being a term in $e^2$;

(2) $e^2$ is developable in terms of the powers of H, the first term being in H;

(3) $\frac{e}{\sqrt{H}}$ is developable in terms of the powers of H;

(4) $i^2$ is equally developable in terms of the powers of $\frac{Z}{\beta G} = \frac{Z}{\Lambda - H}$. /31

(5) $i/\sqrt{Z}$ is developable in terms of the powers of $\frac{Z}{\Lambda - H}$ and consequently in powers of Z and of H.

Now we have

$$\frac{e}{\sqrt{H}} = \frac{e\cos h\sqrt{2}}{\xi} = \frac{e\sin h\sqrt{2}}{\eta}, \qquad \frac{i}{\sqrt{Z}} = \frac{i\cos\zeta\sqrt{2}}{p} = \frac{i\sin\zeta\sqrt{2}}{q}.$$

Therefore e cos h, e sin h, i cos $\zeta$, i sin $\zeta$ are developable in powers of $\xi$, $\eta$, p and q; e' cos h', e' sin h', i' cos $\zeta'$, i' sin $\zeta'$ are developable in powers of $\zeta'$, $\eta'$, p', and q'.

However, the form of the development of the perturbative function is well known.

It is developable in increasing powers of the eccentricities and of the inclinations and in terms of the cosine of the multiples of $\lambda$, $\lambda'$, h, h', $\zeta$ and $\zeta'$, and any term of the development is of the following form (Tisserand, Mécanique céleste, Vol. 1, p. 307)

$$N e^{\mu_1} e'^{\mu_2} i^{\mu_3} i'^{\mu_4} \cos(m_1\lambda + m_2\lambda' + m_3 h + m_4 h' + m_5\zeta + m_6\zeta'),$$

$\mu_i$ being positive integers or zero and $m_i$ being any integers. Moreover, we have

$$\mu_i = |m_i| + \text{ an even number}$$

and, on the other hand,

$$m_1 + m_2 = m_3 + m_4 + m_5 + m_6.$$

We can conclude from this that the perturbative function is developable in terms of the powers of

$$e \cos h, \quad e \sin h, \quad i \cos\zeta, \quad i \sin\zeta,$$
$$e' \cos h', \quad e' \sin h', \quad i' \cos\zeta, \quad i' \sin\zeta',$$

and consequently in terms of the powers of

$$\xi, \ \xi', \ \eta, \ \eta', \ p, \ p', \ q, \ q'. \tag{5}$$

I can observe in addition that the development of /32

$$\frac{e \cos h}{\xi}, \quad \frac{e \sin h}{\eta}, \quad \frac{i \cos\zeta}{p}, \quad \frac{i' \sin\zeta}{q}, \quad \ldots$$

contains only even powers of the variables (5); from this I will conclude that the development of F will be of the following form

$$\sum N \xi^{\mu_3} \eta^{\nu_3} \xi'^{\mu_4} \eta'^{\nu_4} p^{\mu_5} q^{\nu_5} p'^{\mu_6} q'^{\nu_6} \frac{\cos}{\sin} (m_1 \lambda + m_2 \lambda'), \tag{6}$$

N being a coefficient which depends only on $\Lambda$ and $\Lambda'$.

The numbers $\mu_i$, $\nu_i$ are positive or zero integers whose sum

$$\mu_3 + \nu_3 + \mu_4 + \nu_4 + \mu_5 + \nu_5 + \mu_6 + \nu_6$$

is equal to $|m_1 + m_2|$ + a positive even number or zero.

I have allowed the double symbol cos or sin to remain in expression (6); we should take the cosine when the sum

$$\nu_3 + \nu_4 + \nu_5 + \nu_6$$

is even, and the sine for the contrary case.

From this it results that F does not change when we at the same time change the sign of $\lambda$, of $\eta$ and of q, and that it also does not change when we change $\lambda$ and $\lambda'$ to $\lambda + \pi$ and $\lambda' + \pi$, and that at the same time we change the signs of $\xi$, of $\eta$, of p and of q.

The function F enjoys another property to which we must draw attention; it does not change when at the same time we change the sign of p, q, p' and q'.

## General Problem of Dynamics

13. We are therefore led to propose the following problem:

To study the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{1}$$

[illegible] that the function F can be developed in terms of the powers of a very [illegible] $\mu$ in the following manner:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots,$$

[illegible] also that $F_0$ depends only on x and is independent of y, and that $F_1$, [illegible] are periodic functions of period 2π with respect to y. /33

## Reduction of the Canonical Equations

14. We have seen that integration of equations (1) of the preceding arti- [illegible] can be reduced to the integration of a partial differential equation

$$F\left(x_1, x_2, \dots, x_p; \frac{dS}{dx_1}, \frac{dS}{dx_2}, \dots, \frac{dS}{dx_p}\right) = \text{const.} \tag{2}$$

Let us imagine that we know an integral of equations (1) and that this integral is written

$$F_1(x_1, x_2, \dots, x_p; y_1, y_2, \dots, y_p) = \text{const.};$$

this means that we will have identically

$$[F, F_1] = 0. \tag{3}$$

I propose to demonstrate that knowledge of this integral permits lowering the number of degrees of freedom by one.

In effect, equation (3) signifies that there exists an infinity of functions S satisfying at the same time equation (2) and the equation

$$F_1\left(x_1, x_2, \dots, x_p; \frac{dS}{dx_1}, \frac{dS}{dx_2}, \dots, \frac{dS}{dx_p}\right) = \text{const.}$$

This granted, we eliminate $dS/dx_1$ between equations (2) and (4), with the result that

$$\Phi\left(x_1, x_2, \dots, x_p; \frac{dS}{dx_2}, \frac{dS}{dx_3}, \dots, \frac{dS}{dx_p}\right) = 0. \tag{5}$$

The value $dS/dx_1$ does not enter equation (5); then nothing impedes regarding $x_1$ no longer as a variable but as an arbitrary parameter; equation (5) then becomes a partial differential equation with only p-1 independent variables.

The problem is thus reduced to the integration of the equations /34

$$\frac{dx_i}{dt} = \frac{d\Phi}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{d\Phi}{dx_i} \qquad (i = 2, 3, \dots, p),$$

which are canonical equations involving not more than p-1 degrees of freedom.

Thus, if in general we know an integral of a system of differential equations, we will be able to lower the order of the system by unity; but if this system is canonical, we will be able to lower its order by two.

Let us take for example the problem of the motion of a heavy body suspended at a fixed point; we have seen that this problem involves three degrees of freedom; but we know the area integral; the number of degrees of freedom can therefore be lowered to two.

What will now happen if we no longer know only one, but q integrals of equations (1)?

Let

$$F_1, \quad F_2, \quad \ldots, \quad F_q$$

be these q integrals such that

$$[F, F_1] = [F, F_2] = \ldots = [F, F_q] = 0.$$

May we, with the help of these integrals, lower by q units the number of degrees of freedom? Generally this will not take place; to do so, it is required that the q + 1 partial differential equations

$$F = \text{const.}, \qquad F_1 = \text{const.}, \qquad F_2 = \text{const.}, \qquad \ldots, \qquad F_q = \text{const.} \tag{6}$$

be compatible; this demands the conditions

$$[F_i, F_k] = 0 \qquad (i, k = 1, 2, \ldots, q). \tag{7}$$

If conditions (7) are fulfilled, we will eliminate among equations (6)

$$\frac{dS}{dx_1}, \quad \frac{dS}{dx_2}, \quad \ldots, \quad \frac{dS}{dx_q},$$

and we will arrive at a partial differential equation $\Phi = 0$, where these q derivatives no longer enter and which we can consider as dependent only on the p-q independent variables /35

$$x_{q+1}, \quad x_{q+2}, \quad \ldots, \quad x_p,$$

while the first q variables

$$x_1, \quad x_2, \quad \ldots, \quad x_q$$

are regarded as arbitrary parameters.

We will thus be led to a reduced system of canonical equations containing not more than p - q degrees of freedom.

We now reconsider, for example, the Problem of Three Bodies, retaining the notations from the beginning of article 2. We have seen that the number of degrees of freedom is equal to 9.

However, we have the first three integrals of the motion of the center of gravity which can be written

$$\left.\begin{aligned} F_1 &= y_1 + y_4 + y_7 = \text{const.}, \\ F_2 &= y_2 + y_5 + y_8 = \text{const.}, \\ F_3 &= y_3 + y_6 + y_9 = \text{const.} \end{aligned}\right\} \tag{8}$$

It is easy to verify that

$$[F_2, F_3] = [F_3, F_1] = [F_1, F_2] = 0.$$

The number of degrees of freedom can therefore be lowered to 6.

If we restrict ourselves to the case of the Problem of Three Bodies in a plane, the original number of the degrees of freedom is not greater than six. But there are no more than two analogs to 8. After reduction, there will therefore be only four degrees of freedom.

Let us now imagine that we know, other than the q integrals $F_1$, $F_2$, ..., $F_q$, another integral $F_{q+1}$; could we deduce from this an integral of the reduced system? This question can be stated in another manner.

We know a partial differential equation

$$F_{q+1} = \text{const.}$$

compatible with the equation

$$F = \text{const.};$$

will it still be compatible with the system /36

$$F = \text{const.}, \qquad F_1 = \text{const.}, \qquad \ldots, \qquad F_q = \text{const.}? \tag{6}$$

We see immediately that the necessary and sufficient condition for this to be so is that we have

$$[F, F_{q+1}] = [F_1, F_{q+1}] = \ldots = [F_q, F_{q+1}] = 0.$$

Let us return, for example, to the Problem of Three Bodies and let us consider the three area integrals

$$\left.\begin{aligned} F_4 &= x_2 y_3 - x_3 y_2 + x_5 y_6 - x_6 y_5 + x_8 y_9 - x_9 y_8 = \text{const.}, \\ F_5 &= x_3 y_1 - x_1 y_3 + x_6 y_4 - x_4 y_6 + x_9 y_7 - x_7 y_9 = \text{const.}, \\ F_6 &= x_1 y_2 - x_2 y_1 + x_4 y_5 - x_5 y_4 + x_7 y_8 - x_8 y_7 = \text{const.} \end{aligned}\right\} \tag{9}$$

It is easy to verify that we have

$$\begin{array}{lll} [F_1, F_4] = 0, & [F_2, F_4] = +F_3, & [F_3, F_4] = -F_2, \\ [F_1, F_5] = -F_3, & [F_2, F_5] = 0, & [F_3, F_5] = +F_1, \\ [F_1, F_6] = +F_2, & [F_2, F_6] = -F_1, & [F_3, F_6] = 0. \end{array}$$

We do not decrease the generality of the problem by supposing that the center of gravity is fixed, i.e., that all three constants which enter into the last members of equations (8) are zero.

We will then have

$$F_1 = F_2 = F_3 = 0$$

and consequently,

$$[F_i, F_k] = 0 \qquad (i = 1, 2, 3;\ k = 4, 5, 6),$$

which shows that the area integrals are still integrals of the reduced system. To finish, I am going to attempt to reduce the number of degrees of freedom in the Problem of Three Bodies as much as possible, considering at the same time the integrals of the center of gravity and those of area.

In the particular case where the three bodies move in a plane, we have seen that the number of degrees of freedom could be reduced to 4, considering equations (8). The problem thus reduced still involves one integral which is an area /37
integral, which permits reducing the number of degrees of freedom to 3.

In the general case, it is easy to see that we have

$$[F_4, F_5] = F_6, \qquad [F_6, F_6] = F_4, \qquad [F_6, F_4] = F_5.$$

The three brackets not being zero, knowledge of three area integrals does not permit reducing the number of degrees of freedom by 3.

However, it is easy to see that whenever a canonical system admits three integrals

$$F_4, \quad F_5, \quad F_6,$$

it will always be possible to find two combinations of these integrals

$$\varphi(F_4, F_5, F_6),$$
$$\psi(F_4, F_5, F_6),$$

such that

$$[\varphi, \psi] = 0,$$

which permits reducing the number of degrees of freedom by two.

In the case we are concerned with, these combinations are immediately recognized; we need only take $F_4$ and

$$\varphi = F_4^2 + F_5^2 + F_6^2.$$

We will then have identically

$$[\varphi, F_4] = 0.$$

There will thus be, all reduction accomplished, only 4 degrees of freedom.

If we remember that a canonical system containing p degrees of freedom can be reduced to the order of 2p-2, we must conclude that the Problem of Three Bodies in the general case contains 4 degrees of freedom and can be reduced to the sixth order.

In the case of plane motion, it involves 3 degrees of freedom and can be reduced to the fourth order.

In the special case of article 9, it involves 2 degrees of freedom and can be reduced to the second order.

## Reduction of the Problem of Three Bodies

15. It is first a question of effectively making this reduction. /38

Let us first envision the case where the three bodies move in the same plane. We have seen that the number of degrees of freedom could then be reduced to 3. Let us attempt to accomplish this reduction effectively.

We have seen that the equations of motion could be written

$$\frac{dL}{dt}=\frac{dF}{\beta dl},\quad \frac{d\Pi}{dt}=\frac{dF}{\beta d\varpi},\quad \frac{dL'}{dt}=\frac{dF}{\beta' dl'},\quad \frac{d\Pi'}{dt}=\frac{dF}{\beta' d\varpi'},$$
$$\frac{dl}{dt}=-\frac{dF}{\beta dL},\quad \frac{d\varpi}{dt}=-\frac{dF}{\beta d\Pi},\quad \frac{dl'}{dt}=-\frac{dF}{\beta' dL'},\quad \frac{d\varpi'}{dt}=-\frac{dF}{\beta' d\Pi'}.$$

We also have

$$\frac{dF}{d\varpi}+\frac{dF}{d\varpi'}=0,$$

whence the area integral

$$\beta\Pi+\beta'\Pi'=C,$$

C being a constant.

Let us set

$$\beta\Pi=H,\quad \beta'\Pi'=C-H,\quad \varpi-\varpi'=h,$$

whence (if we replace $\Pi$ and $\Pi'$ by their values as functions of C and H)

$$\frac{dF}{d\Pi}=\frac{dF}{\beta d\Pi}-\frac{dF}{\beta' d\Pi'},\quad \frac{dF}{dh}=\frac{dF}{d\varpi}=-\frac{dF}{d\varpi'},\tag{1}$$

and the equations of motion will become

$$\frac{d(\beta L)}{dt}=\frac{dF}{dl},\quad \frac{d(\beta' L')}{dt}=\frac{dF}{dl'},\quad \frac{d\Pi}{dt}=\frac{dF}{dh},$$
$$\frac{dl}{dt}=-\frac{dF}{d(\beta L)},\quad \frac{dl'}{dt}=-\frac{dF}{d(\beta' L')},\quad \frac{dh}{dt}=-\frac{dF}{d\Pi}.$$

There are only 3 degrees of freedom.

16. Let us proceed to the general case where the number of degrees of freedom must be reduced to 4. The equations are then written /39

$$\frac{dL}{dt}=\frac{dF}{\beta dl},\quad \frac{dG}{dt}=\frac{dF}{\beta dg},\quad \frac{d\theta}{dt}=\frac{dF}{\beta d\theta},$$
$$\frac{dL'}{dt}=\frac{dF}{\beta' dl'},\quad \frac{dG'}{dt}=\frac{dF}{\beta d'g'},\quad \frac{d\theta'}{dt}=\frac{dF}{\beta' d\theta'},$$
$$\frac{dl}{dt}=-\frac{dF}{\beta dL},\quad \frac{dg}{dt}=-\frac{dF}{\beta dG},\quad \frac{d\theta}{dt}=-\frac{dF}{\beta d\theta},$$
$$\frac{dl'}{dt}=-\frac{dF}{\beta' dL'},\quad \frac{dg'}{dt}=-\frac{dF}{\beta' dG'},\quad \frac{d\theta'}{dt}=-\frac{dF}{\beta' d\theta'}.$$

Moreover, we have the three area integrals which, if we take the plane of the maximum of the areas as first coordinate plane, are written

$$\beta\theta+\beta'\theta'=C,\quad \theta=\theta',\quad \beta^2(G^2-\theta^2)=\beta'^2(G'^2-\theta'^2).$$

We then have

$$\frac{dF}{d\theta} + \frac{dF}{d\theta'} = 0,$$

which shows that F depends on θ and θ' only by their difference θ - θ'; however, as this difference is zero, by virtue of the area integrals, F can be regarded as no longer depending on either θ or θ'.

We find also

$$\theta = \theta',$$

whence

$$\frac{d\theta}{dt} = \frac{d\theta'}{dt},$$

whence

$$\frac{dF}{\beta d\theta} = \frac{dF}{\beta' d\theta'}. \qquad (2)$$

Let us now set

$$\left.\begin{array}{c} G = \Gamma, \qquad G' = \Gamma', \\ \text{whence} \\ \beta\theta + \beta'\theta' = C, \qquad \beta^2\Gamma^2 - \beta'^2\Gamma'^2 = C(\beta\theta - \beta'\theta') \end{array}\right\} \qquad (3)$$

and

$$\beta\theta = \frac{C}{2} + \frac{\beta^2\Gamma^2}{2C} - \frac{\beta'^2\Gamma'^2}{2C}, \qquad \beta'\theta' = \frac{C}{2} + \frac{\beta'^2\Gamma'^2}{2C} - \frac{\beta^2\Gamma^2}{2C}, \qquad (4)$$

whence /40

$$\frac{dF}{d\Gamma} = \frac{dF}{dG}\frac{dG}{d\Gamma} + \frac{dF}{d\theta}\frac{d\theta}{d\Gamma} + \frac{dF}{d\theta'}\frac{d\theta'}{d\Gamma}$$

or

$$\frac{dF}{d\Gamma} = \frac{dF}{dG} + \frac{dF}{d\theta}\frac{\beta\Gamma}{C} - \frac{dF}{d\theta'}\frac{\beta^2\Gamma}{\beta' C}$$

or finally, by virtue of equation (2),

$$\frac{dF}{d\Gamma} = \frac{dF}{dG}$$

and similarly

$$\frac{dF}{d\Gamma'} = \frac{dF}{dG'}.$$

The area constant C can be regarded as a given quantity in the problem.

If therefore in F we replace G, G', Θ and Θ' by their values (3) and (4), F depends only on L, L', $l$, $l'$, g, g', Γ and Γ', and the equations of motion can be written

$$\frac{dL}{dt} = \frac{dF}{\beta dl}, \quad \frac{d\Gamma}{dt} = \frac{dF}{\beta dg}, \quad \frac{dL'}{dt} = \frac{dF}{\beta' dl'}, \quad \frac{d\Gamma'}{dt} = \frac{dF}{\beta' dg'},$$
$$\frac{dl}{dt} = -\frac{dF}{\beta dL}, \quad \frac{dg}{dt} = -\frac{dF}{\beta d\Gamma}, \quad \frac{dl'}{dt} = -\frac{dF}{\beta' dL'}, \quad \frac{dg'}{dt} = -\frac{dF}{\beta' d\Gamma'},$$

and there are now only 4 degrees of freedom.

## Form of the Perturbative Function

17. It is important to see what form function F assumes when we adopt the variables of the two preceding articles. Let us first suppose that we take the variables of article 15 and that the three bodies move in the same plane; function F, depending only on the distances of the three bodies, will be developable in terms of the cosine and sine of the multiples of $l - l' + h$; the coefficients of this development will themselves be developable in terms of the increasing powers of

$$e\cos l,\quad e\sin l,\quad e'\cos l',\quad e'\sin l',$$

designating the eccentricities by e and e'; finally, the coefficients of these new developments will themselves be uniform functions of L and L'. /41

For brevity, I will set

$$\beta L = \Lambda, \qquad \beta' L' = \Lambda';$$

we will then have, according to the definition of H,

$$e = \frac{1}{\Lambda}\sqrt{\Lambda^2 - H^2}, \qquad e' = \frac{1}{\Lambda'}\sqrt{\Lambda'^2 - (H - C)^2}.$$

Let us add that F does not change when $l$, $l'$ and h change sign; consequently, if we develop F in terms of the cosines and sines of the multiples of these three variables, the development can contain only cosines.

We will therefore finally have

$$F = \Sigma A (\Lambda^2 - H^2)^{\frac{p}{2}} [\Lambda'^2 - (H - C)^2]^{\frac{q}{2}} \cos(m_1 l + m_2 l' + m_3 h),$$

p and q are positive integers, $m_1$, $m_2$ and $m_3$ arbitrary integers, A is a coefficient which depends only on $\Lambda$ and on $\Lambda'$. What is more, $|m_3 - m_1|$ is at most equal to p and can differ from it only by an even number; similarly, $|m_3 + m_2|$ is at most equal to q and can differ from it only by an even number.

Such a development is valid when $\Lambda$ - H and $\Lambda'$ - (C-H) are sufficiently small; we see that for

$$\Lambda = H$$

all terms vanish, except those for which $m_3 = m_1$.

Similarly, if we have

$$\Lambda' = C - H,$$

all terms vanish except those for which $m_3 = -m_2$.

If, consequently, we have at the same time,

$$\Lambda = H, \qquad \Lambda' = C - H,$$

all terms will vanish except those for which $m_3 = m_1 = -m_2$, such that F becomes a function of $l - l' + h$.

If, in one of the terms of the development of F, we make /42

$$\Lambda = -H, \quad \Lambda' = H - C,$$

this term will still vanish, unless

$$m_3 = m_1 = -m_2.$$

We might be tempted to conclude that, for

$$\Lambda = -H, \quad \Lambda' = H - C,$$

F is still a function of $l - l' + h$; but this is not so, for the development is valid only for small values of $\Lambda - H$ and $\Lambda' - C + H$. An analogous reasoning to the one which precedes proves, on the contrary, that for $\Lambda = -H$, $\Lambda' = H - C$, F is a function of $l - l' - h$ and not of $l - l' + h$.

In the case where the value of $\Lambda - H$ is extremely small, it can be advantageous to change the special variables.

We have identically

$$\Lambda l + H h = \Lambda(l + h) - h(\Lambda - H);$$

the canonical form, by virtue of article 5, is not therefore altered when we replace the variables

$$\begin{matrix} \Lambda, & \Lambda', & H, \\ l, & l', & h \end{matrix}$$

by the following

$$\begin{matrix} \Lambda, & \Lambda', & \Lambda - H, \\ l + h, & l', & -h. \end{matrix}$$

Let us now set

$$l + h = \lambda^*, \quad \sqrt{2(\Lambda - H)}\cos h = \xi^*, \quad -\sqrt{2(\Lambda - H)}\sin h = \eta^*;$$

by virtue of article 6 the canonical form of the equations remains when we take as variables

$$\begin{matrix} \Lambda, & \Lambda', & \xi^*, \\ \lambda^*, & l', & \eta^*. \end{matrix}$$

It is of advantage that function F, which remains periodic in $\lambda^*$ and $l'$, is developable in terms of the powers of $\xi^*$ and $\eta^*$ when these two variables are sufficiently small.

18. Let us now take the variables of article 16, i.e., /43

$$\begin{matrix} \beta L = \Lambda, & \beta' L' = \Lambda', & \beta \Gamma = H, & \beta' \Gamma' = H'. \\ l, & l', & g, & g'. \end{matrix}$$

The variables H and H' are manifestly subject to certain inequalities; we have

$$H = \Lambda\sqrt{1 - e^2},$$

whence

$$\Lambda^2 > H^2. \tag{1}$$

Similarly,

$$\Lambda'^2 > H'^2. \tag{2}$$

On the other hand, we have, by virtue of the area equation,

$$H \cos i + H' \cos i' = C, \qquad H \sin i + H' \sin i' = 0,$$

C being the area constant which should be regarded as one of the given quantities of the problem. From this we deduce the inequalities

$$\left.\begin{array}{l} |H| + |H'| > |C|, \\ |H| - |H'| < |C|. \end{array}\right\} \tag{3}$$

Let us now see how function F depends on our variables.

For the values of H near Λ, function F is no longer holomorphic with respect to H; it is no longer developable in the integral powers of $\Lambda - H$, but in those of $\sqrt{\Lambda - H}$.

We can then employ advantageously the following variables. Let us set

$$l + g = \lambda^*, \qquad \sqrt{2(\Lambda - H)} \cos g = \xi^*, \qquad \sqrt{2(\Lambda - H)} \sin g = \eta^*;$$

the equations will retain the canonical form if we take

$$\begin{array}{llll} \Lambda, & \Lambda', & \xi^*, & H', \\ \lambda^*, & l', & \eta^*, & g' \end{array}$$

as independent variables; in addition, function F will then be developable in terms of the integral powers of $\xi^*$ and of $\eta^*$.

We would operate in a similar manner if we had to consider the values of H' very close to Λ'. /44

What will now happen if the values of H and H' are very near the limits that they are assigned by inequalities (3), i.e., if the inclinations are small or zero?

Let us suppose, for example, that $H + H' = C$.

We have seen, in article 12, that F is developable in terms of increasing powers of the variables $\xi, \xi', \eta, \eta', p, p', q, q'$ of these sections; i.e., in terms of increasing powers of

$$\sqrt{\beta L - \beta G}, \quad \sqrt{\beta' L' - \beta' G'}, \quad \sqrt{\beta G - \beta \Theta}, \quad \sqrt{\beta' G' - \beta' \Theta'},$$

if the inclinations are zero; we have

$$G = \Theta, \qquad G' = \Theta',$$

and the last two radicals vanish, but it is not at all the same with the first two; function F is then holomorphic in $G, G', \sqrt{\beta G - \beta \Theta}, \sqrt{\beta' G' - \beta' \Theta'}$.

But we have seen in article 12 that F does not change when p, p', q, q' change sign at the same time, or, what is the same, when the two radicals $\sqrt{\beta G - \beta\Theta}$ and $\sqrt{\beta' G' - \beta'\Theta'}$ change sign at the same time.

Therefore, for very small or zero values of the inclinations, F is holomorphic with respect to G and G' on one hand, and on the other with respect to $\sqrt{(\beta G-\beta\Theta)-(\beta' G'-\beta'\Theta')}$.

However, we have

$$G = \frac{H}{\beta}, \qquad G' = \frac{H'}{\beta'}, \qquad \theta = \frac{1}{2\beta}\left(C + \frac{H^2 - H'^2}{C}\right), \qquad \theta' = \frac{1}{2\beta'}\left(C + \frac{H'^2 - H}{C}\right.$$

whence

$$\sqrt{(\beta G - \beta\theta)(\beta' G' - \beta'\theta')} = \frac{1}{2C}\sqrt{[(H-C)^2 - H'^2][(H'-C)^2 - H^2]}$$

or

$$\sqrt{(\beta G - \beta\theta)(\beta' G' - \beta'\theta')} = \frac{H + H' - C}{2C}\sqrt{(H - C - H')(H' - C - H)}.$$

These equalities show that

$$G, \quad G', \quad \sqrt{(\beta G - \beta\theta)(\beta' G' - \beta'\theta')}$$

and, consequently, F remains holomorphic in H and in H' for H + H' = C. /45

Invariant Relations

19. In article 1 we considered, with regard to system

$$\frac{dx_i}{dt} = X_i, \tag{1}$$

on one hand its solutions, on the other its integrals. But it remains for us to speak of certain equations which relate to this system and which can be regarded as, shall we say, holding the middle between the solutions and the integrals. I am going to define these equations, which I will call invariant relations.

Let $\varphi$ be an arbitrary function of $x_1, x_2, \ldots, x_n$; we will have

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dx_1}X_1 + \frac{d\varphi}{dx_2}X_2 + \ldots + \frac{d\varphi}{dx_n}X_n.$$

Let us now consider a system of equations

$$\left.\begin{array}{l} \varphi_1(x_1, x_2, \ldots, x_n) = 0, \\ \varphi_2(x_1, x_2, \ldots, x_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \varphi_p(x_1, x_2, \ldots, x_n) = 0, \end{array}\right\} \tag{2}$$

and let us suppose that these equations entail as a consequence the following

$$\frac{d\varphi_i}{dx_1}X_1 + \frac{d\varphi_i}{dx_2}X_2 + \ldots + \frac{d\varphi_i}{dx_n}X_n = 0;$$

we will conclude from this that

$$\frac{d\varphi_i}{dt} = 0.$$

Consequently, if equations (2) are satisfied for an arbitrary value of t, they will be so for all values of t; this is why we will call system (2) a system of invariant relations, and we realize the importance that knowledge of a similar system can have.

Let us now suppose that the system is canonical and let us return to system (1) from article 7 and to equation /46

$$F\left(x_1, x_2, \ldots, x_p; \frac{dS}{dx_1}, \frac{dS}{dx_2}, \ldots, \frac{dS}{dx_p}\right) = \text{const.}, \tag{3}$$

which is correlative to it.

Knowledge of a special solution of this equation (3) will furnish us a system of invariant relations.

Let, in fact, S be this solution; let us consider the system

$$y_1 = \frac{dS}{dx_1}, \quad y_2 = \frac{dS}{dx_2}, \quad \ldots, \quad y_p = \frac{dS}{dx_p}; \tag{4}$$

I say that this will be a system of invariant relations in relation to the canonical equations (1).

We find, in fact, by differentiating equation (3), that

$$\frac{dF}{dx_i} + \frac{dF}{dy_1}\frac{d^2S}{dx_1 dx_i} + \frac{dF}{dy_2}\frac{d^2S}{dx_2 dx_i} + \ldots + \frac{dF}{dy_p}\frac{d^2S}{dx_p dx_i} = 0. \tag{5}$$

Let us set

$$\varphi_i = y_i - \frac{dS}{dx_i}$$

in such a manner as to reduce system (4) to the form (2)

$$\varphi_1 = \varphi_2 = \ldots = \varphi_p = 0. \tag{2}$$

We will have

$$\frac{d\varphi_i}{dy_i} = 1, \qquad \frac{d\varphi_i}{dy_k} = 0 \quad (i \gtrless k), \qquad \frac{d\varphi_i}{dx_k} = \frac{d^2S}{dx_i dx_k},$$

whence

$$\frac{d\varphi_i}{dt} = \sum_k \frac{d\varphi_i}{dy_k}\frac{dy_k}{dt} + \sum_k \frac{d\varphi_i}{dx_k} = \sum_k \left(\frac{d\varphi_i}{dx_k}\frac{dF}{dy_k} - \frac{d\varphi_i}{dy_k}\frac{dF}{dx_k}\right),$$

which shows that equations (5) reduce to

$$\frac{d\varphi_i}{dt} = 0 \quad (i = 1, 2, \ldots, p).$$

Thus, this is, according to what we have just seen, precisely the condition under which system (4) is a system of invariant relations. /47

I will add that in the case where there are only two degrees of freedom any system of two invariant relations can be obtained in this manner.

## Definitions and Various Lemmas

20. The method of Cauchy for demonstrating the existence of an integral of differential equations has been applied by other geometers to the demonstration of a great number of theorems. Since this method and these theorems will be useful later, I am here forced to devote a preliminary chapter to it. For this exposition, I will make use of a notation which I have already introduced in another memoir and which will eliminate delay and repetition for me. /48

Let $\varphi(x, y)$ and $\psi(x, y)$ be two series developed in terms of the increasing powers of x and of y; let us suppose that each of the coefficients of the series $\psi$ is real and positive and greater in absolute value than the corresponding coefficient of the $\varphi$ series: we will then write

$$\varphi(x, y) < \psi(x, y)$$

or, if it is necessary to display the varieables with respect to which the development is made,

$$\varphi < \psi \quad (\text{arg.}\, x, y).$$

We easily see that, if $\varphi(x, y)$ is a series which converges for certain values of x and y (consequently representing a function of x and y, holomorphic for x=y=0), we will always be able to find two real and positive numbers M and $\alpha$, such that

$$\varphi(x, y) < \frac{M}{(1-\alpha x)(1-\alpha y)} < \frac{M}{1-\alpha(x+y)}.$$

In the case where the function $\varphi$ variables for x=y=0, we can write /49

$$\varphi < \frac{M\alpha(x+y)}{1-\alpha(x+y)} < \frac{M\alpha(x+y)[1+\alpha(x+y)]}{1-\alpha(x+y)}.$$

Let us suppose that $\varphi$, besides the arguments x and y, which we assume developed, depends in addition on another variable t: the numbers M and $\alpha$ will be generally continuous of t; if these two numbers do not cancel for any of the considered values of t, we will be able to assign them a lower limit; we will therefore be able to give M and $\alpha$ constant values large enough for the preceding inequalities to subsist.

21. Calculation of the inequalities defined in the preceding article rests on the following principles, and I will restrict myself to stating them without demonstration because of their obviousness:

(1) If the series $\psi$ converges, this will be true also for the series $\varphi$ whenever we have

$$\varphi < \psi.$$

(2) We can add any number of inequalities of the same sense

$$\varphi_1 < \psi_1, \quad \varphi_2 < \psi_2, \quad \ldots, \quad \varphi_n < \psi_n.$$

(3) If we have an infinite number of inequalities of the same sense,

$$\varphi_0 < \psi_0, \quad \varphi_1 < \psi_1, \quad \ldots, \quad \varphi_n < \psi_n \ \ldots \text{ ad inf. (arg. } x, y),$$

we may write, introducing a new argument, $\varphi_0 + \lambda\varphi_1 + \lambda^2\varphi_2 + \ldots < \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \ldots$ (arg. $x, y, \lambda$).

(4) We can multiply two inequalities of the same sense.

(5) If we have

$$\varphi(x_1, x_2, \ldots, x_n) < \psi(x_1, x_2, \ldots, x_n) \quad (\text{arg. } x_1, x_2, \ldots, x_n)$$

and, on the other hand,

$$f_1(x,y) < \theta_1(x,y), \quad f_2(x,y) < \theta_2(x,y),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$f_n(x,y) < \theta_n(x,y) \quad (\text{arg. } x, y),$$

in equality (1) in place of $x_1$, $x_2$, ..., $x_n$, we will be able to substitute, in the first member $f_1$, $f_2$, ..., $f_n$ and in the second member $\theta_1$, $\theta_2$, ..., $\theta_n$. We may therefore write $\varphi[f_1(x,y), f_2(x,y), \ldots, f_n(x,y)] < \psi[\theta_1(x,y), \theta_2(x,y) \ldots, \theta_n(x,y)]$ (arg. $x, y$).

(6) It is permissible to differentiate the inequality

$$\varphi(x,y) < \psi(x,y) \quad (\text{arg. } x, y), \tag{1}$$

with respect to one of the two arguments x and y.

(7) It is permissible to integrate an inequality; but this can be understood in two ways; we can first integrate inequality (1) with respect to one of the two arguments x and y, taking zero as the lower integration limit.

We then find

$$\int_0^x \varphi(x,y)\,dx < \int_0^x \psi(x,y)\,dx.$$

It goes without saying that in the calculation of integrals, y must temporarily be considered a constant.

(8) However, it may also happen that the functions $\varphi$ and $\psi$ depend not only on the two arguments x and y, but on another variable t, without our regarding it as developable in terms of the powers of this variable.

Let us assume that inequality (1) is true for all values of t between $t_0$ and $t_1$; we will be able to integrate this inequality with respect to t, regarding x and y as constants, and to write $\int \varphi(x, y, t)\,dt < \int \psi(x, y, t)\,dt$ (arg $x, y$),

provided, of course, that the integration limits are included between $t_0$ and $t_1$.

22. Let us consider a function

$$\varphi(x, y),$$

developed in terms of the powers of x and y. We will find that frequently x and y will depend on a certain parameter $\mu$ and that we will be able to develop them in terms of the powers of this parameter. We therefore write /51

$$\left.\begin{aligned} x &= x_0 + \mu x_1 + \mu^2 x_2 + \ldots, \\ y &= y_0 + \mu y_1 + \mu^2 y_2 + \ldots. \end{aligned}\right\} \quad (1)$$

Let us assume that in the function $\varphi$ we substitute in place of x and y their developments (1); then $\varphi$ will become a function of $\mu$, $x_0$, $x_1$, ..., $x_p$, ... ad inf.; and of $y_0$, $y_1$, ..., $y_p$, ... ad inf.; moreover, it will be developable in terms of the powers of $\mu$, such that we will have

$$\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \ldots.$$

We easily see that $\varphi_0$ depends only on $x_0$ and $y_0$; $\varphi_1$ on $x_0$, $y_0$, $x_1$, and $y_1$, ...; and in general, $\varphi_p$ depends on $x_0$, $x_1$, ..., $x_p$; $y_0$, $y_1$, ..., $y_p$.

Let us now assume that we have

$$\varphi(x, y) < \psi(x, y \qquad (\text{arg. } x, y).$$

In $\psi$ let us substitute, in place of x and y, their developments (1) such that we have

$$\psi = \psi_0 + \mu\psi_1 + \mu^2\psi_2 + \ldots.$$

We easily see that

$$\begin{aligned} &\varphi_0 < \psi_0, && (\text{arg. } x_0, y_0), \\ &\varphi_1 < \psi_1, && (\text{arg. } x_0, y_0; x_1, y_1), \\ &\ldots\ldots && \ldots\ldots\ldots\ldots, \\ &\varphi_p < \psi_p, && (\text{arg. } x_0, x_1, \ldots, x_p; y_0, y_1, \ldots, p_p). \end{aligned}$$

We realize this in applying the fifth principle of the preceding article, which shows that

$$\varphi < \psi \quad (\text{arg. } \mu, x_0, x_1, \ldots \text{ ad inf.}; y_0, y_1, \ldots \text{ ad inf.}).$$

For brevity we will agree to write $\varphi_p$ $(x_1, y_1)$ in place of $\varphi_p$ $(x_0, x_1, \ldots, x_p;$ $y_0, y_1, \ldots, y_p)$.

## Cauchy's Theorem

23. Cauchy's theorem is found today in all classical Treatises; I would therefore restrict myself to stating it without demonstration, if I did not propose to complete it in several points. /52

Let us consider the differential equations

$$\frac{dx}{dt} = \theta(x, y, z, \mu), \quad \frac{dy}{dt} = \varphi(x, y, z, \mu), \quad \frac{dz}{dt} = \psi(x, y, z, \mu). \tag{1}$$

I assume that the functions $\varphi$ and $\psi$ are developed in terms of the increasing powers of the independent variable $x$, of the two unknown functions $y$ and $z$ and of an arbitrary parameter $\mu$.

In assuming that the independent variable $t$ does not enter the second members of equations (1), I do not decrease the generality, because a system of the n order, where the independent variable enters explicitly, can always be replaced by a system of the n+1 order, where this independent variable does not enter.

In fact, for example, let

$$\frac{dx}{dt} = \varphi(x, y, t),$$
$$\frac{dy}{dt} = \psi(x, y, t);$$

it is obvious that these two equations can be replaced by the following three

$$\frac{dx}{dt} = \varphi(x, y, z), \quad \frac{dy}{dt} = \psi(x, y, z), \quad \frac{dz}{dt} = 1.$$

I propose to demonstrate that there exist three convergent series developed in terms of the powers of $t$, $\mu$, $x_0$, $y_0$, $z_0$, which satisfy equations (1) when they are substituted in place of $x$, $y$ and $z$ and which reduce, respectively, to $x_0$, $y_0$, and $z_0$ for $t=0$.

Thus, instead of only developing, as Cauchy did, with respect to the independent variable $x$, I develop in addition with respect to the parameter $\mu$, and with respect to the initial values $x_0$, $y_0$, $z_0$. However, I must first demonstrate two new lemmas.

24. Let

$$\left.\begin{aligned} \frac{dx}{dt} &= \varphi(x, y, t, \mu), \\ \frac{dy}{dt} &= \psi(x, y, t, \mu) \end{aligned}\right\} \tag{1}$$

be two differential equations, where $\varphi$ and $\psi$ are given series, in terms of the /53
powers of unknown functions $x$ and $y$, variable $t$ and an arbitrary parameter $\mu$.

It is easy to verify that there exist two series

$$f(t, \mu), \quad f_1(t, \mu), \tag{2}$$

ordered in terms of the powers of $t$ and $\mu$, vanishing with $t$, and which, substituted into equations (1) in place of $x$ and $y$, according to the ordinary rules of calculus, formally satisfy these equations.

In seeking to determine the coefficients of these series f and $f_1$ by the method of indeterminate coefficients, we find that any coefficient of f (or $f_1$) is an integral polynomial with positive coefficients to the various coefficients of $\varphi$ and of $\psi$.

Let us therefore consider other equations of the same form as (1)

$$\frac{dx}{dt} = \varphi'(x, y, t, \mu), \quad \frac{dy}{dt} = \psi'(x, y, t, \mu), \tag{1a}$$

and which are such that

$$\varphi < \varphi', \quad \psi < \psi' \quad (\text{arg. } x, y, t, \mu);$$

if the series

$$f(t, \mu), \quad f'(t, \mu) \tag{2a}$$

are ordered in terms of the powers of t and $\mu$, vanish with t and formally satisfy equations (1a) when they are substituted in place of x and of y, it is permissible to conclude that

$$f < f', \quad f_1 < f'_1 \quad (\text{arg. } t, \mu).$$

25. Let us again take equations (1) from the preceding article; let us assume that $\varphi$ and $\psi$ are developable in terms of the powers of x, y and $\mu$ for all values /54
of t included between 0 and $t_1 (t_1 > 0)$ (we will agree to consider only the values of t included between these two limits). I do not propose, however, that $\varphi$ and $\Psi$ be developable in terms of the powers of t.

Then there will exist the series

$$f(t, \mu), \quad f_1(t, \mu)$$

which will be ordered in terms of the powers of $\mu$ (the coefficient of any power of $\mu$ being a function of t, which can be nondevelopable in powers of t), which will vanish and which will formally satisfy equations (1).

How can we determine the coefficients of the two series f and $f_1$?

Let $x_m$ be the coefficient of $\mu^m$ in f, and $y_m$ be that of $\mu^m$ in $f_1$. In order to determine $x_m$ and $y_m$, we then find the following equations

$$\frac{dx_0}{dt} = \varphi(x_0, y_0, t_0, 0), \quad \ldots\ldots\ldots, \quad \frac{dy_0}{dt} = \psi(x_0, y_0, t, 0), \quad \ldots\ldots\ldots,$$

$$\frac{dx_1}{dt} = \frac{d\varphi}{dx_0} x_1 + \frac{d\varphi}{dy_0} y_1 + X_1, \quad \frac{dx_m}{dt} = \frac{d\varphi}{dx_0} x_m + \frac{d\varphi}{dy_0} y_m + X_m; \quad \frac{dy_1}{dt} = \frac{d\psi}{dx_0} x_1 + \frac{d\psi}{dy_0} y_1 + Y_1, \quad \frac{dy_m}{dt} = \frac{d\psi}{dx_0} x_m + \frac{d\psi}{dy_0} Y_m + Y_m,$$

$X_m$ and $Y_m$ being developed in terms of the powers of

$$x_1, \; y_1; \; x_2, \; y_2; \; \ldots, \; x_{m-1}, \; y_{m-1},$$

and on the other hand depending on $x_0$, $y_0$ and t.

Furthermore, in $\frac{d\varphi}{dx_0}, \frac{d\varphi}{dy_0}, \frac{d\psi}{dx_0}, \frac{d\psi}{dy_0}$, x, y and $\mu$ must be replaced by $x_0$, $y_0$ and 0.

Now let there be some equations

$$\left.\begin{aligned} \frac{dx}{dt} &= \varphi'(x, y, t, \mu), \\ \frac{dy}{dt} &= \psi'(x, y, t, \mu), \end{aligned}\right\} \tag{1a}$$

such that

$$\varphi < \varphi', \quad \psi < \psi' \quad \text{arg. x, y and } \mu\text{, but not arg. t).}$$

Let

$$f'(t, \mu) = x'_0 + \mu x'_1 + \mu^2 x'_2 + \ldots,$$
$$f'_1(t, \mu) = y'_0 + \mu y'_1 + \mu^2 y'_2 + \ldots$$

be the given series in terms of the powers of $\mu$ and vanishing with t, which formally satisfy equations (1a).

We will have

$$\frac{dx'_0}{dt} = \varphi'(x'_0, y'_0, t, 0), \qquad \frac{dy'_0}{dt} = \psi'(x'_0, y'_0, t, 0),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\frac{dx'_m}{dt} = \frac{d\varphi'}{dx'_0} x'_m + \frac{d\varphi'}{dy'_0} y'_m + X'_m; \qquad \frac{dy'_m}{dt} = \frac{d\psi'}{dx'_0} x'_m + \frac{d\psi'}{dy'_0} y'_m + Y'_m.$$

At the origin of times, we will have

$$x'_0 = x_0 = 0, \qquad y'_0 = y_0 = 0$$

and also

$$|\varphi| < \varphi', \quad |\psi| < \psi', \tag{2}$$

whence

$$\left|\frac{dx_0}{dt}\right| < \frac{dx'_0}{dt}, \qquad \left|\frac{dy_0}{dt}\right| < \frac{dy'_0}{dt}; \tag{3}$$

$x'_0$ and $y'_0$, for the small positive values of t, are therefore positive and greater in absolute value than $x_0$ and $y_0$.

Therefore I write

$$|x_0| < x'_0, \qquad |y_0| < y'_0. \tag{4}$$

Equalities (4) could not cease to be satisfied without inequalities (3) ceasing to do so first. But it will not be so, because inequalities (4), related to inequalities (2), imply inequalities (3) as consequences. Therefore inequalities (4) will subsist whenever

$$0 < t < t_1.$$

I assume that we have also demonstrated that

$$\left.\begin{array}{llll} |x_1| < x'_1, & |x_2| < x'_2, & \ldots, & |x_{m-1}| < x'_{m-1}; \\ |y_1| < y'_1, & |y_2| < y'_2, & \ldots, & |y_{m-1}| < y'_{m-1}, \end{array}\right\} \quad (5).$$

and I propose to demonstrate that /56

$$|x_m| < x'_m, \quad |y_m| < y'_m.$$

In fact, we conclude from inequalitites (5) that

$$\left|\frac{d\varphi}{dx_0}\right| < \frac{d\varphi'}{dx'_0}, \quad \left|\frac{d\varphi}{dy_0}\right| < \frac{d\varphi'}{dy'_0}, \quad \left|\frac{d\psi}{dy_0}\right| < \frac{d\psi'}{dy'_0}, \quad \left|\frac{d\psi}{dx_0}\right| < \frac{d\psi'}{dx'_0}, \quad |X_m| < X'_m, \quad |Y_m| < Y'_m.$$

We must therefore conclude that inequalities

$$|x_m| < x'_m, \quad |y_m| < y'_m$$

imply the following:

$$\left|\frac{dx_m}{dt}\right| < \frac{dx'_m}{dt}, \quad \left|\frac{dy_m}{dt}\right| < \frac{dy'_m}{dt}.$$

A reasoning quite similar to the preceding would then show that we have

$$|x_m| < x'_m, \quad |y_m| < y'_m \quad \text{for} \quad 0 < t < t_1.$$

These inequalities can also be written

$$f \prec f', \quad f_1 \prec f'_1 \quad (\text{arg. } \mu, \text{ but not arg. } t).$$

26. Let us again take equations (1) from article 23

$$\frac{dx}{dt} = \theta(x, y, z, \mu), \quad \frac{dy}{dt} = \varphi(x, y, z, \mu), \quad \frac{dz}{dt} = \psi(x, y, z, \mu). \quad (1)$$

These equations are formally satisfied by certain series

$$\left.\begin{array}{l} x = f_1(t, x_0, y_0, z_0, \mu), \\ y = f_2(t, x_0, y_0, z_0, \mu), \\ z = f_3(t, x_0, y_0, z_0, \mu), \end{array}\right\} \quad (3)$$

developed in terms of the increasing powers of t, $x_0$, $y_0$, $z_0$ and $\mu$ and are reduced respectively to $x_0$, $y_0$ and $z_0$ for $t=0$.

To demonstrate the convergence of these series, let us compare them to the series obtained starting with different equations.

We can still find three real, positive numbers M, $\alpha$ and $\beta$ such that by setting

$$\theta' = \varphi' = \psi' = \frac{M}{(1-\beta\mu)[1-\alpha(x+y+z)]}, \quad (1)$$

we have

$$\left.\begin{array}{l}\theta<\theta' \\ \varphi<\varphi' \\ \psi<\psi'\end{array}\right\} (\arg x, y, z, \mu).$$

Let us consider the equations

$$\left.\begin{array}{l}\frac{dx}{dt}=\theta', \\ \frac{dy}{dt}=\varphi', \\ \frac{dz}{dt}=\psi',\end{array}\right\} \tag{2a}$$

which can also be written

$$\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=\frac{M}{(1-\beta\mu)[1-\alpha(x+y+z)]} \tag{3a}$$

We can satisfy these equations by series analogous to series (3), that is a power series in t, $x_0$, $y_0$, $z_0$ and $\mu$ and reducing in the same way to $x_0$, $y_0$ and $z_0$ for t=0.

The principles of article 24 show that series (3) will converge whenever series (3a) themselves converge.

Now, equations (2a) integrate easily, and we find that equations (3a), which are their integrals, can be written

$$x=x_0+\frac{1}{3\alpha}(S-\sqrt{S^2-ht}),$$

$$y=y_0+\frac{1}{3\alpha}(S-\sqrt{S^2-ht}),$$

$$z=z_0+\frac{1}{3\alpha}(S-\sqrt{S^2-ht}),$$

where we have set, for brevity,

$$S=1-\alpha(x_0+y_0+z_0), \quad h=\frac{6\alpha M}{1-\beta\mu}.$$

These series, developed in terms of the powers of $\mu$, t, $x_0$, $y_0$, $z_0$, converge, provided that

$$|\mu|, \ |t|, \ |x_0|, \ |y_0|, \ |z_0|$$

are sufficiently small.

The same will be true for series (3).

Q. E. D.

## Extension of Cauchy's Theorem

27. The considerations developed in article 26 show the possibility of developing the solutions of a differential equation in terms of the powers of an arbitrary parameter $\mu$, but only for values of the independent variable t with a sufficiently small modulus. We will now attempt to free ourselves of this restriction.

Let us consider the following equations

$$\frac{dx}{dt} = \varphi(x, y, t, \mu), \qquad \frac{dy}{dt} = \psi(x, y, t, \mu). \tag{1}$$

Therefore I once more assume that the variable t enters the equations explicitly.

Let

$$x = \theta(t, \mu), \qquad y = \omega(t, \mu)$$

be that of the solutions of equations (1), which is such that the initial values of x and y, for t=0, are zero.

I assume that for all values of t between 0 and $t_0$ the two functions $\varphi$ and $\psi$ may develop in terms of the powers of

$$\mu, \quad x - \theta(t, o), \qquad y - \omega(t, o)$$

(the coefficients of the developments being any functions of t as well).

This condition can be stated in another manner: when for a certain system of values of x, y, t and $\mu$ one of the functions $\varphi$ and $\psi$ ceases being holomorphic, we say that this system of values corresponds to a singular point of equations (1). Consequently, we can state the condition which precedes by saying, in rather incorrect but useful language, that the particular solution /59

$$\mu = o, \qquad x = \theta(t, o), \qquad y = \omega(t, o)$$

is not going to pass through any singular point.

I say that if this condition is fulfilled, $\theta(t, \mu)$, $\omega(t, \mu)$ will for all values of t between 0 and $t_0$ to be developable in terms of the powers of $\mu$ (I say of $\mu$ and not of t and $\mu$), provided that $|\mu|$ is sufficiently small.

I first observe that we can, without loss of generality, assume that the functions $\varphi$ and $\psi$ vanish identically when we here make

$$x = y = \mu = o$$

or, which is the same, that we have identically

$$\theta(t, o) = \omega(t, o) = o.$$

If, in fact, this were not so, we would change variables by setting

$$x' = x - \theta(t, o), \qquad y' = y - \omega(t, o)$$

and we would be led to the case which we have just stated, for the transformed equations admit as a solution, for $\mu$ = 0,

$$x' = o, \qquad y' = o.$$

Let us therefore make this hypothesis: the functions of $\varphi$ and $\psi$ will be developable in terms of the powers of x, y and $\mu$; but I do assume them developed in terms of the powers of t.

We will be able to find series (3) developed in terms of the powers of $\mu$ and which, substituted in place of x and y, will formally satisfy equations (1). Moreover, these series will vanish for

$$t=0.$$

To demonstrate the convergence of these series, let us form equations analogous to equations (2a) of article 26.

The functions $\varphi$ and $\psi$ are developable in terms of the powers of x, y and $\mu$, provided that

$$0 < t < t_0.$$

When t varies from 0 to $t_0$, the radii of convergence of these developments will vary as well, but we will be able to assign them a lower limit. We will therefore be able, according to article 20, to find two positive numbers M and $\alpha$, such that for all values of t between 0 and $t_0$, we have

$$\varphi < \varphi', \qquad \psi < \psi' \qquad (\text{arg. } x, y, \mu)$$

setting

$$\varphi' = \psi' = \frac{M(x+y+\mu)[1+\alpha(x+y+\mu)]}{1-\alpha(x+y+\mu)}.$$

Let us then form equations

$$\frac{dx}{dt} = \varphi', \qquad \frac{dy}{dt} = \psi'. \tag{2a}$$

We can satisfy these equations by series (3a) in the same form as series (3), and this will formally satisfy these equations.

According to article 25, series (3) will converge provided series (3a) converge.

Now, if we set

$$x + y + \mu = S,$$

our equations give

$$x = y = \frac{S-\mu}{2}$$

and

$$\frac{dS}{dt} = \frac{2MS(S+1)}{1-S}$$

or

$$2M\,dt = \frac{dS}{S} - \frac{2\,dS}{S+1},$$

whence, because $S=\mu$ for $t=0$,

$$2Mt = L\frac{S}{(S+1)^2} - L\frac{\mu}{(\mu+1)^2}.$$

We will easily verify that S and, consequently, x and y can develop in terms of the powers of $\mu$ and that the development converges for all values of t provided that $|\mu|$ is sufficiently small, we can conclude from this that series (3a) and series (3) converge. /61

Q. E. D.

28. The results of the preceding article obviously continue to exist when, in place of only one arbitrary parameter $\mu$, we have several. Here is the use which we are going to make of this result: in article 27 we considered only the particular solution for which the initial values of x and y are zero.

Let us suppose we were to consider the particular solution for which these initial values are $x_0$ and $y_0$, and that we were to propose to develop this solution in terms of the powers of $x_0$, $y_0$ and $\mu$.

However, we can go still farther: let us again take equations (1) from the preceding article, and let us consider this special solution such that

$$x = x_0, \quad y = y_0$$

for t=0; let us then seek to develop the values of x and y for $t=t_0+\tau$ in terms of powers of $x_0$, $y_0$, $\mu$ and $\tau$.

Let us then set

$$x = x' + x_0, \qquad y = y' + y_0, \qquad t = t'\frac{t_0+\tau}{t_0};$$

equations (1) will become

$$\frac{dx'}{dt'} = \frac{t_0}{t_0+\tau}\varphi\left(x'+x_0, y'-y_0, t'\frac{t_0-\tau}{t_0}, \mu\right),$$
$$\frac{dy'}{dt'} = \frac{t_0}{t_0+\tau}\psi\left(x'+x_0, y'+y_0, t'\frac{t_0+\tau}{t_0}, \mu\right)$$

We may regard x', y' and t' as the variables of $\mu$, $\tau$, $x_0$, $y_0$ as four arbitrary parameters.

The particular solution which we consider is such that, for t=0, we have

$$x = x_0, \qquad y = y_0$$

and, consequently,

$$x' = y' = 0.$$

We also have to calculate the values of x' and y' for $t=t_0+\tau$, i.e., for $t'+t_0$. /62

We therefore fall back on the case studied in the preceding article and we see that x and y are developable in terms of the powers of $x_0$, $y_0$, $\tau$ and $\mu$, provided that the moduli of these quantities are sufficiently small. For that there is only one condition: it is the particular solution, for which the initial values of x and y are zero and in which we assume in addition that $\mu=0$, does not pass through any single point.

Let us apply this to the equations of article 13

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

where

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots$$

and where $F_0$ does not depend on $y$.

F will be a function of x and $y$ which will cease being holomorphic only in certain singular points. It may happen that, if we give x the following values

$$x_1 = x_1^0, \quad x_2 = x_2^0, \quad \dots, \quad x_p = x_p^0,$$

the function F remains holomorphic for all the values of $y$.

Let us imagine that the following problem is proposed.

Let us consider the particular solution such that, for $t=0$, we have

$$x_1 = x_1^0 + \xi_1, \quad x_2 = x_2^0 + \xi_2, \quad \dots, \quad x_p = x_p^0 + \xi_p,$$
$$y_1 = y_1^0 + \eta_1, \quad y_2 = y_2^0 + \eta_2, \quad \dots, \quad y_p = y_p^0 + \eta_p,$$

and considering in particular the values of the variables for

$$t = t_0 + \tau,$$

to develop these values according to the powers of $\mu$, $\tau$, $\xi$ and $\varkappa$.

This development will be possible; in fact, if we make at the same time

$$\mu = \tau = \xi_i = \eta_i = 0,$$

the particular solution considered reduces to

$$x_i = x_i^0, \quad y_i = n_i t + y_i^0$$

(where $n_i$ is the value of $\frac{dF_0}{dx_i}$ for $x_k = x_k^0$), and, according to what we have just /6

assumed, this solution passes through no singular point.

Let us see what happens in the special case of the Problem of Three Bodies. The function F can cease to be holomorphic only if two of the three bodies have just collided. The particular solution which we are considering represents, in the case of $\mu=0$, the combination of two Keplerian ellipses described by the two small masses under the attraction of a mass equal to 1 placed at the origin. In order that a shock be produced, the two ellipses must intersect; now, this is what never happens in astronomical applications.

We therefore arrive at this conclusion:

In the Problem of Three Bodies, we will define the state of the system by the twelve variables defined in article 11.

We are given the values $x_i^0 + \xi_i, y_i^0 + \eta_i^0$ of these variables for $t=0$, and we ask what will be the values of these same variables at the epoch $t_0+\tau$.

We have just seen that these values are developable in terms of the powers of the masses, of the $\xi$, of $\varkappa$ and of $\tau$.

There is only one exceptional case, the following: let us assume that for $t=0$, the initial values of the variables are $x_i^0$ and $y_i^0$, and that, the masses being assumed zero, the motion then continues according to Kepler's laws; if under these conditions a shock is produced before the epoch $t_0$, what we have just said would no longer be true.

We would be able to calculate, in this manner, a lower limit of the time during which it is permitted to develop the coordinates of the planets as a power series in the masses; but the limit thus obtained would be much too distant from the precise limit for the calculation to present any interest.

## Use of Trigonometric Series

29. Power series are not the only ones which may serve in the integration of differential equations; trigonometric series are also used. I want to say a few words about them here before beginning the equations with partial derivatives. /64

It is known that a periodic function of x with period $2\pi$ can be developed in a series with the following form

$$\begin{aligned} F(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \ldots + A_n \cos nx + \ldots \\ + B_1 \sin x + B_2 \sin 2x + \ldots + B_n \sin nx + \ldots . \end{aligned}$$

I have shown in the Bulletin astronomique (November, 1886) that if the function $f(x)$ as well as its first derivatives p-2 is finite and continuous, and if its $(p-1)^{th}$ derivative is finite but can become discontinuous in a limited number of points, we can find a positive number K, such that we have, no matter how large n may be,

$$|n^p A_n| < K, \qquad |n^p B_n| < K.$$

If $f(x)$ is an analytical function, it as well as all its derivatives will be finite and continuous. We will therefore be able to find a number K, such that

$$|n^2 A_n| < K, \qquad |n^2 B_n| < K.$$

The result is that the series

$$\begin{aligned} |A_0| + |A_1| + |A_2| + \ldots + |A_n| + \ldots \\ + |B_1| + |B_2| + \ldots + |B_n| + \ldots \end{aligned}$$

converges and, consequently, that series (1) is absolutely and uniformly convergent.

This granted, let us consider a system of linear differential equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \varphi_{1,1} x_1 + \varphi_{1,2} x_2 + \ldots + \varphi_{1,n} x_n, \\ \frac{dx_2}{dt} &= \varphi_{2,1} x_1 + \varphi_{2,2} x_2 + \ldots + \varphi_{2,n} x_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{dx_n}{dt} &= \varphi_{n,1} x_1 + \varphi_{n,2} x_2 + \ldots + \varphi_{n,n} x_n. \end{aligned}\right\} \qquad (2)$$

The $n^2$ coefficients $\varphi_{i,k}$ are periodic functions of t with period $2\pi$.

Equations (2) therefore do not change when we change t to $t+2\pi$. This being so, let /65

$$\left.\begin{array}{llll} x_1=\psi_{1,1}(t), & x_2=\psi_{1,2}(t), & \ldots, & x_n=\psi_{1,n}(t), \\ x_1=\psi_{2,1}(t), & x_2=\psi_{2,2}(t), & \ldots, & x_n=\psi_{2,n}(t), \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, & \ldots, & \ldots\ldots\ldots, \\ x_1=\psi_{n,1}(t), & x_2=\psi_{n,2}(t), & \ldots, & x_n=\psi_{n,n}(t) \end{array}\right\} \qquad (3)$$

be n linearly independent solutions of equations (2).

The equations do not change when we substitute $t+2\pi$ for t and the n solutions will become

$$\begin{array}{lll} x_1=\psi_{1,1}(t+2\pi), & \ldots, & x_n=\psi_{1,n}(t+2\pi), \\ x_1=\psi_{2,1}(t+2\pi), & \ldots, & x_n=\psi_{2,n}(t+2\pi), \\ \ldots\ldots\ldots\ldots, & \ldots, & \ldots\ldots\ldots\ldots \\ x_1=\psi_{n,1}(t+2\pi), & \ldots, & x_n=\psi_{n,n}(t+2\pi). \end{array}$$

They must therefore be linear combinations of the n solutions (3), so that we will have

$$\left.\begin{array}{l} \psi_{1,1}(t+2\pi)=A_{1,1}\psi_{1,1}(t)+A_{1,2}\psi_{2,1}(t)+\ldots+A_{1,n}\psi_{n,1}(t), \\ \psi_{2,1}(t+2\pi)=A_{2,1}\psi_{1,1}(t)+A_{2,2}\psi_{2,1}(t)+\ldots+A_{2,n}\psi_{n,1}(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{n,1}(t+2\pi)=A_{n,1}\psi_{1,1}(t)+A_{n,2}\psi_{2,1}(t)+\ldots+A_{n,n}\psi_{n,1}(t), \end{array}\right\} \qquad (4)$$

values A being constant coefficients.

In addition, similarly (with the same coefficients) we will have

$$\psi_{1,2}(t+2\pi)=A_{1,1}\psi_{1,2}(t)+A_{1,2}\psi_{2,2}(t)+\ldots+A_{1,n}\psi_{n,2}(t),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This granted, let us form the equation with S

$$\begin{vmatrix} A_{1,1}-S & A_{1,2} & \ldots & A_{1,n} \\ A_{2,1} & A_{2,2}-S & \ldots & A_{2,n} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ A_{n,1} & A_{n,2} & \ldots & A_{n,n}-S \end{vmatrix} = 0. \qquad (5)$$

Let $S_1$ be one of the roots of this equation. According to the theory of linear substitutions, there will always exist n constant coefficients

$$B_1, \quad B_2, \quad \ldots, \quad B_n,$$

such that if we set /66

$$\theta_{1,1}(t)=B_1\psi_{1,1}(t)+B_2\psi_{2,1}(t)+\ldots+B_n\psi_{n,1}(t),$$

and furthermore

$$\theta_{1,i}(t)=B_1\psi_{1,i}(t)+B_2\psi_{2,i}(t)+\ldots+B_n\psi_{n,i}(t),$$

we have

$$\theta_{1,1}(t+2\pi)=S_1\theta_{1,1}(t)$$

and furthermore

$$\theta_{1,i}(t+2\pi) = S_1\theta_{1,i}(t).$$

Let us set

$$S_1 = e^{2\alpha_1\pi} \quad ;$$

we will have

$$e^{-\alpha_1(t+2\pi)}\theta_{1,1}(t+2\pi) = S_1 e^{-2\alpha_1\pi}e^{-\alpha_1 t}\theta_{1,1}(t) = e^{-\alpha_1 t}\theta_{1,1}(t).$$

This equation expresses that

$$e^{-\alpha_1 t}\theta_{1,1}(t)$$

is a periodic function which we will be able to develop in a trigonometric series

$$\lambda_{1,1}(t).$$

If the periodic functions $\varphi_{i,k}(t)$ are analytic, the same will result as from solutions of differential equations (2) and from $\lambda_{1,1}(t)$. The series $\lambda_{1,1}(t)$ will therefore be absolutely and uniformly convergent.

Furthermore,

$$e^{-\alpha_1 t}\theta_{1,i}(t)$$

will be a periodic function which we may represent by a trigonometric series

$$\lambda_{1,i}(t).$$

We therefore have a particular solution of equations (2), which is written

$$x_1 = e^{\alpha_1 t}\lambda_{1,1}(t), \quad x_2 = e^{\alpha_1 t}\lambda_{1,2}(t), \quad \ldots, \quad x_n = e^{\alpha_1 t}\lambda_{1,n}(t). \tag{6}$$

A solution of form (6) corresponds to each root of equation (5).

If equation (5) has all its roots distinct, we will have n linearly independent and the general solution may be written /67

$$\left.\begin{aligned} x_1 &= C_1 e^{\alpha_1 t}\lambda_{1,1}(t) + C_2 e^{\alpha_2 t}\lambda_{2,1}(t) + \ldots + C_n e^{\alpha_n t}\lambda_{n,1}(t) \\ x_2 &= C_1 e^{\alpha_1 t}\lambda_{1,2}(t) + C_2 e^{\alpha_2 t}\lambda_{2,2}(t) + \ldots + C_n e^{\alpha_n t}\lambda_{n,2}(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= C_1 e^{\alpha_1 t}\lambda_{1,n}(t) + C_2 e^{\alpha_2 t}\lambda_{2,n}(t) + \ldots + C_n e^{\alpha_n t}\lambda_{n,n}(t). \end{aligned}\right\} \tag{7}$$

The C are integration constants, $\alpha$ are constants, and $\lambda$ are absolutely and uniformly convergent trigonometric series.

Let us now see what occurs when equation (5) has a double root, for example, when $\alpha_1=\alpha_2$. Let us again take formula (7); let us there make $C_3=C_4=\ldots=C_n=0$, and let $\alpha_2$ tend toward $\alpha_1$. It will become

$$x_1 = e^{\alpha_1 t}[C_1\lambda_{1,1}(t) + C_2 e^{(\alpha_2-\alpha_1)t}\lambda_{2,1}(t)]$$

or, setting

$$C_1 = C_1' - C_2,$$

$$C_2 = \frac{C_2'}{\alpha_2 - \alpha_1},$$

it will become

$$x_1 = e^{\alpha_1 t}\left[C'_1 \lambda_{1,1}(t) + C'_2 \frac{e^{(\alpha_2-\alpha_1)t}\lambda_{2,1}(t) - \lambda_{1,1}(t)}{\alpha_2-\alpha_1}\right].$$

It is clear that the difference

$$\lambda_{2,1}(t) - \lambda_{1,1}(t)$$

will vanish for $\alpha_2=\alpha_1$. We will therefore be able to set

$$\lambda_{2,1}(t) = \lambda_{1,1}(t) + (\alpha_2-\alpha_1)\lambda'(t).$$

It will thus occur that

$$x_1 = e^{\alpha_1 t}\left[C'_1\lambda_{1,1} + C'_2\lambda_{1,1}\frac{e^{(\alpha_2-\alpha_1)t}-1}{\alpha_2-\alpha_1} + C'_2\lambda'(t)e^{(\alpha_2-\alpha_1)t}\right],$$

and at the limit (for $\alpha_2=\alpha_1$), $x_1 = C_1 e^{\alpha_1 t}\lambda_{1,1} + C'_2 e^{\alpha_1 t}[t\lambda_{1,1} + \lim \lambda'(t)]$.

We would see that the limit $\lambda'(t)$ for $\alpha_2=\alpha_1$ is still an absolutely and uni- /68
formly convergent trigonometric series.

Thus, the effect of the presence of a double root in equation (5) has been to introduce into the solution terms of the following form

$$e^{\alpha_1 t} t\lambda(t),$$

$\lambda(t)$ being a trigonometric series.

We would easily see that a triple root would introduce terms of the form

$$e^{\alpha_1 t} t^2\lambda(t),$$

and so forth.

I do not stress all these detailed points. These results are well known from the works of Floquet, Callandreau, Bruns and Stieltjes, and if I have here given the demonstration in extenso for the general case it is because its extreme simplicity allowed me to do so in a few words.

## Implicit Functions

30. If we have n+p quantities $y_1, y_2, \ldots, y_n; x_1, x_2, \ldots, x_p$ among which there exist n relations

$$\left.\begin{array}{l} f_1(y_1, y_2, \ldots, y_n; x_1, x_2, \ldots, x_p) = 0, \\ f_2(y_1, y_2, \ldots, y_n; x_1, x_2, \ldots, x_p) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(y_1, y_2, \ldots, y_n; x_1, x_2, \ldots, x_p) = 0, \end{array}\right\} \qquad (7)$$

if values f are developable as power series in x and y and vanish together with these n+p variables;

if finally the functional determinant of f with respect to y is not zero when x and y vanish simultaneously;

we will be able to solve the equations (7) for the n unknowns y in the form of a power series in $x_1, x_2, \ldots, x_n$.

Let us in fact consider $x_1$ as the only independent variable and $x_2$, $x_3$, ..., $x_n$ as arbitrary parameters: we will be able to replace equations (7) by the n partial differential equations

$$\frac{df_i}{dy_1}\frac{dy_1}{dx_1} + \frac{df_i}{dy_2}\frac{dy_2}{dx_1} + \dots + \frac{df_i}{dy_n}\frac{dy_n}{dx_1} + \frac{df_i}{dx_1} = 0 \qquad i = 1, 2, \dots, n).$$

The problem is thus reduced to the case with which we have just concerned ourselves. /69

In particular, if $f(y, x_1, x_2, \dots, x_n)$ is a function that can be developed as a power series in y and x, if for

$$y = x_1 = x_2 = \dots = x_n = 0,$$

we have

$$f = 0, \qquad \frac{df}{dy} \gtrless 0,$$

and if y is defined by the equality

$$f = 0,$$

y will be developable as a power series in x.

31. This result can be stated in another manner; let us in fact consider any algebraic equation

$$f(x) = 0.$$

If, for a certain value $x_0$, of x, f(x) vanishes without its derivative vanishing, we say that $x_0$ is a simple root of the equation; on the other hand, it is a multiple root of the n order if f vanishes together with its first n-1 derivatives.

Furthermore, if we have any system of algebraic equations, three for example, namely

$$f_1(x, y, z) = 0,$$
$$f_2(x, y, z) = 0,$$
$$f_3(x, y, z) = 0,$$

we will say that

$$x = x_0, \qquad y = y_0, \qquad z = z_0$$

is a simple solution of this system if for these values $f_1$, $f_2$, $f_3$ vanish without their Jacobian or functional determinant also vanishing.

We can retain the same definition when $f_1$, $f_2$ and $f_3$ rather than being whole polynomials in x, y, z are holomorphic functions in x, y, z.

The result of the preceding article can then be stated as follows: if we /73
have p equations (where the unknowns are $y_1$, $y_2$, ..., $y_p$)

$$
\begin{aligned}
f_1(y_1, y_2, \ldots, y_p; x_1, x_2, \ldots, x_n) &= 0,\\
f_2(y_1, y_2, \ldots, y_p; x_1, x_2, \ldots, x_n) &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\ldots,\\
f_p(y_1, y_2, \ldots, y_p; x_1, x_2, \ldots, x_n) &= 0,
\end{aligned}
$$

whose first members are holomorphic, if, for

$$x_1 = x_2 = \ldots = x_n = 0$$

the system of values

$$y_1 = y_2 = \ldots = y_p = 0$$

is a simple solution of the equations, y can be developed as a power series in x. If therefore we give x sufficiently small values, our equations will admit a real solution.

## Algebraic Singular Points

32. Let us consider an equation (1)

$$f(y, x) = 0,$$

and let us assume that for

$$x = y = 0,$$

f vanishes together with its first m-1 derivatives with respect to y. Then, for x=0, the value y=0 is a solution of the m order of the equation.

It can be demonstrated that there exist m convergent developments of y in terms of the positive and fractional powers of x, vanishing with x and satisfying the equation (cf. the classical works of Puiseux on algebraic equations).

However, these m convergent developments can be divided into groups in the following manner.

Let

$$y = \alpha_1 x^{\frac{1}{p}} + \alpha_2 x^{\frac{2}{p}} + \ldots + \alpha_n x^{\frac{n}{p}} + \ldots \qquad (2)$$

be one of these developments, and let $\lambda$ be a p-th root of unity.

The development /74

$$y = \alpha_1 \lambda x^{\frac{1}{p}} + \alpha_2 \lambda^2 x^{\frac{2}{p}} + \ldots + \alpha_n \lambda^n x^{\frac{n}{p}} + \ldots$$

will also satisfy equation (1). From the development (2) we will therefore be able to deduce p-1 other developments which with it will form a group.

I will say that this group is of the order p.

The sum of the orders of all groups is manifestly equal to m.

Let us suppose that there are $q_p$ groups of the order p; the sum of their orders will be $q_p p$, and we will have

$$q_1 + 2q_2 + \dots + pq_p + \dots = m.$$

The coefficients of the $pq_p$ developments belonging to groups of the order p will be given by algebraic equations of order $pq_p$.

If $pq_p$ is odd, these equations will have at least one real root and at least one of the developments will have real coefficients; if $pq_p$ is odd, and in addition p is odd, the corresponding value of y will then be real.

However, if m is odd, at least one of the quantities $pq_p$ is odd; at least one of the values of y must therefore be real.

If therefore m is odd, equation (1) will then admit at least one real solution for small values of x.

I add that the number of real solutions for small negative values of x are of the same parity as m; I intend to refer to real solutions which vanish with x.

Elimination

33. Let us now consider an equation

$$f(y, x_1, x_2, \dots, x_n) = 0 \tag{1}$$

and imagine that, when y and x are zero, f vanishes together with its first m-1 derivatives with respect to y, but the m-th derivative does not vanish.

At the beginning of my inaugural Thesis on functions defined by partial differential equations (Paris, Gauthier-Villars, 1879), I demonstrated that such an equation can be transformed into another with the following form /72

$$\varphi(y, x_1, x_2, \dots, x_n) = 0,$$

where $\varphi$ is a polynomial of degree m in y, where the coefficient of $y^m$ is equal to 1, and where the other coefficients are holomorphic with respect to the variables x.

If we suppose that m=1, this equation in x reduces to

y-holomorphic function of x=0,

and we may fall back on the theorem of article 30.

I also showed in this same Thesis (lemma IV, p. 14) that: $\varphi_1, \varphi_2, \dots, \varphi_p$ are p holomorphic functions in $z_1, z_2, \dots, z_p$; $x_1, x_2, \dots, x_p$; if these functions vanish when the variables z and the variables x vanish; if the equations

$$\varphi_1 = \varphi_2 = \ldots = \varphi_p = 0$$

remain distinct when we set all the variables x=0; if finally we define the variables z as functions of the variables x by equations

$$\varphi_1 = \varphi_2 = \ldots = \varphi_p = 0, \tag{2}$$

the p functions thus defined are algebraic; which means, according to the definition the thesis cited, that the equations (2) can be replaced by p other equations

$$\psi_1 = 0, \quad \psi_2 = 0, \quad \ldots, \quad \psi_p = 0$$

of the same form, but whose first members are polynomials in the variables z.

This granted, let there be two simultaneous equations

$$\left.\begin{array}{l}\varphi(x, y, z) = 0,\\ \psi(x, y, z) = 0,\end{array}\right\} \tag{3}$$

defining y and z as a function of x; I assume that the first members are holomorphic in x, y and z and vanish with these three variables.

From two possibilities one: the first that the two equations remain distinct when x=0; we will now be able, according to what we have already seen, to replace /73 the two equations by the two other equivalent ones,

$$\varphi_1(x, y, z) = 0,$$
$$\psi_1(x, y, z) = 0,$$

whose first members will be integral polynomials in y and z; we may then, between these two equations which have become algebraic, with respect to the two variables y and z, eliminate z, for example, and arrive at a unique equation

$$F(x, y) = 0,$$

or, on the other hand, when x=0, equations (3) will cease to be distinct.

However, a second case is then presented.

We may be able to find a number $\alpha$ such that equations (3) remain separate when we make $x = \alpha$.

Then, if we set $x' = x - \alpha$, the equations remain separate from $x' = 0$ and we fall back on the preceding case; we can eliminate z between the two equations (3) and reduce them to a single equation between x' and y or, what comes to the same thing, between x and y.

On the other hand, we may not be able to find a similar number $\alpha$; but that can happen only if equations (3) are not distinct; except for this exceptional case, elimination will therefore always be possible.

More generally, let

$$\left.\begin{array}{l}\varphi_1(z_1, z_2, \ldots, z_p; x) = 0,\\ \varphi_2(z_1, z_2, \ldots, z_p; x) = 0,\\ \ldots\ldots\ldots\ldots\ldots\ldots,\\ \varphi_p(z_1, z_2, \ldots, z_p; x) = 0\end{array}\right\} \quad (4)$$

be p equations whose first members are holomorphic and which define the z as functions of x; if these equations are separate, we will always be able to eliminate $z_2$, $z_3$, ..., $z_p$ among these p equations and reduce them to a single equation of the same form

$$F(x, z_1) = 0. \quad (5)$$

I assume that equations (4) are still distinct for x=0 and, consequently, that F is not divisible by x.

I assume that $\varphi_1$, $\varphi_2$, ..., $\varphi_p$ vanish with z and x, such that /74

$$z_1 = z_2 = \ldots = z_p = 0 \quad (6)$$

is a solution of system (4) for x=0, and that $z_1$=0 is a solution of equation (5).

If $z_1$=0 is a solution of the m order of equation (5), I will say also that solution (6) is a solution of the m order of the system (4).

If the solution is of an odd order, we may affirm that equation (5) and consequently system (4) still admit real solutions for small values of x.

## Theorem of the Maxima

34. Let $F(z_1, z_2, \ldots, z_p)$ be any function holomorphic with respect to z; we know that we will find all maxima of this function in solving the system

$$\frac{dF}{dz_1} = \frac{dF}{dz_2} = \ldots = \frac{dF}{dz_p} = 0, \quad (1)$$

but we also know that not all solutions of this system correspond to maxima.

I say that a necessary but insufficiently well understood condition for a solution to be able to correspond to a maximum of F is that this solution be of an odd order.

This is obvious if there is only one variable $z_1$ and only one equation

$$\frac{dF}{dz_1} = 0.$$

We know, in effect, that there can be no maximum here if the first derivative of F which does not vanish is not of an even order.

Let us extend the same result to the general case and, in order to fix these ideas, let us consider the case of only two variables $z_1$ and $z_2$. Let us regard $z_1$ and $z_2$ as the coordinates of a point in a plane; we can always assume that we have taken as origin that point which corresponds to the maximum, in such a manner that this maximum occurs for

$$z_1 = z_2 = 0.$$

We will then be able to describe around the origin a very small closed curve C, such that in all points we have

$$F(z_1, z_2) < F(0, 0).$$

However, even more, we can assume that this curve has as its equation

$$F(z_1, z_2) = F(0, 0) - \lambda^2,$$

$\lambda$ being a very small constant, and that in the interior of this closed curve C we have

$$F(z_1, z_2) > F(0, 0) - \lambda^2;$$

as a consequence, when we cross the curve C in going from the exterior to the interior, F will be increased.

What is to be established is that

$$z_1 = z_2 = 0$$

is an odd-order solution of the system

$$\frac{dF}{dz_1} = \frac{dF}{dz_2} = 0,$$

which is equivalent to saying the following: let

$$F(z_1, z_2, \mu)$$

be a function of $z_1$ and of $z_2$ which reduced to $F(z_1, z_2)$ for $\mu=0$.

The system

$$\frac{dF}{dz_1} = \frac{dF}{dz_2} = 0 \tag{1}$$

has, for $\mu=0$, a multiple solution which is

$$z_1 = z_2 = 0,$$

but we can always choose the function $F(z_1, z_2, \mu)$ (which is given us only for $\mu=0$, and which remains arbitrary for the other values of $\mu$), in such a manner that for the values of $\mu$ differing from zero this same system now has only simple solutions. Thus, what is to be established is that if $\mu$ is sufficiently small, there is, in the interior of curve C, an odd number of these simple solutions.

In my memoir, "Sur les courbes définies par les équations différentielles"

(Concerning curves defined by differential equations) (IVth part, Chap. XVIII, Journal de Liouville, 4th series, Vol. II, p. 177), I had occasion to study the distribution of singular points of a system of differential equations and to define for it the Kronecker index of a closed curve or of a surface closed with respect to this system of differential equations.

The system which we must consider here is the following

$$\frac{dz_1}{\left(\frac{dF}{dz_1}\right)} = \frac{dz_2}{\left(\frac{dF}{dz_2}\right)} , \qquad (2)$$

and, more generally,

$$\frac{dz_1}{\left(\frac{dF}{dz_1}\right)} = \frac{dz_2}{\left(\frac{dF}{dz_2}\right)} = \ldots = \frac{dz_n}{\left(\frac{dF}{dz_n}\right)} .$$

The singular points of system (2) will be the solutions of system (1).

We must calculate the Kronecker index of the closed curve C with respect to system (2). We can verify that it is equal to 1 for $\mu=0$, and from this conclude that it will be equal to 1 for small values of $\mu$, because it can vary only if one of the solutions of system (1) happens to cross this curve C.

The number of singular positive points of system (2) situated inside C is therefore equal to the number of the singular negative points plus one.

The total number of singular points, i.e., the total number of solutions of system (1), assumed simple, situated inside C, is therefore odd.

Q. E. D.

This reasoning is applicable without change to the case where there are more than two variables.

## New Definitions

35. For the moment I will not speak of the application of Cauchy's methods to differential partial equations, so as not to prolong these preliminary remarks, although I reserve the right to return to this question later. /77

I will end this chapter by giving a new extension to the notation $<$ from article 20.

Let $\varphi(x, y, t)$, $\Psi(x, y, t)$ two power series in x and y, such at the coefficients are periodic functions of t, developed in terms of the sine or cosine of multiples of t or, what amounts to the same thing, in positive and negative powers of $e^{it}$.

Let us therefore consider the development of $\varphi$ and $\psi$ in terms of the powers of x, y and $e^{it}$; if each coefficient of $\psi$ is real, positive and greater in absolute value than the corresponding coefficient of $\varphi$, we will write

$$\varphi < \psi \quad (\arg. x, y, e^{\pm it}).$$

If the series $\Psi$ is convergent for

$$x = |x_0|, \quad y = |y_0|, \quad t = 0,$$

the series $\varphi$ will converge for

$$x = x_0, \quad y = y_0, \quad t = \text{any real quantity.}$$

I add that it suffices that series $\psi$ converge when t=0 in order for it to converge whatever t may be.

If the series $\varphi(x, y, t)$ converges and if it represents an analytic function, the convergence is absolute and uniform, as we have seen in the preceding article.

We can therefore find a real positive constant $\alpha$ and a function M of t, periodic and of period $2\pi$, which are such that:

(1) the development of M, in positive and negative powers of $e^{it}$, has all its coefficients real and positive;

(2) we have

$$\varphi < \frac{M}{1 - \alpha(x + y)} \quad (\arg. x, y, e^{\pm it}).$$

We will therefore have a fortiori, whatever t may be, /73

$$\varphi < \frac{M_0}{1 - \alpha(x + y)} \quad (\arg. x, y),$$

$M_0$ being the value of M for t=0.

In fact, let

$$\varphi = \Sigma A x^m y^n e^{pit};$$

it will follow that

$$\frac{d^2\varphi}{dt^2} = -\Sigma A p^2 x^m y^n e^{pit}.$$

This series must converge, by hypothesis, for all real values of t and for the values of x and y which are within the circle of convergence. Let us assume, for example, that convergence takes place for

$$x = y = \frac{1}{\alpha}.$$

The terms of the series must be limited in absolute value, so that we will be able to write, calling K a positive constant,

$$|A| < \frac{\alpha^{m+n}}{p^2} K.$$

If we set

$$M = \sum \frac{K e^{pit}}{p^2},$$

it will follow that

$$\varphi < \frac{M}{(1 - \alpha x)(1 - \alpha y)} < \frac{M}{1 - \alpha(x + y)}.$$

# CHAPTER 3. PERIODIC SOLUTIONS

36. Let /79

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of differential equations where values X are uniform given functions of $x_1, x_2, \ldots, x_n$.

Now let

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t), \tag{2}$$

be a particular solution of this system. Let us imagine that at the epoch T the n variables $x_i$ take on their initial values, so that we have

$$\varphi_i(0) = \varphi_i(T).$$

It is clear that at this epoch we will have exactly the same conditions as at the epoch 0 and, consequently, we will have, whatever t may be,

$$\varphi_i(t) = \varphi_i(t + T).$$

In other words the functions $\varphi_i$ will be periodic functions of t.

We say then that solution (2) is a periodic solution of equations (1).

Let us now assume that the functions $X_i$ depend not only on the $x_i$, but on time t. I imagine, moreover, that values $X_i$ are periodic functions of t and that the period is equal to T. Then, if the functions $\varphi_i$ are such that

$$\varphi_i(0) = \varphi_i(T),$$

we will still be able to conclude that /80

$$\varphi_i(t) = \varphi_i(t + T),$$

and solution (2) will still be periodic.

Here is another case somewhat more complicated. Let us assume once again that functions $X_i$ no longer depend only on x but are periodic functions of the first p of x, namely $x_1, x_2, \ldots, x_p$, in such a way that functions $X_i$ do not change $x_1$ into $x_1+2\pi$, or $x_2$ into $x_2+2\pi$, ..., or $x_p$ into $x_p+2\pi$.

Let us now imagine that we have

$$\varphi_1(T) = \varphi_1(0) + 2k_1\pi, \quad \varphi_2(T) = \varphi_2(0) + 2k_2\pi, \quad \dots \quad \varphi_p(T) = \varphi_p(0) + 2k_p\pi$$
$$\varphi_{p+1}(T) = \varphi_{p+1}(0), \quad \varphi_{p+2}(T) = \varphi_{p+2}(0), \quad \dots \quad \varphi_n(T) = \varphi_n(0),$$

$k_1, k_2, \dots, k_p$ being integers.

At the epoch T, the first p variables of x will have increased by a multiple of $2\pi$, the last n-p will not have changed; $X_i$ will therefore not have changed, and we will find again the same conditions as at epoch 0. We will therefore have

$$\varphi_i(t+T) = \varphi_i(t) + 2k_i\pi \qquad (i = 1, 2, \dots, p),$$
$$\varphi_i(t+T) = \varphi_i(t) \qquad (i = p+1, p+2, \dots, n).$$

We will still agree to say that solution (2) is a periodic solution.

Finally it can happen that a convenient change of variables causes the appearances of periodic solutions, which we would not encounter with the old variables.

Let us again take, for example, equations (2) from article 2

$$\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} = \frac{dV}{d\xi} + n^2\xi,$$
$$\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} = \frac{dV}{d\eta} + n^2\eta.$$

It is a question, we recall, of the motion of a point related to two moving axes $O\xi$ and $O\eta$ and subject to a force whose components relative to these two axes are $\frac{dV}{d\xi}$ and $\frac{dV}{d\eta}$. In many applications, V depends only on $\xi$ and on $\eta$ and the /81 equations admit particular solutions such that $\xi$ and $\eta$ are periodic functions of t, the period being equal to T.

If we had related the point to the fixed axes Ox and Oy, we would have had

$$x = \xi\cos nt - \eta\sin nt,$$
$$y = \xi\sin nt + \eta\cos nt,$$

and x and y would not have been periodic functions of t unless T is commensurable with $\frac{2\pi}{n}$.

A periodic solution may thus appear in passing from the fixed axes to moving axes.

The problem which we are going to treat here is the following.

Let us assume that in equations (1) functions $X_i$ depend on a certain parameter $\mu$; let us assume that in the case of $\mu=0$ we have been able to integrate the equations, and that we have thus found the existence of a certain number of periodic solutions. Under what conditions will we have the right to conclude from this that the equations will still admit periodic solutions for small values of $\mu$.

Let us take for example the Problem of Three Bodies: we agreed above (art.11) to call $\alpha_2\mu$ and $\alpha_3\mu$ the masses of the two smallest bodies, $\mu$ being very small and $\alpha_2$ and $\alpha_3$ finite. For $\mu=0$, the problem is integrable, each of the two small bodies describing a Keplerian ellipse about the third; it is easy to see then that there exists an infinity of periodic solutions. We will see later that it is permissible to conclude from this that the Problem of Three Bodies still admits an infinity of periodic solutions, provided that $\mu$ is sufficiently small.

It seems at first that this fact can be of no interest whatever for practice. In fact, there is a zero probability for the initial conditions of the motion to be precisely those corresponding to a periodic solution. However, it can happen that they differ very little from them, and this takes place precisely in the case where the old methods are no longer applicable. We can then advantageously take the periodic solution as first approximation, as intermediate orbit, to use Gyldén's language.

There is even more: here is a fact which I have not been able to demonstrate /82
rigorously, but which seems very probable to me, nevertheless.

Given equations of the form defined in art. 13 and any particular solution of these equations, we can always find a periodic solution (whose period, it is true, is very long), such that the difference between the two solutions is as small as we wish, during as long a time as we wish. In addition, these periodic solutions are so valuable for us because they are, so to say, the only breach by which we may attempt to enter an area heretofore deemed inaccessible.

37. Let us again take the equations

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n), \tag{1}$$

assuming that values $X_i$ are functions of the n unknowns $x_1, x_2, \ldots, x_n$, of time t and of an arbitrary parameter $\mu$.

Let us assume, in addition, that these functions are periodic with respect to t and that the period is $2\pi$.

Let us imagine that, for $\mu=0$, these equations admit a periodic solution of period $2\pi$

$$x_i = \varphi_i(t)$$

in such a way that

$$\varphi_i(0) = \varphi_i(2\pi).$$

Let us attempt to see if equations (1) will still admit a periodic solution of period $2\pi$ when $\mu$ is no longer zero, but very small.

Let us consider now any solution.

Let $\varphi_i(0)+\beta_i$ be the value of $x_i$ for $t=0$; let $\varphi_i(0)+\beta_i+\psi_i$ be the value of $x_i$ for $t=2\pi$.

The $\psi_i$ will be, according to the theorem from art. 27, holomorphic functions of $\mu$ and of the $\beta_i$, and these functions will vanish for

$$\mu = \beta_1 = \beta_2 = \ldots = \beta_n = 0.$$

In order to express the fact that the solution is periodic, it is necessary to write the equations

$$\psi_1 = \psi_2 = \ldots = \psi_n = 0. \tag{1}$$

If the functional or Jacobian determinant of the $\psi$, with respect to $\beta$, is not zero /83 for $\mu=\beta_i=0$, the theorem from art. 30 tells us that we can solve these n equations with respect to $\beta$ and that we find

$$\beta_i = \theta_i(\mu),$$

$\theta_i(\mu)$ being developable in powers of $\mu$ and vanishing with $\mu$.

We must conclude from this that for values of $\mu$ sufficiently small, the differential equations admit one more periodic solution.

This is true, if the Jacobian of the $\psi$ is not zero or, in other words, if for $\mu=0$ equations (1) admit the system

$$\beta_1 = \beta_2 = \ldots = \beta_n = 0$$

as a simple solution.

What will now happen if this solution is multiple?

Let us assume that it is multiple of the order m. Let $m_1$ be the number of solutions of system (1) for small positive values of $\mu$, and $m_2$ the number of solutions of this same system for small negative values of $\mu$; I intend to refer to solutions such that $\beta_1, \beta_2, \ldots, \beta_n$ tend to 0 with $\mu$.

According to what we have seen in articles 32 and 33, the three numbers m,

$m_1$ and $m_2$ are of the same parity. If therefore m is odd, we will be assured that there still exist periodic solutions for small values of $\mu$ both positive and negative.

If $m_1$ is not equal to $m_2$, the difference can only be an even number; it can therefore result that, when we cause $\mu$ to increase in a continuous manner, a certain number of periodic solutions disappear at the moment when $\mu$ changes sign (or more generally, when nothing distinguishes the value $\mu=0$ from other values of $\mu$, at the moment when $\mu$ passes by some $\mu_0$); however, this number must always be even.

A periodic solution can therefore disappear only after being fused with another periodic solution.

In other words, the periodic solutions disappear by pairs in the manner of the real roots of algebraic equations.

According to art. 33, we can eliminate between equations (1) the n-1 variables $\beta_1, \beta_2, \beta_3, \ldots, \beta_{n-1}$, and obtain a single equation /84

$$\Phi(\beta_n, \mu) = 0 \tag{2}$$

whose first member is holomorphic in $\beta_n$ and $\mu$ and vanishes with these variables.

If we consider for a moment $\beta_n$ and $\mu$ as the coordinates of a point in a plane, this equation represents a curve passing through the origin; a periodic solution corresponds to each of the points of this curve.

We will therefore be able to realize all of the circumstances which can present themselves by studying the form of this curve in the neighborhood of the origin.

An interesting special case is the one where, for $\mu=0$, the differential equations admit an infinity of periodic solutions.

Let

$$x_1 = \varphi_1(t, h), \quad x_2 = \varphi_2(t, h), \quad \ldots, \quad x_n = \varphi_n(t, h)$$

be a system of periodic solutions containing an arbitrary constant h. No matter what this constant may be, the functions $\varphi_i$ are periodic of period $2\pi$ with respect to t, and they satisfy the differential equations when we substitute them there in place of the x, and make $\mu=0$.

In this case, for $\mu=0$, equations (1) are no longer distinct, and equation (2) must be reduced to an identity.

Then the function $\Phi$ must contain $\mu$ as a factor and reduce to $\mu\Phi_1$, in such a manner that curve (2) decomposes into a straight line $\mu=0$ and another curve $\Phi_1=0$.

A periodic solution corresponds to each point of this curve $\Phi_1=0$, so that the study of this curve will allow us to know the various circumstances which may present themselves.

However, this curve $\Phi_1=0$ does not always pass through the origin.

We must therefore, above all, make use of the arbitrary constant h in such a way that this curve passes through the origin.

Another special case which seems to me to be worthy of interest is the following: Let us assume that we have found by any means that curve $\Phi=0$ presents a branch B passing through the origin. A periodic solution will correspond to each of the points of this branch. Let us imagine, in addition, that we know in any manner that branch B is not tangent to the straight line $\mu=0$; let us assume finally that the functional determinant of the $\psi$ with respect to the $\beta$ is zero. From this we will conclude that /85

$$\frac{d\Phi}{d\beta_n}=0$$

and, as branch B is hypothetically not tangent to the straight line $\mu=0$, we will have to have

$$\frac{d\Phi}{d\mu}=0.$$

This shows that the curve $\Phi=0$ presents at the origin a multiple point: consequently one or several branches of the curve other than B will pass through the origin. Save for exceptional cases to which we will return later, at least one of these branches is real.

There will exist, therefore, apart from periodic solutions corresponding to branch B, another system of periodic solutions, and the solutions of the two systems will blend into only one for $\mu=0$. Here is a circumstance where this case will appear.

Above we called

$$\varphi_i(0)+\beta_i$$

the value of $x_i$ for t=0 and

$$\varphi_i(0)+\beta_i+\psi_i$$

the value of $x_i$ for $t=2\pi$.

In the same way, let us call

$$\varphi_i(o) + \beta_i + \psi'_i$$

the value of $x_i$ for $t=2k\pi$, k being an integer.

I suppose that for $\mu=\beta_1=\beta_2=\ldots=\beta_n=0$ the functional determinant of $\psi$ with respect to $\beta$, which I call $\Delta$, does not vanish, whereas the functional determinant of the $\psi'$ with respect to the $\beta$, which I call $\Delta'$, does vanish.

From the fact that $\Delta$ does not vanish, we can conclude that there exists a periodic solution, of period $2\pi$, which reduces to /86

$$x_i = \varphi_i(t)$$

for $\mu=0$. If we construct the curve

$$\Phi=0$$

corresponding to the periodic solutions so defined, this curve will pass through the origin, and its tangent will not be the straight line $\mu=0$, because $\Delta$ is not zero.

However, a solution of period $2\pi$ can be equally regarded as a periodic solution of period $2k\pi$.

Let us therefore seek the periodic solutions of period $2k\pi$. To do so, we will have to solve the equations

$$\psi'_1 = \psi'_2 = \ldots = \psi'_n = o.$$

Be eliminating among these equations $\beta_1$, $\beta_2$, ..., $\beta_{n-1}$, we will obtain a single equation

$$\Phi'(\beta_n, \mu) = o$$

which, according to our conventions, will represent a curve passing through the origin.

We must again find our solutions of period $2\pi$; therefore curve $\Phi=0$ will be one of the branches of the curve $\Phi'=0$ ($\Phi'$ will therefore be divisible by $\Phi$), and this branch will not touch the straight line $\mu=0$.

Moreover, since $\Delta'$ is zero, we will have

$$\frac{d\Phi'}{d\beta_n} = o.$$

Therefore the origin is a multiple point of curve $\Phi'=0$. There exist, therefore, solutions of period $2k\pi$, distinct from the solution of period $2\pi$ and merging with it for $\mu=0$.

There are several exceptional cases to which we will return later.

I still have to speak of the case where equations (1) from art. 36 admit an integral

$$F(x_1, x_2, \ldots, x_n, t) = \text{const}.$$

whose first member (which I will for brevity write as $F[x_i, t]$) is a periodic function of t of period $2\pi$. /87

I say that in this case equations

$$\psi_1 = \psi_2 = \ldots = \psi_n = 0 \qquad (1)$$

will in general not be distinct.

In effect, we will have identically

$$F[\varphi_i(0) + \beta_i; 0] = F[\varphi_i(0) + \beta_i + \psi_i; 2\pi] = F[\varphi_i(0) + \beta_i + \psi_i; 0]. \qquad (2)$$

Let us therefore consider equation

$$F[\varphi_i(0) + \beta_i + \psi_i, 0] - F[\varphi_i(0) + \beta_i, 0] = 0. \qquad (3)$$

The first member can be developed in terms of the powers of $\psi_i$, $\beta_i$ and $\mu$; in addition, it vanishes when the $\psi_i$ vanish.

Let us assume that we do not have

$$\frac{dF}{dx_n} = 0$$

for $x_i=\varphi_i(0)$, $\mu=0$.

The derivative of the first member of (3) with respect to $\psi_n$ does not vanish for

$$\psi_i = 0, \qquad \beta_i = 0, \qquad \mu = 0.$$

Therefore, by virtue of the theorem from art. 30, we will be able to eliminate from equation (3)

$$\psi_n = \theta(\psi_1, \psi_2, \ldots, \psi_{n-1}; \beta_1, \beta_2, \ldots, \beta_n, \mu),$$

$\theta$ being a series developed in terms of the powers of $\psi_1$, $\psi_2$, ..., $\psi_{n-1}$; $\beta_1$, $\beta_2$, ..., $\beta_n$ and $\mu$ vanishing when we have, at the same time,

$$\psi_1 = \psi_2 = \ldots = \psi_{n-1} = 0.$$

The n-th of equations (1) is therefore a consequence of the first n-1.

If we had
$$\frac{dF}{dx_n} = 0, \qquad \frac{dF}{dx_1} \gtrless 0$$

for $x_i=\varphi_i(0)$, it would be the first of equations (1), a result of the last n-1. In any case, equations (1) will not be distinct. There would be an exception only if we had at the same time /88

$$\frac{dF}{dx_1}=\frac{dF}{dx_2}=\ldots=\frac{dF}{dx_n}=0$$

for $x_i=\varphi_i(0)$, $\mu=0$.

We will therefore suppress one of equations (1), for example,

$$\psi_n=0,$$

(if $\frac{dF}{dx_n}\gtrless 0$), and then solve, with respect to $\beta$, the system $\psi_1=\psi_2=\ldots=\psi_{n-1}=0$, to which we will add an n-th arbitrarily chosen equation, for example,

$\beta_i$=arbitrary const. or F=C,

(C being a given constant).

For each value of $\mu$ there is therefore an infinity of periodic solutions of period $2\pi$; if, nevertheless, we regard the constant C (to which F is equated) as given condition of the problem, there are not generally more than one.

If instead of one uniform integral, we had two

$$F(x_1, x_2, \ldots, x_n, t)=\text{const.},$$
$$F_1(x_1, x_2, \ldots, x_n, t)=\text{const.},$$

the last two equations (1) would be a result of the first n-2, provided that the Jacobian

$$\frac{dF}{dx_n}\frac{dF_1}{dx_{n-1}}-\frac{dF}{dx_{n-1}}\frac{dF_1}{dx_n}$$

is not zero for $x_i=\varphi_i(0)$, $\mu=0$.

We could then eliminate these two last equations

$$\psi_{n-1}=\psi_n=0,$$

and replace them by two other arbitrarily chosen equations.

### Case When Time Does Not Enter Explicitly Into the Equations

38. In the preceding we assumed that the functions $X_1$, $X_2$, ..., $X_n$, which /89 enter differential equations (1), depend on time t. The results would be modified if time t does not enter these equations.

First there is a difference in the two cases which cannot be ignored. We had assumed in the preceding that values $X_i$ were periodic functions of time and that this period was $2\pi$; this gave us the results that if the equations admit a periodic solution, the period of this solution would have to be equal to $2\pi$ or to a multiple of $2\pi$. If, on the contrary, values $X_i$ are independent of t, the period of a periodic solution and be of any duration.

In the second place, if equations (1) admit one periodic solution (and if values X do not depend on t), they admit an infinity of such solutions.

If, in fact,

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \dots, \quad x_n = \varphi_n(t)$$

is a periodic solution of equations (1), it will be the same case, no matter what the constant h is, for

$$x_1 = \varphi_1(t+h), \quad x_2 = \varphi_1(t+h), \quad \dots, \quad x_n = \varphi_n(t+h).$$

Therefore, the case with which we first concerned ourselves, and for which, for $\mu=0$, equations (1) admit one and only one periodic solution, cannot occur if values X do not depend on t.

Let us therefore concern ourselves with the case where time t does not enter explicitly into equations (1), and let us assume that for $\mu=0$ these equations admit a periodic solution of period T

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \dots, \quad x_n = \varphi_n(t). \tag{4}$$

Let $\varphi_i(0)+\beta_i$ be the value of $x_i$ for t=0; let $\varphi_i(0)+\beta_i+\psi_i$ be the value of $x_i$ for $t=T+\tau$.

The $\psi_i$ will be holomorphic functions of $\mu$, $\beta_1$, $\beta_2$, ..., $\beta_n$ and of $\tau$ vanishing with these variables.

We therefore have to resolve with respect to the n+1 unknowns /90

$$\beta_1, \beta_2, \dots, \beta_n, \tau$$

the n equations

$$\psi_1 = \psi_2 = \dots = \psi_n = 0. \tag{5}$$

We have one unknown too many; we can therefore arbitrarily assume a value, for example,

$$\beta_n = 0.$$

We will then derive from equations (5) $\beta_1$, $\beta_2$, ..., $\beta_{n-1}$ and $\tau$ as holomorphic functions of $\mu$ vanishing with $\mu$. This is possible unless the determinant

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_1} & \cdots & \frac{d\psi_1}{d\beta_{n-1}} & \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_2}{d\beta_{n-1}} & \frac{d\psi_2}{d\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_1} & \cdots & \frac{d\psi_n}{d\beta_{n-1}} & \frac{d\psi_n}{d\tau} \end{vmatrix}$$

is not zero for $\mu=\varphi_i=\tau=0$.

If this determinant were zero, instead of arbitrarily setting $\beta_n=0$, we would set, for example, $\beta_i=0$, and the method would not fail if all the determinants contained in the matrix

$$\begin{Vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} & \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_2}{d\beta_n} & \frac{d\psi_2}{d\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n} & \frac{d\psi_n}{d\tau} \end{Vmatrix}$$

were zero at the same time. (It is to be noted that the determinant obtained by eliminating the last column of this matrix is always zero for $\mu=\beta_i=\tau=0$.)

Since in general all these determinants are not zero at the same time, equations (1) admit, for small values of $\mu$, a periodic solution of period $T+\tau$.

Let us call /91

$$\Delta_1,\ \Delta_2,\ \ldots,\ \Delta_n,\ \Delta_{n+1}$$

the determinants contained in this matrix; $\Delta_i$ will be the determinant obtained by eliminating the i-th column in it.

The periodic solution, which has served us as a starting point and which belongs to equations (1) for $\mu=0$, was written, we remember,

$$x_i = \varphi_i(t).$$

I designate by $\varphi_i'(t)$ the derivative of this function $\varphi_i(t)$ and here is what I propose to demonstrate:

If $\varphi'_n(0)$ is not zero, the determinant $\Delta_n$ cannot vanish without all the determinants

$$\Delta_1,\ \Delta_2,\ \ldots,\ \Delta_n,\ \Delta_{n+1}$$

vanishing at the same time.

In fact, if we assume that all these determinants are not zero at the same time and that $\Delta_n$ is zero, I say that $\varphi'_n(0)$ will be zero.

The differential equations not containing time explicitly will still admit for $\mu = 0$ the periodic solution

$$x_i = \varphi_i(t + h),$$

no matter what the constant h may be.

If therefore we make

$$\tau = 0, \qquad \mu = 0, \qquad \beta_i = \varphi_i(h) - \varphi_i(0),$$

the $\psi$ will vanish, no matter what h may be.

This will continue to be true if h is infinitely small, which gives us the relations

$$\frac{d\psi_i}{d\beta_1}\varphi'_1(0) + \frac{d\psi_i}{d\beta_2}\varphi'_2(0) + \dots + \frac{d\psi_i}{d\beta_n}\varphi'_n(0) \qquad (i = 1, 2, \dots, n). \tag{6}$$

These relations (6) first will show that $\Delta_{n+1}$ is zero.

Additionally, there cannot be among the quantities

$$\frac{d\psi_i}{d\beta_k}, \quad \frac{d\psi_i}{d\tau}$$

other linear relations of the same form, i.e., of the form /92

$$A_1 \frac{d\psi_i}{d\beta_1} + A_2 \frac{d\psi_i}{d\beta_2} + \dots + A_n \frac{d\psi_i}{d\beta_n} + A_{n+1} \frac{d\psi_i}{d\tau} = 0 \qquad (i = 1, 2, \dots, n). \tag{2}$$

Without this, in fact, all determinants $\Delta_i$ would vanish at the same time.

We have assumed that $\Delta_n$ is zero. However, this determinant is nothing other than the functional determinant of $\psi_1, \psi_2, \dots, \psi_n$ and $\beta_n$ with respect to $\beta_1, \beta_2, \dots, \beta_n$ and $\tau$. To say that this determinant is zero is therefore to say that we have among the derivatives of the $\psi$ relations of the form (2), and in addition that

$$A_1 \frac{d\beta_n}{d\beta_1} + A_2 \frac{d\beta_n}{d\beta_2} + \dots + A_n \frac{d\beta_n}{d\beta_n} + A_{n+1} \frac{d\beta_n}{d\tau} = 0,$$

i.e.,

$$A_n = 0.$$

However, here there cannot be relations of the form (2) other than the relations (1). We have therefore

$$A_n = \varphi'_n(o)$$

and consequently

$$\varphi'_n(o) = o.$$

If therefore $\varphi'_n(0)$ is not zero (and we can always assume so; for if it were not so, a change of appropriate variables would suffice to lead us to this case), it is unnecessary to consider all determinants $\Delta_i$: considering $\Delta_n$ suffices.

If $\Delta_n$ is not zero, we will solve with respect to the $\beta$ the equations

$$\psi_1 = \psi_2 = \ldots = \psi_n = \beta_n = o. \tag{3}$$

It appears at first that the arbitrary introduction of the equation $\beta_n=0$ limits generality and that we can thus find only the periodic solutions, which are such that $\beta_n$ is zero for t=0. However, we will obtain the others by changing t into t+h, h being any constant.

If, on the contrary, $\Delta_n$ is zero, we will eliminate $\beta_2$, $\beta_3$, ..., $\beta_n$ and $\tau$ among equations (3), and will obtain a single equation /93

$$\Phi(\beta_1, \mu) = o$$

analogous to the equation of the same form from the preceding article.

This equation may be regarded as representing a curve passing through the origin, and the study of this curve will permit us to become familiar with all the circumstances which might present themselves.

We will meet as well absolutely the same particular cases as in the preceding article.

For example, the periodic solutions, when we make $\mu$ vary in a continuous manner, can only vanish in pairs, in the manner of roots of algebraic equations.

It might also happen that, if we make $\mu$=0 and $\beta_n$=0, there exists an infinity of periodic solutions. Then $\Phi$ is divisible by $\mu$, and we may write

$$\Phi = \mu\Phi_1$$

in such a manner that curve $\Phi$=0 has two segments, the straight line $\mu$=0 and the curve $\Phi_1$=0. It will in this case be advantageous to replace the equation

$$\Phi = o$$

by the equation

$$\Phi_1 = 0.$$

It will even happen that some of the functions $\psi_i$ are divisible by $\mu$, in such a manner that, for example,

$$\psi_1 = \mu\psi'_1, \quad \psi_2 = \mu\psi'_2, \quad \psi_3 = \mu\psi'_3.$$

$\psi'_1$, $\psi'_2$, $\psi'_3$ being holomorphic functions of $\mu$, of $\beta$ and $\tau$.

It will then be advantageous to replace equations (3) by the following:

$$\beta_n = 0, \quad \psi'_1 = \psi'_2 = \psi'_3 = 0, \quad \psi_4 = \psi_5 = \ldots = \psi_n = 0.$$

We will see examples of this in the following text.

If we suppose that there exists an integral

$$F(x_1, x_2, \ldots, x_n) = \text{const.},$$

equations (3) are no longer distinct and we replace them advantageously by the following

$$\beta_n = 0, \quad F = C + \lambda\mu, \quad \psi_2 = \psi_3 = \ldots = \psi_n = 0,$$

where /94

$$C = F[\varphi_1(0), \varphi_2(0), \ldots, \varphi_n(0)],$$

while $\lambda$ is an arbitrary constant.

We will also be able to replace equations (3) by

$$\beta_n = 0, \quad \tau = 0, \quad \psi_2 = \psi_3 = \ldots = \psi_n = 0,$$

from which there results this important consequence: in the general case, for small values of $\mu$ there is no periodic solution having the same period T as for $\mu$=0; on the contrary, if there exists an integral F=const., we will be able to find, providing that $\mu$ is sufficiently small, a periodic solution having precisely this value T as a period.

In fact, if we do not have

$$\frac{dF}{dx_1} = 0$$

for

$$x_i = \varphi_i(0),$$

the equations

$$\psi_2 = \psi_3 = \ldots = \psi_n = 0$$

imply $\psi_1=0$.

Here is another circumstance that we have encountered in the preceding article and which we again will find here.

Let $\beta_i$ be the value of $x_i$ for $t=0$, $\beta_i+\psi_i$ be the value of $x_i$ for $t=T+\tau$, and $\beta_i+\psi_i'$ be the value of $x_i$ for $t=kT+\tau$, k being an integer.

Let us imagine that the functional determinant of the $\psi_i$, with respect to $\beta_1, \beta_2, \ldots, \beta_{n-1}, \tau$ is not zero, but that the functional determinant of $\psi_i'$ is zero.

Let us eliminate $\beta_2, \beta_3, \ldots, \beta_n$ and $\tau$ among the equations

$$\psi_i = 0, \qquad \beta_n = 0;$$

we will obtain the single equation

$$\Phi(\beta_1, \mu) = 0$$

which we will regard as representing a curve; this curve has a simple point at the origin.

Let us now eliminate $\beta_2, \beta_3, \ldots, \beta_n$ and $\tau$ among the equations

$$\psi_i' = 0, \qquad \beta_n = 0;$$

we will have /95

$$\Phi'(\beta_1, \mu) = 0.$$

We would see, as in the preceding article, that $\Phi'$ is divisible by $\Phi$. The curve $\Phi=0$ can therefore be regarded as one of the branches of the curve $\Phi'=0$; since the functional determinant of $\psi_i'$ is zero, we should have

$$\frac{d\Phi'}{d\beta_1} = 0.$$

Therefore, either the curve $\Phi'=0$ has several branches passing through the origin, or on the other hand, it must be tangent to the straight line $\mu=0$.

But already we are familiar with one of the branches of the curve $\Phi'=0$, namely $\Phi=0$, and we know that the tangent to this branch is not the straight line $\mu=0$. Therefore the curve $\Phi'=0$ has other branches passing through the origin.

This means that these differential equations admit periodic solutions of which the period is little different from kT, which are distinct from the periodic solutions of period T for small values of $\mu$, but which converge to them for $\mu=0$.

Application to the Problem of Three Bodies

39. Does the Problem of Three Bodies admit periodic solutions?

Let us again take the notations of article 11 and let us designate the three masses $m_1$, $\alpha_2\mu$ and $\alpha_3\mu$. If we make $\mu=0$, i.e., if the two small masses are regarded as zero, the large mass will be fixed and each of the two small ones will describe about the large one a Keplerian ellipse.

It is clear, then, that if the average motion of these two small masses are commensurable to each other, at the end of a certain time the entire system will again be found in its initial position, and consequently the solution will be periodic.

This is not all: instead of relating the three masses to fixed axes (or to movable axes which remain constantly parallel to the fixed axes, as in article 11), we can relate them to movable axes moving with a uniform rotational motion.

It can happen that the coordinates of the three masses, with respect to the fixed axes, are not periodic functions of time, although the coordinates with respect to the movable axes will on the contrary be periodic functions of time (cf. article 36). /96

Let us assume now that $\mu=0$; the two small masses will describe Keplerian ellipses; let us assume that these two ellipses are in the same plane, in the plane of the $x_1$, $x_2$, for example, and that their eccentricity is zero. The motion of the two small masses will then be circular and uniform; let n and n' be the mean motions of these two masses (n'>n).

Let us assume that the origin of the time has been chosen at the moment of conjunction, such that the initial longitude of the two masses is zero.

At the end of time $\frac{2\pi}{n'-n}$ these longitudes will have become, respectively,

$$\frac{2\pi n}{n'-n} \quad \text{and} \quad \frac{2\pi n'}{n'-n}$$

and their difference will be equal to $2\pi$.

The two masses are again in conjunction, and the three bodies will again be in the same relative position. The entire system will have turned through an angle equal to $\frac{2\pi n}{n'-n}$.

If, therefore, we relate the system to two movable axes turning with a uniform motion with an angular velocity equal to n, the coordinates of the three bodies with respect to these movable axes will be periodic functions of the time of period $\frac{2\pi}{n'-n}$.

From this point of view, and according to what we have said at the end of article 36, this solution can still be considered as periodic.

Thus, in the limiting case where $\mu=0$, the Problem of Three Bodies admits periodic solutions. Do we have the right to conclude from this that it will still admit them for small values of $\mu$? This is what the principles of articles 37 and 38 will permit us to decide.

The first periodic solution which has been indicated for the case where $\mu=0$ is that discovered by Lagrange and where the three bodies describe similar Keplerian ellipses, while their mutual distances remain in a constant relationship /97 (cf. Laplace, Mécanique céleste, Book X, Chapter VI). This case has been too well studied for us to repeat it.

Hill, in his quite remarkable researches on lunar theory (American Journal of Mathematics, Vol. 1), has studied another one, whose importance is much greater from the practical point of view.

I again took up the question in the Bulletin astronomique (Vol. 1, p.65) and I have been led to distinguish three types of periodic solutions: for those of the first type, the inclinations are zero and the eccentricities quite small; for those of the second type, the inclinations are zero and the eccentricities finite; finally, for those of the third type, the inclinations are no longer zero.

For these as for those, the mutual distances of the three bodies are periodic functions of time; at the end of a period, the three bodies again are found in the same relative position, the entire system having turned only by a certain angle. It is therefore necessary for the coordinates of the three bodies to be periodic functions of time, that they be related to a system of moving axes moving with a uniform rotational motion.

The speed of this rotational motion is finite for first-type solutions and very small for those of the two latter types.

## First-Type Solutions

40. I am going to reproduce here what I have already presented on the subject of these three types of solution. I will begin with those of the first type, which contain, as a particular case, that of Hill.

Let us again take up the notations from article 11. Let A,B,C be the three masses, which I will assume to remain constantly in the same plane. Let D be the center of gravity of A and of B. Let $x_1$, and $x_2$ be the coordinates of B with respect to the fixed axes having their origin in A; let $x_3$ and $x_4$ be the coordinates of C with respect to axes parallel to the fixed axes and having their origin in D.

Let us adopt the variables from article 12, i.e., the variables /98

$$\Lambda,\ \Lambda',\ \xi,\ \xi',\ p,\ p',$$
$$\lambda,\ \lambda',\ \eta,\ \eta',\ q,\ q'.$$

Here, the motion occurring in a plane, we will have

$$p = p' = q = q' = 0.$$

The mutual distances of the three bodies and the derivatives of these distances with respect to time are functions of

$$\left.\begin{array}{l}\Lambda, \quad \Lambda', \quad \xi\cos\lambda - \eta\sin\lambda, \quad \xi\sin\lambda + \eta\cos\lambda, \\ \xi'\cos\lambda' - \eta'\sin\lambda', \quad \xi'\sin\lambda' + \eta'\cos\lambda'\end{array}\right\} \tag{1}$$

and of $\lambda'-\lambda$.

In order for the solution to be periodic, it is therefore necessary that at the end of a period the variables (1) resume their initial values and that $\lambda'-\lambda$ increase to a multiple of $2\pi$, that is, $\lambda'-\lambda$ will increase by $2\pi$.

If we have $\mu=0$, the motion is Keplerian; let us assume that, moreover, the initial values of $\lambda, \lambda', \xi, \eta, \xi', \eta'$ are zero; then the motion will be circular and uniform.

If the initial values of $\Lambda_0$ and $\Lambda'_0$ of $\Lambda$ and of $\Lambda'$ are chosen in such a manner that the mean motions are n and n', the solution will be periodic of the period $\frac{2\pi}{n'-n}$.

Let us no longer assume now that $\mu$ is zero, and let us consider an arbitrary solution; we will be able to choose the origin of time at the moment of a conjunction and to take for the origin of the longitudes the longitude of this conjunction.

The initial values of $\lambda$ and $\lambda'$ will be zero.

Let $\Lambda_0+\beta_1$, $\Lambda'_0+\beta_2$ be the initial values of $\Lambda$ and $\Lambda'$.

Let $\xi_0$, $\eta_0$, $\xi'_0$, $\eta'_0$, be the initial values of $\xi, \eta$, and $\xi'$, $\eta'$.

These will also be the initial values of the four last variables (1).

Now let $2\pi+\psi_0$ be the value of $\lambda'-\lambda$ at the end of the period

$$\frac{2\pi}{n'-n}.$$

At the end of this same period, let

$$\Lambda_0 + \beta_1 + \psi_1, \quad \Lambda'_0 + \beta_2 + \psi_2$$

be the values of $\Lambda$ and $\Lambda'$, and let /99

$$\xi_0 + \psi_3, \quad \eta_0 + \psi_4, \quad \xi'_0 + \psi_5, \quad \eta'_0 + \psi_6$$

be the values of the last four variables (1).

In order for the solution to be periodic, it is necessary that

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = \psi_5 = \psi_6 = 0.$$

These equations are not distinct; the differential equations of motion admit in fact two integrals: the vis viva integral and the area integral. The Jacobian of these two integrals with respect to $\Lambda$ and $\Lambda'$ is not zero for

$$\mu = 0, \quad \xi = \eta = \xi' = \eta' = 0.$$

The equations $\psi_1 = \psi_2 = 0$ are therefore a consequence of the other five.

We therefore have to solve the system

$$\psi_0 = \psi_3 = \psi_4 = \psi_5 = \psi_6 = 0, \tag{2}$$

to which we will adjoin the vis viva equation F=C, where we consider the constant as a given condition of the problem.

It is therefore necessary that we consider the functional determinant of the first members of these six equations with respect to the six variables

$$\beta_1, \quad \beta_2, \quad \xi_0, \quad \eta_0, \quad \xi'_0, \quad \eta'_0$$

and that we demonstrate that this determinant does not vanish for

$$\mu = \beta_1 = \beta_2 = \xi_0 = \eta_0 = \xi'_0 = \eta'_0 = 0.$$

Now, for $\mu=0$, we have

$$F = F_0 = \frac{\gamma}{(\Lambda_0 + \beta_1)^2} + \frac{\gamma'}{(\Lambda'_0 + \beta_2)^2},$$

$\gamma$ and $\gamma'$ being constants depending on the masses,

$$\psi_0 = \frac{2\pi}{n' - n}\left[n'\left(1 + \frac{\beta_2}{\Lambda'_0}\right)^{-3} - n\left(1 + \frac{\beta_1}{\Lambda_0}\right)^{-3}\right],$$

$$\psi_3 = \xi_0(\cos\lambda_0 - 1) - \eta_0 \sin\lambda_0, \quad \psi_4 = \xi_0 \sin\lambda_0 + \eta_0(\cos\lambda_0 - 1),$$

$$\psi_5 = \xi'_0(\cos\lambda'_0 - 1) - \eta'_0 \sin\lambda'_0, \quad \psi_6 = \xi'_0 \sin\lambda'_0 + \eta'_0(\cos\lambda'_0 - 1),$$

where

$$\lambda_0 = \frac{2n\pi}{n' - n}\left(1 + \frac{\beta_1}{\Lambda_0}\right)^{-3}, \qquad \lambda'_0 = \frac{2n'\pi}{n' - n}\left(1 + \frac{\beta_2}{\Lambda'_0}\right)^{-3},$$

$\lambda_0$ and $\lambda'_0$ therefore designating the values of the two longitudes at the end of the period, in such a manner that /100

$$2\pi + \psi_0 = \lambda'_0 - \lambda_0.$$

We thus see that for $\mu=0$, F and $\psi_2$ depend only on $\beta_1$ and $\beta_2$; $\psi_3$ and $\psi_4$ on $\beta_1$, $\xi_0$ and $\eta_0$; $\psi_5$ and $\psi_6$ on $\beta_2$, $\xi'_0$ and $\eta'_0$.

Our functional determinant is therefore the product of three others:

(1) that of $F_0$ and $\psi_2$ with respect $\beta_1$ and $\beta_2$;

(2) that of $\psi_3$ and $\psi_4$ with respect to $\xi_0$ and $\eta_0$;

(3) that of $\psi_5$ and $\psi_6$ with respect to $\xi'_0$ and $\eta'_0$.

The first of these three determinants vanishes only for $\Lambda_0=\Lambda'_0$, $n=-n'$, this not being of importance, because if it vanishes, instead of adjoining to the vis viva equation system (2), we will adjoin there any other equation arbitrarily chosen between $\beta_1$ and $\beta_2$. No matter what it may be, the case of $n=-n'$ presenting difficulties of a diverse nature and having no importance from the point of view of applications, we will leave it aside.

(2) The second determinant reduces to

$$(1-\cos\lambda_0)^2+\sin^2\lambda_0.$$

It therefore can vanish only if $\lambda_0$ is a multiple of $2\pi$.

For

$$\beta_1=\beta_2=\xi_0=\eta_0=\xi'_0=\eta'_0=0$$

we have

$$\lambda_0=\frac{2n\pi}{n'-n}.$$

Our determinant therefore will vanish only if n is a multiple of n'-n.

(3) In the same way, the third determinant will vanish only if n', and consequently n, is a multiple of n'-n.

As a consequence:

For all values of the vis viva constant C, which is equal to

$$\left(\frac{n}{2}\right)^{\frac{2}{3}}\gamma^{\frac{1}{3}}+\left(\frac{n'}{2}\right)^{\frac{2}{3}}\gamma'^{\frac{1}{3}},$$

and for small values of $\mu$, the Problem of Three Bodies will admit a periodic solution of the first type whose period will be $\frac{2\pi}{n'-n}$. /11

There will be an exception only if n is a multiple of n'-n or if n=-n'.

There is a four-fold infinity of periodic solutions of the first type; we can in fact, if $\mu$ is small enough, arbitrarily choose:

(1) the period $\frac{2\pi}{n_0'-n_0}=T$;

(2) the constant C;

(3) the moment of conjunction, which we had taken in the preceding calculation for the origin of time;

(4) the longitude of conjunction, which we had taken for the origin of the longitudes, such that we have, for each value of $\mu$, $\infty^4$ periodic solutions.

We can obtain these solutions in the following manner.

Let us assume that at the origin of time we have

$$\lambda = \lambda' = \eta = \eta' = 0;$$

the three bodies will be in conjunction and their velocities will be perpendicular to the straight line which joins them; this straight line will be the axis $Ax_1$, which at this instant becomes the same with the axis $Dx_3$. From this symmetry of the position of the three bodies at the instant 0 the following consequences result immediately:

The values of the vector radii, at instant t and at instant -t will be the same; the values of the longitudes at instant t and at instant -t will be equal and of opposite sign.

We will then say that at the epoch 0 the three bodies are found in symmetrical conjunction.

We have assumed that there is symmetrical conjunction at time 0 and that at this moment the common longitude of the three bodies is zero; we have thus determined four of the osculating elements $\lambda$, $\lambda'$, $\eta$, $\eta'$. We will dispose of them in such a manner that at instant $\frac{T}{2}$ there is again symmetrical conjunction and that the common longitude of the three bodies is $\frac{2\pi}{n'-n}$ or, more exactly, that we have (calling $\nu$ and $\nu'$ the true longitudes) /102

$$v = \frac{n\pi}{n'-n}, \qquad v' = \frac{n\pi}{n'-n} + \pi.$$

It is therefore not a problem, properly speaking, of a symmetrical conjunction, but of a symmetrical opposition.

In order that there be symmetrical conjunction (or opposition), it is necessary, as we have just seen, to have four conditions; we will therefore have four equations to determine our four elements, which have remained arbitrary. These four equations may be resolved if the corresponding functional determinant is not zero; now, this is not true in general: it is, as we would see by easy calculation, completely similar to the one which precedes and which it is useless to reproduce here.

Thus, the vector radii have the same value at the epoch t and at the epoch -t; the same value still at the epoch t and at the epoch T-t (since there is still symmetrical conjunction at epoch $\frac{T}{2}$. As for the difference of the longitudes, its values at epochs t and -t (or still at epochs t and T-t) are equal and of opposite sign. Therefore the mutual distances of the three bodies are periodic functions whose period is T. These solutions, which present alternately symmetrical conjunctions and oppositions are therefore periodic solutions.

We might believe that the periodic solutions thus defined are less general than those when existence we had first demonstrated. This does not matter; there is also a four-fold infinity of these for we can arbitrarily choose the epoch of conjunction and opposition, and the longitude of the three bodies at the moment of this conjunction and of this opposition. Therefore there remain four parameters, which shows that all the solutions of the first type are included in this same category. If we conveniently choose epoch 0, there is for all solutions of the first type a symmetrical conjunction at the beginning of each period and a symmetrical opposition in the middle of each period.

We can make this clear in the following manner:

It is always permissible to assume that the origin of the times has been chosen in such a manner that the initial values of $\lambda$ and $\lambda'$ are zero. It suffices here to take the epoch of a conjunction as origin of the times and the longitude of this conjunction as origin of the longitudes. /12

On the other hand, the equations of the Problem of Three Bodies present a symmetry such that they do not change when we change t to -t, or when we change simultaneously $\lambda$ to $-\lambda$ and $\lambda'$ to $-\lambda'$.

If therefore there is periodic solution when the initial values of the variables $\Lambda$, $\Lambda'$, $\lambda$, $\lambda'$, $\xi$, $\eta$, $\xi'$, $\eta'$ will be $\Lambda_0'+\beta_1$, $\Lambda_0'+\beta_2$, 0, 0, $\xi_0$, $\eta_0$, $\xi_0'$, $\eta_0'$, there will still be periodic solution when these initial values are

$$\Lambda_0+\beta_1,\quad \Lambda_0'+\beta_2,\quad 0,\quad 0,\quad \xi_0,\quad -\eta_0,\quad \xi_0'-\eta_1'.$$

Equations (3) therefore do not change when we change $\eta_0$ and $\eta_0'$ into $-\eta_0$ and $-\eta_0'$.

Now these equations (3) have only one solution; we must therefore have

$$\eta' = \eta_0' = 0,$$

which means that at the origin of time there is symmetrical conjunction.

Q. E. D.

The $\infty^4$ periodic solutions of the first type are related one to another by simple relationships. We can pass from one to the other: (1) by changing the origin of time; (2) by changing the origin of the longitudes; (3) by changing simultaneously the units of length and time in such a manner that the unit of length is multiplied by $k^{2/3}$ when that of time is multiplied by k. All these changes do not alter the form of the equations and, consequently, can only change these periodic solutions into the others. There is therefore, in reality, only one simple infinity of truly distinct periodic solutions; each of these truly distinct solutions is characterized by the relationship $\frac{n_0'}{n_0'-n_0}$, or, which is the same, by the difference between the longitude of a symmetrical conjunction and that of the opposition which follows it.

## Hill's Researches Concerning the Moon

41. There is a particular case where the solutions of the first type are simplified: it is the one where one of the masses, the mass $m_2$ for example, is infinitely small. The motion of C with respect to A then remaining Keplerian, there can be a symmetrical conjunction there only when C passes at the perihelion or at the aphelion, at least that the motion of C is not circular. But the longitude of a symmetrical conjunction should therefore differ from the longitude of the symmetrical opposition which follows it immediately by an angle which should be a multiple of π. Now this will not be true unless $\frac{n_0'}{n_0'-n_0}$ is integral, a case which we have precisely excluded. We must therefore conclude that the motion of C is circular. /104

The simplicity is greater if we assume that the mass of C is much greater than that of A and that the distance of AC is very great (which is the case in lunar theory). If we assume AC infinitely large and the mass of C infinitely large, such that the angular velocity of C in its orbit remains finite; if, at the same time, we relate the mass B to two moving axes, namely to an axis A$\xi$ coinciding with AC and to an axis A$\eta$ perpendicular to the first, the equations of motion will become, as Hill has demonstrated,

$$\left.\begin{aligned} &\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + \left(\frac{\mu}{r^3} - 3n^2\right)\xi = 0 \\ &\frac{d^2\eta}{dt^2} - 2n\frac{d\xi}{dt} + \frac{\mu}{r^3}\eta = 0; \end{aligned}\right\} \quad (1)$$

n designating the angular velocity of C.

Periodic solutions of the first type still subsist in this case and they are those whose existence Hill first recognized, as I have said above.

They include symmetrical conjunctions and oppositions which can only take place on the axis of $\xi$. However, they include still other notable situations which we could call symmetrical quadratures; under these circumstances BAC is the right angle and the velocity of point B with respect to point A is perpendicular to BA. /105

In fact, the equations include a symmetry which does not change when we change $\xi$ into $-\xi'$; neither, therefore, should the periodic solutions change when we change $\xi$ into $-\xi$. If, therefore, we consider the relative trajectory of point B with respect to the system of movable axes $A\xi$ and $A\eta$, this trajectory is a closed curve (since the solution is periodic), which is symmetrical at the same time with respect to $A\xi$ and with respect to $A\eta$.

If, on the contrary, assuming the motion of C circular and taking for the axis of $\xi$ the straight line AC, we had not assumed the distance AC infinite (if, in other words, in constructing the lunar theory, we had taken account of the Sun's parallax in continuing to neglect the inclination of the orbits and the Sun's eccentricity), this relative trajectory would still have been a closed symmetrical curve with respect to the axis of $\xi$, but it would no longer have been symmetrical with respect to the axis of $\eta$.

Equations (1) admit an integral which is written

$$\frac{1}{2}\left(\frac{d\xi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\eta}{dt}\right)^2-\frac{\mu}{2}-\frac{3}{2}n^2\xi^2=C.$$

Hill has studied how the solutions of the first type vary when we increase C; he recognized that the relative trajectory is a symmetrical closed curve, the form roughly approximating that of an ellipse, of which the major axis would be the axis of the $\eta$. When C is very small, this type of ellipse differs very little from a circle, and its eccentricity increases rapidly with C. For large values of C, the curve begins to differ greatly from an ellipse, but the relation of the major axis to the minor continues to increase with C. Finally, for a certain value of C, which I will call $C_0$, the curve presents two points of regress situated on the axis of $\eta$. Hill terms this the orbit of the "Moon of maximum lunation." His calculation, based sometimes on the use of series, sometimes on the use of mechanical quadratures, is much too long to find a place here; I will state only that Hill has exactly constructed the curve point by point for various values of C, and in particular for $C=C_0$. He can have here, therefore, no doubt whatever concerning the exactitude of his results. /106

It is easy to realize the significance of these points of regress. I assume that at any instant the relative velocity of mass B with respect to the moving axes becomes zero such that we have at the same time

$$\frac{d\xi}{dt}=\frac{d\eta}{dt}=0;$$

it is clear that the relative trajectory will present a point of regress.

This is how he arrived at his "Moon of maximum lunation."

Hill expresses himself as follows:

"The Moon of the last line (i.e., the Moon of maximum lunation) is, of the class of satellites considered in this Chapter, that which, having the longest lunation, is still able to appear at all angles with the Sun and then undergo all possible phases. Whether this class of satellites is properly to be prolonged beyond this Moon, can only be decided by further employment of mechanical quadratures. But it is at least certain that the orbits, if they do exist, do not intersect the line of quadratures and that the Moons describing them would make oscillations to and fro, never departing as much as 90° from the points of conjunction or of opposition."

There is here, on the author's part, only a simple intuition not resting on any calculation or reasoning whatever. Simple considerations of analytical continuity permit me to affirm that this intuition failed him.

We first wonder if the solutions of the first type still exist for $C > C_0$, or, in other words, if the class of satellites studied by Hill can be prolonged beyond the Moon of maximum lunation. Let us assume, to this end, that at the origin of the times mass B (i.e., the Moon) is in quadrature (on the axis of the $\eta$), and that its relative velocity with respect to the moving axes is perpendicular to the axis of the $\eta$.

I call $\xi_0$, $\xi'_0$, $\eta_0$, $\eta'_0$ the initial values of $\xi, \frac{d\xi}{dt} = \xi', \eta$ and $\frac{d\eta}{dt} = \eta'$ . In the case of Hill's Moon of maximum lunation, we have $\xi_0 = \xi'_0 = \eta'_0 = 0$, and I call $\eta_0^0$ the corresponding value of $\eta_0$.

At the end of a time T, equal to the quarter of a period, this Moon will be found in symmetrical conjunction, and we will have /107

$$\eta = 0, \qquad \xi' = 0.$$

Let us consider now another particular solution from our differential equations, and let

$$0, \quad \xi'_0, \quad \eta_0, \quad 0$$

be the initial values of

$$\xi, \quad \xi', \quad \eta, \quad \eta',$$

in such a manner that at the origin of the times one is in symmetrical quadrature.

Let us consider the values of $\eta$ and of $\xi'$ at the end of time $T+\tau$; and let

$$\eta = f_1(T + \tau, \xi'_0, \eta_0),$$
$$\xi' = f_2(T + \tau, \xi'_0, \eta_0).$$

The values $f_1$ and $f_2$ are developable in terms of the powers of $\tau$, $\xi_0$ and $\eta_0 - \eta_0^0$, and vanish for

$$\tau = \xi'_0 = 0, \qquad \eta_0 = \eta_0^0.$$

If we have

$$f_1 = f_2 = 0, \tag{2}$$

one will be, at the end of the time T+τ, in symmetrical conjunction, and the solution will be periodic of period 4T+4τ.

We can extract from equations (2) $\tau$ and $\eta_0$ as functions of $\xi'_0$, and $\tau$ and $\eta_0$ are developable in terms of the powers of $\xi_0$.

There would only be an exception by virtue of article 30, if the functional determinant of $f_1$ and $f_2$ with respect to $\tau$ and $\eta_0$ vanished precisely for

$$\tau = \xi'_0 = 0, \qquad \eta_0 = \eta_0^0.$$

It is extremely improbable that this will be true; some doubts may still subsist, however, if Hill's mechanical quadratures do not clearly prove the contrary. Here is, in fact, how Hill proceeded to determine $\eta_0^0$. He calculated, for different values of T and of $\eta_0$, the functions /1

$$f_1(T, 0, \eta_0), \quad f_2(T, 0, \eta_0),$$

and he then determined by interpolation the values of T and of $\eta_0$, for which these two functions vanish. If the functional determinant of $f_1$ and of $f_2$ vanished precisely for these values, interpolation would have become impossible by ordinary processes. We must therefore conclude that the class of satellites discovered by Hill can be extended beyond the Moon of maximum lunation.

What therefore becomes the form of the orbit beyond this Moon? The values of $\xi$ and of $\eta$ depend on time t and on parameter $\xi'_0$, because the other initial value $\eta_0$ is given as a function of $\xi'_0$ by equations (2).

If $\xi'_0$ and t are sufficiently small, $\xi$ and $\eta$ can be developed in terms of the powers of these two variables. In addition, by reasons of symmetry, $\xi$ will only contain odd powers of t, and $\eta$ will only contain even powers of t. We therefore will have

$$\xi = \xi'_0 t + \frac{\xi'''_0}{6} t^3 + \frac{\xi^{V}_0}{120} t^5 + \dots,$$

$\xi_0^{(n)}$ being the initial value of the n-th derivative of $\xi$.

If $\xi'_0$ and t are sufficiently small, I may, without substantial error, reduce $\xi$ to its two first terms; in addition, $\xi'''_0$ is developable in terms of the increasing powers of $\xi_0'$; but, since $\xi_0'$ is very small, I may reduce $\xi_0'''$ to the value which this quantity take for $\xi_0'=0$. Now, for $\xi_0'=0$, we have

$$\xi_0''' = \frac{-2\mu n}{(\eta_0^0)^2};$$

we therefore have

$$\xi = \xi_0' t - \frac{\mu n}{3(\eta_0^0)^2} t^3. \tag{3}$$

for the Moons considered by Hill and of which the lunation is less than that of the Moon of maximum lunation, $\xi_0'$ is negative, the two terms of the second member of (3) are of the same sign, and $\xi$ cannot vanish for very small values of t, if it is not for t=0.

On the contrary, for the new satellites in question and which we encounter after the Moon of maximum lunation, $\xi_0'$ is positive and $\xi$ vanishes for /109

$$t = 0, \qquad t = \pm \eta_0^0 \sqrt{\frac{3\xi_0'}{\mu n}}.$$

There are therefore three very small values of t for which $\xi$ vanishes, i.e., three quadratures having very closely related epochs.

The relative trajectory for $C > C_0$ therefore presents the form represented by figure 1.

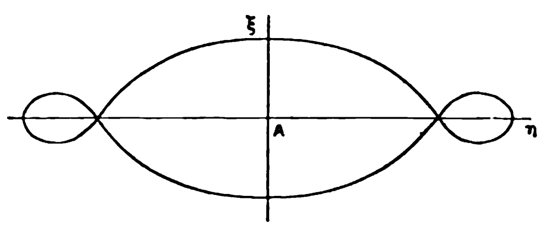


Figure 1.

In the course of a period, mass B is found in quadrature six times, for its relative trajectory intersects the axis of $\eta$ in two double points and in two simple points.

Thus Hill errs in assuming that this type of satellite will never be in quadrature; on the contrary, there will be three quadratures between two consecutive syzygies.

It is not that there do not exist any periodic solutions for which mass B can never be in quadrature: we will study them later, in article 52; but these solutions are not the analytical continuation of those which Hill so authoritatively studied in the American Journal.

The same results are still true when we do not neglect the Sun's parallax, except that the symmetry with respect to the axis of the $\eta$ disappears.

## Application to the General Problem of Dynamics

42. Now, before instituting the study of second and third kinds of periodic solutions, we are going to study the periodic solutions of the equations of Dynamics in a more general manner. /11

Let us return to the equations of article 13

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{1}$$

and the hypotheses of this article. The function F is developed in terms of the powers of a very small parameter $\mu$, such that

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots$$

Here F is a periodic function of y; $F_0$ is a function only of x. I will assume, to establish the ideas, that there are only 3 degrees of freedom. It is easy to integrate these equations when $\mu$=0 and when F=$F_0$.

In fact, $F_0$ not depending on y, these equations reduce to

$$\frac{dx_i}{dt} = 0, \qquad \frac{dy_i}{dt} = -\frac{dF_0}{dx_i} = n_i.$$

Values $x_i$ and consequently $n_i$ are therefore constants.

Thus, equations (1) admit the solution, when $\mu$=0,

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3,$$
$$y_1 = n_1 t + \varpi_1, \qquad y_2 = n_2 t + \varpi_2, \qquad y_3 = n_3 t + \varpi_3,$$

$\alpha$ and $\varpi$ being integration constants, and n being functions of $\alpha$.

It is clear that if

$$n_1 T, \quad n_2 T, \quad n_3 T$$

are multiples of $2\pi$, this solution is periodic of period T.

Let us now assume that $\mu$ ceases to be zero, and let us imagine that in a certain solution the values of x and y for t=0 are, respectively,

$$x_1 = a_1 + \beta_1, \qquad x_2 = a_2 + \beta_2, \qquad x_3 = a_3 + \beta_3,$$
$$y_1 = \varpi_1 + \beta_4, \qquad y_2 = \varpi_2 + \beta_5, \qquad y_3 = \varpi_3 + \beta_6.$$

Let us assume that in this same solution the values of x and y for t=T are /11

$$
\begin{aligned}
x_1 &= a_1 + \beta_1 \quad + \psi_1, \\
x_2 &= a_2 + \beta_2 \quad + \psi_2, \\
x_3 &= a_3 + \beta_3 \quad + \psi_3, \\
y_1 &= \varpi_1 + n_1 \mathrm{T} + \beta_4 + \psi_4, \\
y_2 &= \varpi_2 + n_2 \mathrm{T} + \beta_5 + \psi_5, \\
y_3 &= \varpi_2 + n_2 \mathrm{T} + \beta_6 + \psi_6.
\end{aligned}
$$

The condition for which this solution is periodic of period T is that we have

$$\psi_1 = \psi_2 = \psi_3 = \psi_4 = \psi_5 = \psi_6 = 0. \tag{12}$$

The six equations (12) are not distinct. In fact, as F=const. is an integral of equations (1) and in addition F is periodic with respect to y, we have

$$
\begin{aligned}
\mathrm{F}(a_i + \beta_i, \varpi_i + \beta_{i+3}) &= \mathrm{F}(a_i + \beta_i + \psi_i, \varpi_i + n_i \mathrm{T} + \beta_{i+3} + \psi_{i+3}) \\
&= \mathrm{F}(a_i + \beta_i + \psi_i, \varpi_i + \beta_{i+3} + \psi_{i+3}).
\end{aligned}
$$

It will suffice us therefore to satisfy five of equations (12). I will assume, in addition,

$$\varpi_1 = \beta_4 = 0.$$

For this it suffices to choose the origin of the time in such a manner that $y_1$ is zero for t=0.

It is easy to see that the $\psi_i$ and the $\psi_{i+3}$ are holomorphic functions of $\mu$ and of $\beta$, vanishing when all these variables vanish.

It therefore is a question of demonstrating that one can determine from the last five equations (12) the $\beta_i$ as functions of $\mu$.

Let us note that, when $\mu$ is zero, we have identically

$$\psi_1 = \psi_2 = \psi_3 = 0.$$

Consequently, $\psi_1$, $\psi_2$ and $\psi_3$, developed in terms of the powers of $\mu$ and of $\beta$, contain $\mu$ as a factor. We will isolate this factor $\mu$, and consequently write the five equations (12) which we have to solve in the form

$$\frac{\psi_2}{\mu} = \frac{\psi_3}{\mu} = \psi_4 = \psi_5 = \psi_6 = 0. \tag{13}$$

For $\mu=0$, we know the general solution of equations (1); we therefore easily find /112

$$
\begin{aligned}
\psi_4 &= -\mathrm{T} \frac{\partial}{\partial \beta_1} \mathrm{F}_0(a_1 + \beta_1, a_2 + \beta_2, a_3 + \beta_3), \\
\psi_5 &= -\mathrm{T} \frac{\partial}{\partial \beta_2} \mathrm{F}_0(a_1 + \beta_1, a_2 + \beta_2, a_3 + \beta_3), \\
\psi_6 &= -\mathrm{T} \frac{\partial}{\partial \beta_3} \mathrm{F}_0(a_1 + \beta_1, a_2 + \beta_2, a_3 + \beta_3).
\end{aligned}
$$

The functional determinant of $\psi_4$, $\psi_5$ and $\psi_6$ with respect to $\beta_1$, $\beta_2$ and $\beta_3$ is therefore equal, to within a factor $-T^3$, to the Hessian of $F_0$ with respect to x.

I now propose to express $\frac{\psi_1}{\mu}$, $\frac{\psi_2}{\mu}$ and $\frac{\psi_3}{\mu}$ functions of $\beta_4$, $\beta_5$ and $\beta_6$, assuming $\mu=0$ and at the same time

$$\beta_1 = \beta_2 = \beta_3 = 0.$$

Now we find

$$\frac{d}{dt}\left(\frac{x_i - a_i}{\mu}\right) = \frac{dF_1}{dy_i} + \mu \frac{dF_2}{dy_i} + \mu^2 \frac{dF_3}{dy_i} + \dots,$$

from which

$$\frac{\psi_i}{\mu} = \int_0^T \frac{dF_1}{dy_i} dt + \mu \int_0^T \frac{dF_2}{dy_i} dt + \dots \qquad (i = 1, 2, 3),$$

or, for $\mu=0$,

$$\frac{\psi_i}{\mu} = \int_0^T \frac{dF_1}{dy_i} dt. \tag{2}$$

Since we assume $\mu=0$ and at the same time

$$\beta_1 = \beta_2 = \beta_3 = 0,$$

and if we remember that $\omega = \beta_4 = 0$, we must, in the second member of equation (3), replace $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$ by respectively

$$a_1, \quad a_2, \quad a_3, \quad n_1 t, \quad n_2 t + \varpi_2 + \beta_5, \quad n_3 t + \varpi_3 + \beta_6.$$

Then $\frac{dF_1}{dy_i}$ becomes a periodic function of t.

We can write

$$F_1 = \Sigma A \sin(m_1 y_1 + m_2 y_2 + m_3 y_3 + h),$$

$m_1$, $m_2$, $m_3$ being positive integers, whereas A and h are functions of the independents x of y. /11

We then have

$$F_1 = \Sigma A \sin\omega, \qquad \frac{dF_1}{dy_i} = \Sigma A m_i \cos\omega = \frac{dF_1}{d\varpi_i},$$

where we have set, for brevity,

$$\omega = t(m_1 n_1 + m_2 n_2 + m_3 n_3) + h + m_2(\varpi_2 + \beta_5) + m_3(\varpi_3 + \beta_6);$$

$F_1$ thus becomes a periodic function of t of period T; it is equally a periodic function of period $2\pi$ with respect to $\varpi_2+\beta_5$ and to $\varpi_3+\beta_6$.

I will designate by $[F_1]$ the mean value of the periodic function $F_1$, such that

$$[F_1] = \frac{1}{T}\int_0^T F_1\,dt = SA\sin\omega,$$

the sign S signifying that summation must be extended over all terms, such that

$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0.$$

We then have

$$\frac{\psi_i}{\mu} = T\frac{d[F_1]}{d\varpi_i}, \qquad \frac{d}{d\beta_{k+3}}\left(\frac{\psi_i}{\mu}\right) = T\frac{d^2[F_1]}{d\varpi_i\,d\varpi_k}.$$

From this we conclude:

(1) that it is always possible to choose $\varpi_2$ and $\varpi_3$ in such a manner that the equations

$$\frac{\psi_2}{\mu} = \frac{\psi_3}{\mu} = 0$$

are satisfied for $\beta_5=\beta_6=0$.

In fact, function $[F_1]$, which is finite, is periodic in $\varpi_2$ and $\varpi_3$: it therefore admits a maximum and a minimum; we will have, for this maximum or this minimum,

$$\frac{d[F_1]}{d\varpi_2} = \frac{d[F_1]}{d\varpi_3} = 0,$$

and consequently,

$$\frac{\psi_2}{\mu} = \frac{\psi_3}{\mu} = 0.$$

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(2) that the functional determinant of $\frac{\psi_2}{\mu}$ and $\frac{\psi_3}{\mu}$, with respect to $\beta_5$ and /114
$\beta_6$, is equal to $T^2$ multiplied by the Hessian of $[F_1]$ with respect to $\varpi_2$ and $\varpi_3$.

From this it follows that we can choose the constants $\bar{\omega}_2$ and $\bar{\omega}_3$ so as to satisfy equations (13). It remains, in order to establish the existence of periodic solutions, to show that the functional determinant of these equations, i.e.,

$$\frac{\partial\left(\frac{\psi_2}{\mu}, \frac{\psi_3}{\mu}, \psi_4, \psi_5, \psi_6\right)}{\partial(\beta_1, \beta_2, \beta_3, \beta_5, \beta_6)},$$

is not zero.

Now, for $\mu=0$, $\psi_4$, $\psi_5$ and $\psi_6$ depend only on $\beta_1$, $\beta_2$, $\beta_3$ and not on $\beta_5$ and $\beta_6$. This functional determinant is therefore the product of two others

$$\frac{\partial\left(\frac{\psi_2}{\mu}, \frac{\psi_3}{\mu}\right)}{\partial(\beta_5, \beta_6)} \text{ and } \frac{\partial(\psi_4, \psi_5, \psi_6)}{\partial(\beta_1, \beta_2, \beta_3)}.$$

Now we have just calculated these two functional determinants, and we have seen they are equal, to within a constant factor, one to the Hessian of $[F_1]$ with respect to $\varpi_2$ and to $\varpi_3$, the other to the Hessian of $F_0$ with respect to the x.

Therefore, if neither of these two Hessians is zero, equations (1) will admit periodic solutions for small values of $\mu$.

We are now going to seek to determine no longer only the periodic solutions of period T, but the solutions of a period differing but little from T. We have taken for a point of departure the three numbers $n_1$, $n_2$, $n_3$; we would have been equally able to choose three other numbers, $n_1'$, $n_2'$, $n_3'$, provided that they are commensurable among themselves, and we would have arrived at a periodic solution whose period T' would have been the smallest common multiple of $\frac{2\pi}{n_1'}, \frac{2\pi}{n_2'}, \frac{2\pi}{n_3'}$.

If we take in particular

$$n_1' = n_1(1+\varepsilon), \quad n_2' = n_2(1+\varepsilon), \quad n_3' = n_3(1+\varepsilon),$$

the three numbers $n_1'$, $n_2'$, $n_3'$ will be commensurable among themselves, since they /1
are proportional to the numbers $n_1$, $n_2$ and $n_3$.

They will therefore lead us to a periodic solution of period

$$T + \tau = \frac{T}{1+\varepsilon},$$

such that we will have

$$x_i = \varphi_i(t, \mu, \varepsilon), \quad y_i = \varphi_i'(t, \mu, \varepsilon), \tag{14}$$

the $\varphi_i$ and the $\varphi_i'$ being functions developable in terms of the powers of $\mu$ and of

$\epsilon$, and periodic in t, but in such a manner that the period depends on $\epsilon$.

If in F we replace $x_i$ and $y_i$ by their values (14), F must become a constant independent of time (since F=const. is one of the integrals of equations (1)). But this constant, which is called the vis viva constant, will depend on $\mu$ and $\epsilon$ and can be developed in terms of increasing powers of these variables.

If the vis viva constant B is a given quantity of the problem, equation

$$F(\mu, \epsilon) = B$$

can perhaps be regarded as a relation which unites $\epsilon$ to $\mu$. If therefore we have chosen B arbitrarily, there will still exist a periodic solution, no matter what the value chosen for this constant; but the period will depend on $\epsilon$ and consequently on $\mu$.

A more particular case than the one which we have just dealt with in detail is that where there are only 2 degrees of freedom. Then F depends only on four variables, $x_1$, $y_1$, $x_2$, $y_2$, and function $[F_1]$ depends only on a single variable $\varpi_2$.

Relations (6) then reduce to

$$\frac{d[F_1]}{d\varpi_2} = 0, \qquad (15)$$

and the Hessian of $[F_1]$ reduces to $\frac{d[F_1]}{d\varpi_2^2}$; from this we conclude:

To each of the simple roots of equation (15) there corresponds a periodic solution of equations (1), which exists for all sufficiently small values of $\mu$.

I might even add that the case is still the same for each of the roots of odd order. /116

The existence of periodic solutions once demonstrated, it remains to show that these solutions can be developed in terms of the powers of $\mu$ and can be written

$$x_i = \theta_{i,0}(t) + \mu\theta_{i,1}(t) + \mu^2\theta_{i,2}(t) + \dots \qquad (i = 1, 2, \dots, n),$$

$\theta_{i,0}(t)$, $\theta_{i,1}(t)$, ..., being periodic functions of t which are developable in terms of the sine and cosine of the multiples of

$$\frac{2\pi t}{T+\tau}.$$

According to the theorem from article 28, we will have

$$x_i = H_i[t - t_1, \mu, x_1^0 - \varphi_1(0), x_2^0 - \varphi_2(0), \dots, x_n^0 - \varphi_n(0)],$$

if $x_1^0, x_2^0, \dots, x_n^0$ are the initial values of $x_1, x_2, \dots, x_n$ for t=0.

The value $H_i$ is developable in terms of the powers of

$$t - t_1, \quad \mu \quad \text{et} \quad x_i^0 - \varphi_i(0),$$

if $\mu$ is sufficiently small and if t is sufficiently close to $t_1$ and $x_1^0$ to $\varphi_i(0)$.

We will take

$$t = t_1 + \frac{t_1 \tau}{T}.$$

In addition, we will take

$$x_i^0 - \varphi_i(0) = \beta_i.$$

We will choose the $\beta_i$ and $\tau$ so as to obtain a periodic solution, i.e., to satisfy equations (9). We have just seen that, if $\tau$ and $\beta_i$ satisfy these equations (9), we will be able to develop $\tau$, $\beta_1$, $\beta_2$, ..., $\beta_n$ in terms of increasing powers of $\mu$, and that $\tau$ and the $\beta_i$ will vanish with $\mu$.

We will therefore have

$$x_i = H_i\left(\frac{t_1 \tau}{T}, \mu, \beta_1, \beta_2, \ldots, \beta_n\right) = K_i(\mu),$$

$K_i$ being a function developed in terms of the powers of $\mu$.

$K_i$ does not depend only on $\mu$, it depends also on $t_1$; we will therefore write /11

$$x_i = K_i(t_1, \mu),$$

remembering, nevertheless, that $K_i$ may be developed in terms of the powers of $\mu$, but not of those of $t_1$.

This granted, when we increase $t_1$ to T, we increase t to T+$\tau$, and since we have arranged to obtain a periodic solution of period T+$\tau$, $x_i$ may not change; we have therefore

$$K_i(t_1 + T, \mu) = K_i(t_1, \mu). \tag{10}$$

Since $K_i$ is developable in terms of the powers of $\mu$, we can write

$$K_i(t_1, \mu) = \theta_{i,0} + \theta_{i,1}\mu + \theta_{i,2}\mu^2 + \ldots,$$

$\theta_{i,0}, \theta_{i,1}, \theta_{i,2}, \ldots,$ depending only on $t_1$. Identity (10) then shows that $\theta_{i,k}$ does not change when we change $t_1$ into $t_1$+T. Therefore $\theta_{i,k}$ is a periodic function and can be developed in terms of the sine and cosine of the multiples of

$$\frac{2\pi t_1}{T} = \frac{2\pi t}{T+\tau}.$$

Q. E. D.

## Case Where the Hessian Is Zero

43. There can be difficulty when the Hessian of $F_0$ is zero.

This is how it is permissible, in a rather large number of cases, to overcome the difficulty.

Let us assume that the Hessian of $F_0$ with respect to the variables x is zero, but that we may find a function of $F_0$, which we will call $\varphi(F_0)$ and whose Hessian is not zero.

We are going to transform equations (1) in the following manner.

These equations admit the vis viva integral which is written

$$F=C.$$

If $\varphi'$ is the derivative of the function $\varphi$, we will have for F=C

$$\varphi'(F) = \varphi'(C),$$

and $\varphi'(C)$ will be a constant which may be regarded as known, if we assume that the initial conditions of motion are given and permitting, consequently, the calculation of the constant C. /118

Equations (1) can then be written

$$\frac{dx_i}{dt} = \frac{d[\varphi(F)]}{\varphi'(C)\,dy_i}, \qquad \frac{dy_i}{dt} = -\frac{d[\varphi(F)]}{\varphi'(C)\,dx_i}.$$

They retain the same form, but function $F_0$ is replaced by $\varphi(F_0)$ whose Hessian is not zero.

For example, let us take the particular case of the Problem of Three Bodies studied in article 6, the one where one of the masses is zero and where the other two move circularly.

In this case, we have found

$$F_0 = \frac{1}{2x_1^2} + x_2;$$

we therefore have

$$\frac{d^2F_0}{dx_2^2} = \frac{d^2F_0}{dx_2\,dx_1} = 0.$$

Our Hessian is therefore identically zero; however, if we take

$$\varphi(F_0) = F_0^2 = \frac{1}{4x_1^4} + \frac{x^2}{x_1^2} + x_2^2,$$

the Hessian of $\varphi(F)$ is equal to

$$\frac{6}{x_1^5}$$

and is different from 0.

Thus, all the preceding is applicable to this particular case of the Problem of Three Bodies which possesses periodic solutions for small values of $\mu$.

Let us, on the contrary, consider the general case of the Problem of Three Bodies treated in article 11.

We have found that this problem could be reduced to the canonical form, the two series of variables being

$$\begin{matrix} \beta L, & \beta G, & \beta \theta, & \beta' L', & \beta' G', & \beta' \theta', \\ l, & g, & \theta, & l', & g', & \theta'. \end{matrix}$$

The function F can be developed in terms of the powers of $\mu$

$$F = F_0 + F_1 \mu + F_2 \mu^2 + \dots,$$

and we have /11

$$F_0 = \frac{\beta^3}{2(\beta L)^2} + \frac{\beta'^3}{2(\beta' L')^2}.$$

If, to take up again the notations used in this chapter, we designate the two series of conjugate variables by

$$\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6, \\ y_1, & y_2, & y_3, & y_4, & y_5, & y_6, \end{matrix}$$

such that

$$x_1 = \beta L, \qquad x_4 = \beta' L,$$

we will have

$$F_0 = \frac{\beta^3}{2x_1^2} + \frac{\beta'^3}{2x_4^2} \quad ;$$

the Hessian of $F_0$ will be manifestly zero.

If we consider any function $\varphi(F_0)$, this function will still depend only on $x_1$ and $x_4$ and its Hessian will still be zero. The artifice that we used above is therefore no longer applicable, and the arguments of the present article no longer suffice to establish the existence of periodic solutions.

This is the origin of the difficulties which we will attempt to overcome in articles 46 to 48.

These difficulties are still the result, as we have just seen, of the fact that $F_0$ depends only on $x_1$ and $x_4$, i.e., because we have

$$\frac{dF_0}{dx_2} = \frac{dF_0}{dx_3} = \frac{dF_0}{dx_5} = \frac{dF_0}{dx_6} = 0 ,$$

or even, if $\mu=0$,

$$\frac{dy_2}{dt} = \frac{dy_3}{dt} = \frac{dy_5}{dt} = \frac{dy_6}{dt} = 0 .$$

These equations signify only that in Keplerian motion the perihelions and nodes are fixed.

Now, with any other law of attraction than that of Newton the perihelions and nodes will no longer be fixed.

Therefore, with a different law from the Newtonian, we would no longer encounter, in seeking periodic solutions of the Problem of Three Bodies, the difficulty which I have just indicated and to which articles 46 to 48 will be devoted.

## Direct Calculation of Series

44. We have just demonstrated that equations (1) from article 43 admit periodic solutions, and that these solutions can be developed in terms of the powers of $\mu$. /120

Let us now attempt to effectively form these developments, whose existence and convergence we have thus demonstrated in advance.

I begin by observing that we can introduce an important modification into the calculation of these developments. Above we introduced three numbers:

$$n_1, \quad n_2, \quad n_3,$$

such that

$$n_1 T, \quad n_2 T, \quad n_3 T$$

are multiples of $2\pi$, and consequently commensurable among themselves. These three numbers characterize the periodic solution considered.

I say that we can always, when studying a particular periodic solution, assume that

$$n_2 = n_3 = 0.$$

Let us assume, in fact, that this is not true. We will change variables by setting

$$\begin{aligned} y_1 &= \alpha_1 y'_1 + \alpha_2 y'_2 + \alpha_3 y'_3, & x'_1 &= \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3, \\ y_2 &= \beta_1 y'_1 + \beta_2 y'_2 + \beta_3 y'_3, & x'_2 &= \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3, \\ y_3 &= \gamma_1 y'_1 + \gamma_2 y'_2 + \gamma_3 y'_3, & x'_3 &= \alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3. \end{aligned}$$

The equations (with the new variables x' and y') will retain the canonical form.

If, in addition, $\alpha$, $\beta$ and $\gamma$ are integers and their determinant is equal to 1, function F, periodic with respect to y, will also be periodic with respect to y'.

If we call $n'_1$, $n'_2$, $n'_3$ the transformed values of the three characteristic numbers $n_1$, $n_2$, $n_3$ after the change of variables, these three numbers will be given /121
us by the equations

$$n_1 = \alpha_1 n'_1 + \alpha_2 n'_2 + \alpha_3 n'_3,$$
$$n_2 = \beta_1 n'_1 + \beta_2 n'_2 + \beta_3 n'_3,$$
$$n_3 = \gamma_1 n'_1 + \gamma_2 n'_2 + \gamma_3 n'_3;$$

since $n_1$, $n_2$, $n_3$ are commensurable among themselves, we can obviously choose the integers $\alpha$, $\beta$ and $\gamma$ such that

$$n'_2 = n'_3 = 0.$$

It is therefore permissible to assume

$$n_2 = n_3 = 0,$$

which is what we will do hereafter.

We are therefore going to seek to satisfy equations (1) by making

$$\left.\begin{aligned} x_1 &= x_1^0 + \mu x_1^1 + \mu^2 x_1^2 + \dots, \\ x_2 &= x_2^0 + \mu x_2^1 + \mu^2 x_2^2 + \dots, \\ x_3 &= x_3^0 + \mu x_3^1 + \mu^2 x_3^2 + \dots, \\ y_1 &= y_1^0 + \mu y_1^1 + \mu^2 y_1^2 + \dots, \\ y_2 &= y_2^0 + \mu y_2^1 + \mu^2 y_2^2 + \dots, \\ y_3 &= y_3^0 + \mu y_3^1 + \mu^2 y_3^2 + \dots, \end{aligned}\right\} \qquad (2)$$

$x_i^k$ and $y_i^k$ being periodic functions of time of period T. The values $x_i^0$ are constants such that

$$\frac{d}{dx_i^0} F_0(x_1^0, x_2^0, x_3^0) = -n_i, \qquad n_2 = n_3 = 0,$$

and, on the other hand, we have

$$y_i^0 = n_i t + \varpi_i,$$

from which

$$y_2^0 = \varpi_2, \qquad y_3^0 = \varpi_3,$$

$\varpi_1$, $\varpi_2$ and $\varpi_3$ being constants that we reserve for later, more complete determination.

The origin of time remaining arbitrary, we will be able to choose it such that $y_1=0$, whatever $\mu$ may be for $t=0$. From this it follows that $y_1^0$, $y_1^1$, $y_1^2$, ... will be zero at the same time for $t=0$ and $\varpi_1=0$.

In F, instead of x and y, we substitute their values (2) then we develop F in terms of the increasing powers of $\mu$, as was stated in article 22. It follows that /122

$$F = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \ldots$$

and we will have

$$\Phi_0 = F_0(x_1^0, x_2^0, x_3^0).$$

It will then follow (if we remember that $\frac{dF_0}{dx_i^0} = -n_i$ and that $n_2=n_3=0$) that

$$\Phi_1 = F_1(x_1^0, x_2^0, x_3^0, y_1^0, y_2^0, y_3^0) - n_1 x_1^1. \tag{3}$$

More generally, we will have

$$\Phi_k = \Theta_k - n_1 x_1^k = \Theta_k + x_1^k \frac{dF_0}{dx_1^0} + x_2^k \frac{dF_0}{dx_2^0} + x_3^k \frac{dF_0}{dx_3^0},$$

and $\Theta_k$ will only depend

on $x_i^0$, on $x_i^1$, ... and on $x_i^{k-1}$,
on $y_i^0$, on $y_i^1$, ... and on $y_i^{k-1}$.

With respect to the $y_i^0$, it is periodic of period $2\pi$.

This granted, the differential equations can be written, by equating the terms of the same powers in $\mu$,

$$\frac{dx_1^0}{dt} = \frac{dx_2^0}{dt} = \frac{dx_3^0}{dt} = 0, \quad \frac{dy_1^0}{dt} = n_1, \quad \frac{dy_2^0}{dt} = n_2, \quad \frac{dy_3^0}{dt} = n_3.$$

We then find

$$\frac{dx_1^1}{dt} = \frac{dF_1}{dy_1^0}, \quad \frac{dx_2^1}{dt} = \frac{dF_1}{dy_2^0}, \quad \frac{dx_3^1}{dt} = \frac{dF_1}{dy_3^0} \tag{4}$$

and

$$\frac{dy_1^1}{dt} = -\frac{d\Phi_1}{dx_1^0}, \quad \frac{dy_2^1}{dt} = -\frac{d\Phi_1}{dx_2^0}, \quad \frac{dy_3^1}{dt} = -\frac{d\Phi_1}{dx_3^0}, \tag{5}$$

and more generally

$$\frac{dx_i^k}{dt} = \frac{d\Phi_k}{dy_i^0} \tag{4'}$$

and

$$\frac{dy_i^k}{dt} = -\frac{d\Phi_k}{dx_i^0} = -\frac{d\theta_k}{dx_i^0} - x_1^k \frac{d^2 F_0}{dx_1^0\, dx_i^0} - x_2^k \frac{d^2 F_0}{dx_2^0\, dx_i^0} - x_3^k \frac{d^2 F_0}{dx_3^0\, dx_i^0}. \tag{5'}$$

Let us first integrate equations (4). In $F_1$ we will replace $y_1^0$, $y_2^0$, $y_3^0$ by their values /12

$$n_1 t, \quad \varpi_2, \quad \varpi_3.$$

Then the second members of equations (4) are periodic functions of t of period T; these second members can therefore be developed in series in terms of the sine and cosines of the multiple of $\frac{2\pi t}{T}$. In order for the values of $x_1^1$, $x_2^1$, $x_3^1$, taken from equations (4), to be periodic functions of t, it is necessary and sufficient that these series are not composed solely of given terms.

I may in fact write

$$F_1 = \Sigma A \sin(m_1 y_1^0 + m_2 y_2^0 + m_3 y_3^0 + h),$$

where $m_1$, $m_2$, $m_3$ are positive or negative integers and where A and h are functions of $x_1^0$, $x_2^0$, $x_3^0$. I write, for brevity,

$$F_1 = \Sigma A \sin \omega,$$

setting

$$\omega = m_1 y_1^0 + m_2 y_2^0 + m_3 y_3^0 + h.$$

I will then find

$$\frac{dF_1}{dy_1^0} = \Sigma A m_1 \cos\omega, \qquad \frac{dF_1}{dy_2^0} = \Sigma A m_2 \cos\omega, \qquad \frac{dF_1}{dy_3^0} = \Sigma A m_3 \cos\omega$$

and

$$\omega = t m_1 n_1 + h + m_2 \varpi_2 + m_3 \varpi_3.$$

Among the terms of these series, I will distinguish those for which

$$m_1 = 0$$

and which are independent of t.

$F_1$ being a periodic function of t, I will call $[F_1]$ the mean value of this function and I will have

$$[F_1] = SA \sin\omega, \qquad (m_1 = 0,\ \omega = h + m_2\varpi_2 + m_3\varpi_3),$$

the summation represented by the sign S extending to all terms of $F_1$ for which the coefficient of t is zero. We will then have /124

$$\frac{d[F_1]}{d\varpi_2} = SA\, m_2 \cos\omega, \qquad \frac{d[F_1]}{d\varpi_3} = SA\, m_3 \cos\omega.$$

If, therefore, we have

$$\frac{d[F_1]}{d\varpi_2} = \frac{d[F_1]}{d\varpi_3} = 0. \qquad (6)$$

it follows, since moreover $m_1$ is zero, that

$$SA\, m_1 \cos\omega = 0, \qquad SA\, m_2 \cos\omega = 0, \qquad SA\, m_3 \cos\omega = 0. \qquad (7)$$

If, therefore, relations (6) are satisfied, the series $\Sigma\, Am_i \cos\omega$ will not be composed solely of given terms, and equations (4) will give us

$$x_1^1 = \sum \frac{A_1 \sin\omega}{n_1} + C_1^1, \qquad x_2^1 = \sum \frac{A\, m_2 \sin\omega}{m_1 n_1} + C_2^1,$$

$$x_3^1 = \sum \frac{A\, m_3 \sin\omega}{m_1 n_1} + C_3^1,$$

$C_1^1$, $C_2^1$ and $C_3^1$ being three new integration constants.

It remains for me to demonstrate that we can choose the constants $\omega_2$ and $\varpi_3$ so as to satisfy relations (6). The function $[F_1]$ is a periodic function of $\varpi_2$ and $\varpi_3$, which does not change when one of these two variables increases by 2π. In addition, it is finite; it will therefore have at least one maximum and one minimum. There are therefore at least two means of choosing $\varpi_2$ and $\varpi_3$ so as to satisfy relations (6).

I could even add that there are at least four, still without being able to state when the number of degrees of freedom is greater than 3.

I shall now attempt to determine, with the help of equations (5), the three functions $y_i^1$ and the three constants $C_i^1$ .

We can consider $x_i^0$ and $y_i^0$ known; $x_i^1$ are also known to be constants near $C_i^1$. I can therefore write equations (5) in the following form

$$\frac{dy_i^1}{dt} = H_i - C_1^1 \frac{d^2F_0}{dx_1^0\,dx_i^0} - C_2^1 \frac{d^2F_0}{dx_2^0\,dx_i^0} - C_3^1 \frac{d^2F_0}{dx_3^0\,dx_i^0}, \tag{8}$$

where $H_i$ represents entirely known functions developed in series in terms of the /12 sines and cosines of multiples of $\frac{2\pi t}{T}$. The coefficients of $C_1^1$, $C_2^1$, $C_3^1$ are constants which we may regard as known.

In order for the value of $y_i^1$ obtained from this equation to be a periodic function of t, it is necessary and sufficient that in the second member the completely known term is zero. If, therefore, $H_i^0$ designates the completely known term of the trigonometric series $H_i$, I must have

$$C_1^1 \frac{d^2F_0}{dx_1^0\,dx_i^0} + C_2^1 \frac{d^2F_0}{dx_2^0\,dx_i^0} + C_3^1 \frac{d^2F_0}{dx_3^0\,dx_i^0} = H_i^0. \tag{9}$$

The three linear equations (9) determine the three constants $C_1^1$, $C_2^1$ and $C_3^1$.

There would only be an exception if the determinant of these three equations were zero, i.e., if the Hessian of $F_0$ with respect to $x_1^0$, $x_2^0$ and $x_3^0$ were zero; we will exclude this case.

Equations (8) will therefore give me

$$y_i^1 = \int_0^t \frac{dy_i^1}{dt}\,dt + k_i^1$$

or

$$y_1^1 = \eta_1^1 + k_1^1, \quad y_2^1 = \eta_2^1 + k_2^1, \quad y_3^1 = \eta_3^1 + k_3^1,$$

$\eta_i^1$ being completely known periodic functions of t and $k_i^1$ being three new constants of integration. It follows moreover, from the equations which I have just written, that $\eta_1^1 = \eta_2^1 = \eta_3^1$ for t=0.

Let us now come to equations (4') by here setting k=2 and i=1,2,3 and let us attempt to determine, with the aid of the three equations thus obtained, the three functions $x_i^2$ and the three constants $k_i^1$.

It is easy to see that we have

$$\Theta_2 = \Omega_1 + y_1^1 \frac{dF_1}{dy_1^0} + y_2^1 \frac{dF_1}{dy_2^0} + y_3^1 \frac{dF_1}{dy_3^0} \;;$$

depends only on the $x_i^0$, $y_i^0$ and $x_i^1$ and where we have, as above,

$$\frac{dF_1}{dy_i^0} = \Sigma\, A m_i \cos\omega.$$

Equations (4') are then written /126

$$\frac{dx_i^2}{dt} = \frac{d\Omega_2}{dy_i^0} + \sum_k y_k^1 \frac{d^2F_1}{dy_k^0\, dy_i^0}$$

or

$$\frac{dx_i^2}{dt} = H_i^1 - k_1^1 \Sigma A m_1 m_i \sin\omega - k_2^1 \Sigma A m_2 m_i \sin\omega - k_3^1 \Sigma A m_3 m_i \sin\omega, \tag{10}$$

$H_i^1$ being a periodic function of t, which we may regard as entirely known. In order for us to be able to extract from this equation $x_i^2$ in the form of a periodic function, it is necessary and sufficient that the second members of equations (10), developed in trigonometric series, possess no completely given terms. We must therefore choose the quantities $k_i^1$ so as to eliminate these completely given terms. We would thus be led to three linear equations among the three quantities $k_i^1$; however, since the determinant of these three equations is zero, there is a slight difficulty and I am forced to go into some details.

Since we assumed above that $y_1^1=0$ for $t=0$, we will have

$$k_1^1 = 0;$$

we will have only two unknowns, $k_2^1$ and $k_3^1$, and three equations to satisfy; but these three equations are not distinct, as we shall see.

Let us in fact call $E_i$ the completely known term of $H_i^1$; these three equations are written (if we remember that the summation sign $S$ refers to terms such that $m_1=0$)

$$\left.\begin{aligned} E_1 &= 0, \\ E_2 &= k_2^1\, S\, A m_2^2 \sin\omega + k_3^1\, S\, A m_2 m_3 \sin\omega, \\ E_3 &= k_2^1\, S\, A m_2 m_3 \sin\omega + k_3^1\, S\, A m_3^2 \sin\omega; \end{aligned}\right\} \tag{11}$$

the latter two of equations (11) might also be written

$$-E_2 = k_2^1 \frac{d[F_1]}{d\varpi_2^2} + k_3^1 \frac{d^2[F_1]}{d\varpi_2\, d\varpi_3},$$

$$-E_3 = k_2^1 \frac{d^2[F_1]}{d\varpi_2\, d\varpi_3} + k_3^1 \frac{d^2[F_1]}{d\varpi_3^2}.$$

From these two equations we obtain $k_2^1$ and $k_3^1$, unless the Hessian of $[F_1]$, with respect to $\omega_2$ and $\omega_3$, is zero. If we give $k_i^1$ the values thus obtained, the two latter equations (10) will give us $x_2^2$ and $x_3^2$ in the following form /127

$$x_2^2 = \xi_2^2 + C_2^2, \qquad x_3^2 = \xi_3^2 + C_3^2,$$

$\xi_i^2$ being the completely known, periodic functions of t and the $C_i^2$ being the integration constants.

In order to find $x_1^2$, we can, instead of using the first of equations (10), make use of the following considerations:

Equations (1) admit an integral

$$F=B,$$

B being an integration constant which I will assume developed in terms of the powers of $\mu$ when writing

$$B = B_0 + \mu B_1 + \mu^2 B_2 + \dots,$$

such that we have

$$\Phi_0 = B_0, \qquad \Phi_1 = B_1, \qquad \Phi_2 = B_2, \qquad \dots,$$

$B_0$, $B_1$, $B_2$, ... being so many different constants.

The first member of the equation

$$\Phi_2 = B_2$$

depends on $x_i^0$, $y_i^0$, $x_i^1$, $y_i^1$, $x_2^2$ and $x_3^2$, which are continuous functions of t, and on $x_1^2$ which we have not yet calculated. From this equation we will therefore be able to obtain $x_1^2$ in the following form

$$x_1^2 = \xi_1^2 + C_1^2;$$

$\xi_1^2$ will be a periodic function of completely determined t and $C_1^2$ is a constant which depends on $B_2$, $C_2^2$ and $C_2^3$.

We can conclude from this that the first of equations (11) are not distinct.

Let us now take equations (5') and here make k=2; we will obtain three equations which will permit us to determine the constants $C_1^1$, $C_1^2$ and $C_1^3$ and from which we will furthermore obtain $y_i^2$ in the form /128

$$y_1^2 = \eta_1^2 + k_1^2, \quad y_2^2 = \eta_2^2 + k_2^2, \quad y_3^2 = \eta_3^2 + k_3^2,$$

$\eta$ being entirely known periodic functions of t and k being three new integration constants.

Let us then return to equations (4'), making k=3; if we assume $k_1^2=0$, we will be able to obtain from the three equations thus obtained first the two constants $k_2^2$ and $k_3^2$, then $x_i^3$ in the form

$$x_i^3 = \xi_i^3 + C_i^3,$$

$\xi$ being known, periodic functions of t and $C_i^3$ being three new integration constants.

And so forth.

There is a process to find the power series of $\mu$, periodic of period T with respect to time and satisfying equations (1). This process would fail only if the Hessian of $F_0$ with respect to the $x_i^0$ was zero or if the Hessian of $[F_1]$ with respect to $\bar{\omega}_2$ and $\bar{\omega}_3$ was zero.

## Direct Demonstration of Convergence

45. It might be useful to be familiar with a direct demonstration of convergence of the series which we have just formed and whose existence and convergence we had previously demonstrated in article 28. I will first give this demonstration in a particular case.

Let

$$\frac{d^2y}{dx^2} + \mu f(x, y) \tag{1}$$

be a differential equation; we have seen in article 2 that this equation (considered by Gyldén and then by Lindstedt in their researches on Celestial Mechanics) can be regarded as a particular case of the equations of Dynamics with only 2 degrees of freedom.

I will assume that f(x,y) can be developed in terms of the increasing powers of y, and that we have

$$f = f_0 + f_1 y + f_2 y^2 + \dots,$$

$f_0$, $f_1$, $f_2$, ... being functions of x which I will assume to be periodic and of /129
period $2\pi$. I will assume in addition that the mean value of $f_0$ is zero:

$$[f_0] = 0.$$

This granted, I am going to attempt to develop y in terms of the powers of $\mu$, such that

$$y = y_1\mu + y_2\mu^2 + \ldots + y_n\mu^n + \ldots.$$

By substituting this value of y in $\varphi$ it follows that

$$\varphi = \varphi_0 + \mu\varphi_1 + \ldots + \mu^n\varphi_n + \ldots,$$

and the differential equations will become

$$\frac{d^2y_0}{dx^2} = 0, \quad \frac{d^2y_1}{dx^2} = \varphi_0, \quad \frac{d^2y_2}{dx^2} = \varphi_1, \quad \ldots, \quad \frac{d^2y_n}{dx^2} = \varphi_{n-1}, \quad \ldots.$$

We want $y_1$, $y_2$, ... to be periodic functions of x. This will be possible provided the mean values of the second members are zero, i.e., that we have

$$[\varphi_0] = 0, \quad [\varphi_1] = 0, \quad \ldots, \quad [\varphi_n] = 0.$$

The condition is satisfied by the first member, because we have

$$\varphi_0 = f_0, \quad [\varphi_0] = [f_0] = 0.$$

On the other hand, it follows that

$$\varphi_n = \theta_n + f_1 y_n,$$

$\theta_n$ depending only on $y_1$, $y_2$, ..., $y_{n-1}$.

Let $[y_n]$ be the mean value of $y_n$, and let us set

$$y_n = \eta_n + [y_n],$$

such that $\eta_n$ is a periodic function of x, whose mean value is zero.

This granted, let us imagine that we have determined by previous calculation /1

$$\left.\begin{matrix} \eta_1, & \eta_2, & \ldots, & \eta_n, \\ [y_1], & [y_2], & \ldots, & [y_{n-1}], \end{matrix}\right\} \qquad (2)$$

and consequently also $y_1$, $y_2$, ...,$y_{n-1}$, and that we propose to calculate $\eta_{n+1}$ and $[y_n]$.

The relation $[\varphi_n]=0$ can be written

$$[\theta_n] + [f_1\eta_n] + [f_1][y_n] = 0.$$

In this equation $[\theta_n]$ and $[f_1\eta_n]$ can be regarded as known, because the quantities (2) are known; $[f_1]$ is a given constant; we can therefore obtain $[y_n]$ from it.

We then have

$$\frac{d^2 y_{n+1}}{dx^2} = \frac{d^2 \eta_{n+1}}{dx^2} = \varphi_n.$$

If I set

$$\varphi_n = \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx,$$

it will follow that

$$\eta_{n+1} = -\sum \frac{A_m}{m^2} \cos mx - \sum \frac{B_m}{m^2} \sin mx.$$

Values $\eta$, $[y]$ and y can therefore be calculated in this way by recurrence.

There results from this that if $\psi$ is a periodic function of x, such that we have, taking on the notation from article 20 completed in article 35,

$$\varphi_n < \psi, \quad (\arg e^{\pm ix}) \;;$$

we will have a fortiori

$$\eta_{n+1} < \psi, \quad (\arg e^{\pm ix}).$$

We will write in the following

$$\theta = f - f_0 - f_1 y = f_2 y_2 + f_3 y_3 + \ldots$$

such that

$$\mu(\theta y_1 + \mu^2 y_2 + \mu^3 y_3 + \ldots) = \theta_2 \mu^2 + \theta_3 \mu^3 + \ldots + \theta_n \mu^n + \ldots.$$

This granted, let f' be a function of x and of y in the same form as f', i.e., such that /131

$$f' = f'_0 + f'_1 y + f'_2 y^2 + \ldots,$$

$f'_0$, $f''_1$, $f'_2$, ... being periodic functions of x, and let us assume additionally that we have

$$f < f'(\arg y, e^{\pm ix}).$$

If we then set

$$f'(\mu y_1 + \mu^2 y_2 + \mu^3 y_3 + \ldots) = \varphi'_0 + \mu \varphi'_1 + \ldots + \mu^n \varphi'_n + \ldots,$$

it will follow that

$$\varphi_n < \varphi'_n(\arg y_1, y_2, \ldots, y_n, e^{\pm ix}).$$

We will similarly set

$$\theta' = f' - f'_0 - f'_1 y,$$

$$\theta'(\mu y_1 + \mu^2 y_2 + \mu^3 y_3 + \ldots) = \theta'_2 \mu^2 + \theta'_3 \mu^3 + \ldots + \theta'_n \mu^n + \ldots,$$

from which

$$\theta_n < \theta'_n.$$

We will finally write

$$\left| \frac{1}{[f_1]} \right| = \lambda.$$

Now let y', $\eta'$ and z be three unknown functions connected by the relation

$$y' = \eta' + z$$

and developed in terms of the powers of $\mu$, such that

$$y' = \mu y'_1 + \mu^2 y'_2 + \ldots,$$
$$\eta' = \mu \eta'_1 + \mu^2 \eta'_2 + \ldots,$$
$$z = \mu z_1 + \mu^2 z_2 + \ldots.$$

Let us define these functions by the following equations

$$\left.\begin{aligned} \eta' &= \mu f'(x, \eta' + z) \\ z &= \lambda f_1 \eta' + \lambda \theta'(x, \eta' + z); \end{aligned}\right\} \tag{3}$$

we will first find

$$\eta'_1 = \varphi'_0,$$

and since we have, on the other hand, /132

$$\frac{d^2 \eta_1}{dx^2} = \varphi_0,$$

we will conclude from this

$$\eta_1 < \eta'_1(\arg e^{\pm ix}).$$

We then find

$$z_1 = \lambda f'_1 \eta'_1$$

and, on the other hand,

$$[y_1] = \frac{-1}{[f_1]}[f_1\eta_1],$$

from which

$$[y_1] < z_1,$$
$$y_1 < y'_1.$$

It then follows that

$$\eta'_2 = \varphi'_1(x, y'_1)$$

and, on the other hand,

$$\frac{d^2\eta_2}{dx^2} = \varphi_1(x, y_1)$$

from which

$$\eta_2 < \eta'_2;$$

then

$$z_2 = \lambda f_1 \eta'_2 + \lambda\theta'_2(x, y'_1)$$

and, on the other hand,

$$[y_2] = \frac{-1}{[f_1]}[f_1\eta_2] - \frac{1}{[f_1]}[\theta_2],$$

from which

$$[y_2] < z_2, \quad y_2 < y'_2,$$

and so forth; the law being manifest, we will have

$$y_n < y'_n \quad (\arg e^{\pm i x})$$

and

$$y < y' \quad (\arg \mu, e^{\pm i x}).$$

If, therefore, the series

$$y' = \mu y'_1 + \mu^2 y'_2 + \mu^3 y'_3 + \ldots$$

converges, the series

$$y = \mu y + \mu^2 y_2 + \ldots \qquad (4)$$

converges a fortiori. It remains, therefore, for me to establish that the series y' converges, or, what is the same, that equations (3) can be solved with respect to $\eta'$ and to z. /13

Now the functional determinant relative to these equations (3) can be written

$$\frac{\partial(\eta_1' - \mu f',\ z - \lambda f_1,\ \eta_1' - \lambda\theta')}{\partial(\eta',\ z)},$$

and its value for $\eta'=z=\mu=0$ is equal to 1. It is therefore not zero, and consequently according to the theorem from article 30 equations (3) can be solved.

Therefore series (4) converges. Q. E. D.

The equations treated in this article represent a particular case of those which were the subject of the preceding article. A direct demonstration completely analogous could be given in the general case. We will return to it later.

## Examination of an Important Exceptional Case

46. According to what we have just seen, the principles of article 42 are found to be faulty when the Hessian of $F_0$ with respect to x is zero.

Let us therefore examine the case where the Hessian is zero, and more particularly the case where $F_0$ is independent of any of the variables x.

I will assume, to fix the ideas, that there are four degrees of freedom, that two of the variables $x_1$ and $x_2$ enter into $F_0$, that the two others $x_3$ and $x_4$ do not enter, and finally that the Hessian of $F_0$ with respect to $x_1$ and to $x_2$ is not zero (the Hessian with respect to $x_1$, $x_2$, $x_3$ and $x_4$ is zero, since $\frac{dF_0}{dx_3} = \frac{dF_0}{dx_4} = 0$). For $\mu=0$, the general solution of the differential equations is written

$$\left.\begin{array}{c} x_1 = x_1^0, \quad x_2 = x_2^0, \quad x_3 = x_3^0, \quad x_4 = x_4^0, \quad y_1 = n_1^0 t + \varpi_1, \\ y_2 = n_2^0 t + \varpi_2, \quad y_3 = \varpi_3, \quad y_4 = \varpi_4, \\ n_1^0 = -\frac{dF_0}{dx_1^0}, \quad n_2^0 = -\frac{dF_0}{dx_2^0}, \quad n_3^0 = n_4^0 = 0, \end{array}\right\} \qquad (1)$$

$x_i^0$ and $\varpi_i$ being constants.

If $x_1^0$ and $x_2^0$ have been chosen such that $n_1^0 T$ and $n_2^0 T$ are multiples of $2\pi$, the /13 solution will be periodic of period T, and this will be true no matter what the initial values $x_3^0$, $x_4^0$, $\varpi_1$, $\varpi_2$, $\varpi_3$ and $\varpi_4$.

Let us consider any arbitrary solution for any arbitrary value of $\mu$ and let

$$x_i = x_i^0 + \beta_i, \quad y_i = \varpi_i + \beta'_i \tag{2}$$

be the initial values of $x_i$ and of $y_i$ for t=0. Let

$$x_i = x_i^0 + \beta_i + \psi_i, \quad y_i = n_i^0 T + \beta'_i + \psi'_i + \varpi_i$$

be the values of $x_i$ and $y_i$, for t=T.

In order for the solution to be periodic, it is necessary and sufficient that we have

$$\left.\begin{aligned} \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0, \\ \psi'_1 = \psi'_2 = \psi'_3 = \psi'_4 = 0. \end{aligned}\right\} \tag{3}$$

I will note:

(1) that I may always choose the origin of time such that the initial value of $y_1$ is zero, as well for the periodic solution (1) as for the solution which corresponds to the initial values (2). We will therefore have

$$\varpi_1 = \beta'_1 = 0;$$

(2) that F=C is an integral of our differential equations and that $\frac{dF}{dx_1}$ is not zero ($\frac{dF}{dx_1}$ is equal to $n_1^0$). Equations (3) are therefore not distinct, and I may suppress the first from among them,

$$\psi_1 = 0;$$

(3) that for $\mu$=0, one has identically

$$\psi_2 = \psi_3 = \psi_4 = \psi'_3 = \psi'_4 = 0;$$

that consequently $\psi_2$, $\psi_4$, $\psi'_1$, $\psi'_2$, $\psi'_3$, $\psi'_4$ are divisible by $\mu$. I may therefore replace system (3) by the following

$$\frac{\psi_2}{\mu} = \frac{\psi_3}{\mu} = \frac{\psi_4}{\mu} = \psi'_1 = \psi'_2 = \frac{\psi'_3}{\mu} = \frac{\psi'_4}{\mu} = 0. \tag{4}$$

I propose: /135

(1) to determine

$$x_3^0, \ x_4^0, \ \varpi_2, \ \varpi_3 \text{ and } \varpi_4$$

($x_1^0$ and $x_2^0$ are already assumed determined and $\varpi_1$ is assumed zero), such that equations (4) are satisfied for

$$\mu = 0, \quad \beta_i = 0, \quad \beta'_i = 0;$$

(2) to inquire into whether the functional determinant of the first members of system (4) is zero, or, in other words, if for $\mu=0$ the solution

$$\beta_i = 0, \quad \beta'_i = 0$$

is a simple solution of this system, or at least a solution of odd order.

To do so, it is necessary for us to determine what equations (4) become for $\mu=0$.

We have

$$\psi_2 = \int_0^T dx_2 = \int_0^T \frac{dF}{dy_2} dt$$

or, since $\frac{dF_0}{dy_2}=0$,

$$\frac{\psi_2}{\mu} = \int_0^T \frac{d}{dy_2}\left(\frac{F - F_0}{\mu}\right) dt,$$

or, for $\mu=0$,

$$\frac{\psi_2}{\mu} = \int_0^T \frac{dF_1}{dy_2} dt; \tag{5}$$

for $\mu=0$, we have

$$x_i = x_i^0 + \beta_i, \quad y_i = n_i t + \varpi_i + \beta'_i,$$

$$n_i = -\frac{dF_0(x_1^0 + \beta_1, x_2^0 + \beta_2)}{d(x_i^0 + \beta_i)}, \quad n_3 = n_4 = 0.$$

Let us substitute these values of the $x_i$ and of the $y_i$ into the second member of equation (5).

If we make, in addition, $\beta_1=\beta_2=0$, $n_1$ and $n_2$ reduce to $n_1^0$ and $n_2^0$, and the function $F_1$ becomes a periodic function of t of period T; it is besides, a function $\beta'_2$, of $\varpi_2 + \varpi_3 + \beta'_3$, $\varpi_4 + \beta'_4$ which is periodic and of period $2\pi$; finally it still depends on x $x_3^0 + \beta_3$ and $x_4^0 + \beta_4$. We can write

$$F_1 = \Sigma A \cos(m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4 + k),$$

$m_1$, $m_2$, $m_3$ and $m_4$ being integers, A and k being functions of $x_i$. In fact, the function $F_1$ is by hypotheses periodic of period $2\pi$ with respect to $y_i$.

After substitution, it follows that

$$F_1 = \Sigma A \cos(\alpha t + \beta)$$

where

$$\alpha = m_1 n_1^0 + m_2 n_2^0, \quad \beta = k + m_2(\varpi_2 + \beta'_2) + m_3(\varpi_3 + \beta'_3) + m_4(\varpi_4 + \beta'_4).$$

Among the terms of the development of $F_1$, I distinguish those for which $\alpha$ is zero, and I call R the combination of these terms, such that

$$R = \Sigma A \cos \beta,$$

the summation being extended to all terms for which we have

$$m_1 n_1^0 + m_2 n_2^0 = 0.$$

The function $F_1$ is a periodic function of time of period T, and R is nothing other than the mean value of this function, such that we have

$$TR = \int_0^T F_1 \, dt,$$

or, by differentiation with respect to $\varpi_2$,

$$T \frac{dR}{d\varpi_2} = \int_0^T \frac{dF_1}{d\varpi_2} dt;$$

but we have

$$\frac{dF_1}{d\varpi_2} = \frac{dF_1}{dy_2} \frac{dy_2}{d\varpi_2} = \frac{dF_1}{dy_2}.$$

Equation (5) therefore becomes

$$\frac{\psi_2}{\mu} = \frac{dR}{d\varpi_2} T;$$

we would similarly find /137

$$\frac{\psi_3}{\mu} = \frac{dR}{d\varpi_3} T, \quad \frac{\psi_4}{\mu} = \frac{dR}{d\varpi_4} T.$$

We find, by a completely similar calculation,

$$\frac{\psi_3'}{\mu} = -\frac{1}{\mu}\int_0^T \frac{dF}{dx_3}\,dt = -\int_0^T \frac{d}{dx_3}\left(\frac{F-F_0}{\mu}\right)dt$$

or, for $\mu=0$,

$$\frac{\psi_3'}{\mu} = -\int_0^T \frac{dF_1}{dx_3}\,dt = -T\,\frac{dR}{dx_3^0}$$

and similarly

$$\frac{\psi_4'}{\mu} = -T\,\frac{dR}{dx_4^0}.$$

On the other hand, it follows that

$$n_1^0 T + \psi_1' = -\int_0^T \frac{dF}{dx_1}\,dt$$

or, for $\mu=0$,

$$n_1^0 T + \psi_1' = -\frac{dF_0}{d(x_1^0+\beta_1)}T, \qquad \psi_1' = (n_1 - n_1^0)T.$$

Similarly we find

$$\psi_2' = (n_2 - n_2^0)T.$$

We first want, for

$$\mu = 0, \qquad \beta_i = 0, \qquad \beta_i' = 0\ ,$$

that system (4) be satisfied. Now, if we have $\beta_1=\beta_2=0$, $n_1$ and $n_2$ reduce to $n_2^0$, $\psi_1'$ and $\psi_2'$ reduce to 0, such that two of equations (4) are themselves satisfied. System (4) reduces simply to

$$\frac{dR}{dx_3^0} = \frac{dR}{dx_4^0} = \frac{dR'}{d\varpi_2} = \frac{dR}{d\varpi_3} = \frac{dR}{d\varpi_4} = 0. \qquad (6)$$

Thus, in the function R, let us cancel $\beta_i$ and $\beta_i'$; let us then consider R as a function of $x_3^0, x_4^0, \varpi_2, \varpi_3, \varpi_4$; if this function admits a maximum or a minimum and one gives to the variables $x_i^0$ and $\varpi_i$ the values which correspond to this maximum or to this minimum, we will satisfy equations (6). /138

Does this solution of system (6) lead us to periodic solutions which exist still for small values of $\mu$?

For this it suffices that the functional determinant of equations (4) not vanish for

$$\mu = \beta_i = \beta'_i = 0.$$

Now $\psi_1$ and $\psi_2$ depend (when we make $\mu$=0) only on $\beta_1$ and $\beta_2$, for $F_0$ and its two divisors $-n_1$ and $-n_2$ are functions only of $x_1^0+\beta_1$ and $x_2^0+\beta_2$.

This functional determinant is therefore the product of two others:

(1) of that of $\psi'_1$ and $\psi'_2$ with respect to $\beta_1$ and $\beta_2$ (but this is nothing other than the Hessian of $F_0$ with respect to $x_1$ and $x_2$, which we assume different from 0).

(2) of that of

$$\frac{\psi_2}{\mu}, \quad \frac{\psi_3}{\mu}, \quad \frac{\psi_4}{\mu}, \quad \frac{\psi'_3}{\mu}, \quad \frac{\psi'_4}{\mu} \tag{7}$$

with respect to

$$\beta_3, \quad \beta_4, \quad \beta'_2, \quad \beta'_3, \quad \beta'_4.$$

Now quantities (7) are functions of

$$x_3^0+\beta_3, \quad x_4^0+\beta_4, \quad \varpi_2+\beta'_2, \quad \varpi_3+\beta'_3, \quad \varpi_4+\beta'_4.$$

The derivative of any one of the quantities (7) with respect to $\beta_i$ or to $\beta'_i$ is equal to its derivative with respect to $x_i^0$ or to $\varpi_i$.

The determinant sought is thus the functional determinant of quantities (7) with respect to

$$x_3^0, \quad x_4^0, \quad \varpi_2, \quad \varpi_3, \quad \varpi_4. \tag{8}$$

However we must calculate the values of this determinant for

$$\mu = \beta_i = \beta'_i = 0.$$

But, when $\mu$, $\beta_i$ and $\beta'_i$ cancel, quantities (7) reduce to the first members of equations (6).

Our determinant is therefore nothing other than the Hessian of R with respect to the variables (8).

If the Hessian is not zero, our differential equations will still admit periodic solutions for small values of $\mu$. /139

This result can still be stated in another manner.

There will exist periodic solutions for small values of $\mu$, provided that equations (6) admit a simple solution. But there is more: there will still exist periodic solutions provided that equations (6) admit a solution of odd order.

However, according to article 34, to a maximum of the function R there will always correspond a solution of odd order of equations (6).

Therefore, if function R admits a maximum or a minimum our differential equations will admit periodic solutions for small values of $\mu$.

Solution of the Second Kind

47. Let us apply the preceding to the Problem of Three Bodies, assuming first that these three bodies move in the same plane, and let us attempt to determine the periodic solutions of the second kind.

Let us adopt the variables of article 15, i.e., the variables

$$\beta L = \Lambda, \quad \beta' L' = \Lambda', \quad H.$$
$$l', \quad h.$$

A solution will be periodic if at the beginning of a period, $\Lambda$, $\Lambda'$ and H have resumed their primitive values, and if $l$, $l'$ and h have increased by a multiple of $2\pi$.

The function F is equal to

$$F_0 + \mu F_1 + \mu^2 F_2 + \ldots,$$

and $F_0$ depends only on $\Lambda$ and on $\Lambda'$.

If therefore we assume $\mu=0$ and call

$$\Lambda_0, \quad \Lambda'_0, \quad H_0,$$
$$l_0, \quad l'_0, \quad h_0$$

the initial values of our six variables, four of these six variables, $\Lambda$, $\Lambda'$, H and h will be constants and we will have

$$\Lambda = \Lambda_0, \quad \Lambda' = \Lambda'_0, \quad H = H_0, \quad h = h_0.$$

If in addition we set /14

$$n = -\frac{dF_0}{d\Lambda_0}, \quad n' = -\frac{dF_0}{d\Lambda'_0},$$

we will have

$$l = nt + l_0, \quad l' = n't + l'_0.$$

Therefore, for $\mu=0$, if $\Lambda_0$ and $\Lambda'_0$ have been chosen such that $nT$ and $n'T$ are multiples of $2\pi$, the solution will be periodic of period $2\pi$, whatever the constants $H_0$, $l_0$, $l'_0$, $h_0$ may be.

Here is the question which we present:

Is it possible to choose the constants $H_0$, $l_0$, $l'_0$ and $h_0$ such that, for small values of $\mu$, the equations of motion admit a periodic solution of period T and which is such that the initial values of the six variables are, respectively,

$$\begin{array}{lll} \Lambda_0+\beta_1, & \Lambda'_0+\beta_2, & H_0+\beta_3, \\ l_0+\beta_4, & l'_0+\beta_5, & h_0+\beta_6, \end{array}$$

$\beta_i$ being functions of $\mu$ which vanish with $\mu$?

In order to resolve this question, it suffices to apply the principles of the preceding article.

Value $F_1$ being periodic in $l$, $l'$ and h, we are able to write

$$F_1 = \Sigma A \cos(m_1 l + m_2 l' + m_3 h + k),$$

A and k being functions of $\Lambda$, $\Lambda'$ and H.

Let us replace in $F_1$ the six variables

$$\begin{array}{lll} \Lambda, & \Lambda', & H, \\ l, & l', & h \end{array}$$

by

$$\begin{array}{lll} \Lambda_0, & \Lambda'_0, & H_0, \\ l_0+nt, & l'_0+n't, & h_0 \end{array};$$

it will follow that

$$F_1 = \Sigma A \cos(\alpha t + \beta),$$

where

$$\alpha = m_1 n + m_2 n', \qquad \beta = k + m_1 l_0 + m_2 l'_0 + m_3 h_0.$$

$F_1$ is a periodic function of t; let R be the mean value of this function, such that

$$R = \Sigma A \cos\beta,$$

the summation being extended to all terms, such that

$$\alpha = 0, \quad \text{or} \quad m_1 n + m_2 n' = 0.$$

According to the principles of the preceding article, we will find the sought values of $H_0$, $l_0$, $l'_0$ and $h_0$ by solving the system /14

$$\frac{dR}{dH_0} = \frac{dR}{dl_0} = \frac{dR}{dl'_0} = \frac{dR}{dh_0} = 0.$$

We can always assume that the origin of time has been chosen such that $l_0=0$.

On the other hand, according to the definition of the function R, we have

$$n\frac{dR}{dl_0} + n'\frac{dR}{dl'_0} = 0.$$

We can, therefore, replace the preceding system by the simpler system

$$\frac{dR}{dH_0} = \frac{dR}{dl'_0} = \frac{dR}{dh_0} = 0. \tag{1}$$

It can occur that not all solutions of system (1) are suitable; but there are solutions which will certainly be satisfactory: they are those whose order of multiplicity is odd, and particularly those which correspond to a true maximum or minimum of R.

In order to establish the existence of these solutions of the second kind, it is sufficient for me to show that the function R has a maximum.

Now, this function R is essentially finite; in addition, it is periodic in $l'_0$ and $h_0$: it still depends on $H_0$; I will add that it can be developed in terms of the powers of

$$\sqrt{A_0^2 - H_0^2} \quad \text{and} \quad \sqrt{A_0'^2 - (H_0 - C)^2}, \tag{2}$$

C being the areal constant.

The function R will therefore be real only if we have

$$H_0^2 < A_0^2, \quad (H_0 - C)^2 < A_0'^2, \tag{3}$$

and $H_0$ must always lie between these two limits. I may always choose a variable /14 $\varphi$ such that $H_0$ and the two radicals (2) are doubly periodic functions of $\varphi$.

Thus R is a uniform function, periodic and finite, of only three variables (since $\Lambda_0$, $\Lambda'_0$ and C are considered as given, and $l=0$), namely of $l'_0$, $h_0$ and $\varphi$.

This function therefore admits at least one maximum and one minimum, such that there are always at least two periodic solutions of the second kind.

We know that the development of the perturbing function $F_1$ contains only cosines, so that the quantity which we have just called k is always zero.

If therefore we make

$$l_0 = l'_0 = h_0 = 0,$$

we will have

$$\frac{dR}{dl'_0} = \frac{dR}{dh_0} = 0;$$

it will remain to satisfy the equation

$$\frac{dR}{dH_0} = 0$$

where, what comes back to the same thing,

$$\frac{dR}{d\varphi} = 0.$$

This will always be possible, for R is a periodic function of $\varphi$ which must have at least one maximum and one minimum.

There therefore exist always at least two solutions of the second kind, for which

$$l_0 = l'_0 = h_0 = 0.$$

If $\mu=0$, the initial values of $l$, $l'$ and h are therefore zero, which amounts to saying that there is symmetrical conjunction.

By a completely similar reasoning to that of article 40 (p.77), we might conclude that there is again symmetrical conjunction for small values of $\mu$, and that if at the beginning of the period we have symmetrical conjunction, it is the same in the middle of the period.

Among the periodic solutions of the second kind, there are always some which admit symmetrical conjunctions (or oppositions) at the beginning and at the middle of each period. /143

One difficulty might, nevertheless, present itself, and I am obliged to say a few words concerning it.

Function R depends on $l$, $h_0$, $H_0$, since we will consider $\Lambda_0$ and $\Lambda'_0$ as given quantities and we choose $\iota_0$ to be zero.

The function R is periodic in $\iota_0'$ and in $h_0$; in addition, the third variable $H_0$ is subject to certain inequalities, for example, to the following

$$\Lambda_0 > H_0.$$

From this we have concluded that function R always admits a maximum and a minimum.

But we can wonder what would happen if this maximum were precisely attained when $H_0$ reaches one of the limits assigned to it by inequalities (3).

Would the conclusions from article 46 still be applicable?

We might doubt this, for, if R reaches its maximum for $H_0=\Lambda_0$, for example, the derivative $\frac{dR}{dH_0}$ is not zero, it is on the contrary infinite.

It is true that for the Problem of Three Bodies we could without difficulty verify that the maximum of R does not take place for this value of $H_0$; however, as this case might present itself with other disturbing forces than those which we consider in the Problem of Three Bodies, it is not without interest to examine it more closely.

Let us assume, for example, that we consider the values of $H_0$ very close to $\Lambda_0$; we will be able to adopt the variables of article 17, i.e., the variables

$$\Lambda, \quad \Lambda', \quad \xi^*,$$
$$\lambda^*, \quad l', \quad \eta^*.$$

Let us then call

$$\Lambda_0+\beta_1, \quad \Lambda_0'+\beta_2, \quad \xi_0^*+\beta_3,$$
$$\lambda_0^*+\beta_4, \quad l_0'+\beta_5, \quad \eta_0^*+\beta_6$$

the initial values of these six variables and let us attempt to find if we can choose these initial values such that the solution is periodic, the $\beta_i$ will be functions of $\mu$ which will have to vanish with $\mu$.

To do so, we have seen, it suffices to choose /144

$$\lambda_0^*, \quad \xi_0^* \text{ et } \eta_0^*,$$

such that R is a maximum or minimum; we know, on the other hand, that $\Lambda_0$ and $\Lambda_0'$ must be regarded as given values and that we can always assume that $\iota_0'$ is zero.

If R reaches its maximum for $\Lambda_0=H_0$ with the new variables, this maximum will be attained for

$$\xi_0^* = \eta_0^* = 0.$$

However, this time there is no more difficulty, because R is a holomorphic function of $\xi_0^*$ and of $\eta_0^*$ developable in terms of the powers of these variables, while the difficulty arose from the former variables for which R ceases to be a holomorphic function of $H_0$ for $\Lambda_0=H_0$ being developable, not in terms of the integral powers of $\Lambda_0=H_0$, but in powers of $\sqrt{\Lambda_0 - H_0}$.

The results of the present article would therefore subsist even though the function R would attain its maximum for $\Lambda_0=H_0$, or more generally when $H_0$ attains one of the limits which are assigned to it by the inequalities (3).

## Solution of the Third Kind

48. Let us now attempt to determine the periodic solutions of the third kind, i.e., those for which the inclinations are not zero.

Let us adopt the variables from article 16, i.e.,

$$\begin{matrix} \beta L = \Lambda, & \beta' L' = \Lambda', & \beta \Gamma = H, & \beta' \Gamma' = H', \\ l, & l', & g, & g'. \end{matrix}$$

Let us first assume that $\mu=0$ and let

$$\begin{matrix} \Lambda_0, & \Lambda'_0, & H_0, & H'_0, \\ l_0, & l'_0, & g_0, & g'_0 \end{matrix}$$

be the initial values of these eight variables. If $\Lambda_0$ and $\Lambda'_0$ are chosen such that

$$n\mathrm{T} = -\mathrm{T}\frac{d\mathrm{F}_0}{d\Lambda_0}, \qquad n'\mathrm{T} = -\mathrm{T}\frac{d\mathrm{F}_0}{d\Lambda'_0}$$ /145

are multiples of $2\pi$, the solution will be periodic whatever the six constants may be

$$H_0, \; H'_0, \; l_0, \; l'_0, \; g_0, \; g'_0.$$

Can we choose these six constants such that, for small values of $\mu$, the equations of the Problem of Three Bodies admit a periodic solution of period T which is such that the initial values of the eight variables are functions of $\mu$ which reduce to

$$\begin{matrix} \Lambda_0, & \Lambda'_0, & H_0, & H'_0, \\ l_0, & l'_0, & g_0, & g'_0. \end{matrix}$$

for $\mu=0$?

We will operate as in the preceding article. We will first assume that the origin of time has been chosen such that $l_0=0$.

Then we will form, as in the preceding article, the functions $F_1$ and R.

According to the results of the two preceding articles, we must determine the five constants $H_0$, $H'_0$, $l'_0$, $g_0$, $g'_0$ so as to make R maximum or minimum.

A periodic solution will correspond to each maximum or each minimum R.

Value R considered as a function of $l'_0$, $g_0$ and $g'_0$ is a periodic function of period $2\pi$. On the other hand, $H_0$ and $H'_0$ are subject to certain inequalities (3) which I will write, as in article 18,

$$\left.\begin{array}{c} |\Lambda_0|>|H_0|, \quad |\Lambda'_0|>|H'_0|, \\ |H_0|-|H'_0|<C<|H_0|+|H'_0|. \end{array}\right\} \qquad (3)$$

The two variables $H_0$ and $H'_0$ can therefore only vary in a limited field.

The function R is therefore forced to admit one maximum and one minimum to which periodic solutions must correspond.

One difficulty can, nevertheless, present itself, as in the preceding article. Might it not happen that function R reaches it maximum at the moment when the variables $H_0$ and $H'_0$ reach the limits which are assigned to them by inequalities (3)? What will then happen? /1

Let us assume first that the maximum is reached for

$$H_0 = \Lambda_0.$$

We will then adopt the variables of article 18, i.e.,

$$\begin{array}{cccc} \Lambda, & \Lambda', & \xi^*, & H', \\ \lambda^*, & l', & \eta^*, & g'. \end{array}$$

We will set, consequently,

$$\lambda_0^* = l_0 + g_0, \qquad \xi_0^* = \sqrt{2(\Lambda_0 - H_0)}\cos g_0, \qquad \eta_0^* = \sqrt{2(\Lambda_0 - H_0)}\sin g_0.$$

We will then see that R reaches its maximum for

$$\xi_0^* = \eta_0^* = 0,$$

and, as R can be developed in terms of the powers of $\xi_0^*$ and $\eta_0^*$, the difficulty will be removed.

If, therefore, the maximum is reached for $H_0=\Lambda_0$, it will not be less true that a periodic solution will correspond to this maximum; it will still be the same for the same reason if the maximum is reached for $H'_0=\Lambda'_0$.

It remains to examine the case when the maximum would be reached for values of $H_0$ and $H'_0$ satisfying the condition

$$C = \pm H_0 \pm H'_0;$$

but this case is one where the inclinations are zero; if therefore the maximum is reached for similar values of $H_0$ and $H'_0$, we fall back on the case of the solutions of the second kind studied in the preceding article. A periodic solution will therefore still correspond to a similar maximum.

To summarize, we have demonstrated that function R always admits at least one maximum and one minimum and that to each of these maxima and minima there corresponds a periodic solution; however, one difficulty still subsists.

The solutions of the third kind which we have studied here include as a par- /147
ticular case the solutions of the second kind whose existence we demonstrated above.

We can wonder if there exist others; this is what a more exhaustive examination is going to teach us. We will see that function R has other maxima and minima than those which are produced when the inclinations are zero, and consequently there exist solutions of the third kind distinct from those of the second kind.

For this purpose, let us examine more closely the form of function R. We have to satisfy, on the one hand, relations

$$\frac{dR}{dl_0} = \frac{dR}{dg_0} = \frac{dR}{dg'_0} = 0; \qquad (4)$$

on the other, relations

$$\frac{dR}{dH_0} = \frac{dR}{dH'_0} = 0. \qquad (5)$$

I say that we will satisfy conditions (4) by making

$$l_0 = g_0 = g'_0 = 0;$$

such that we will have only to satisfy equations (5), i.e., to seek the maxima and minima of R considered as a function of $H_0$ and $H'_0$ only.

I in fact observe that R is of the following form (if we assume, as we do, $l'_0=0$, $\theta=\theta'$),

$$R = \Sigma A \cos(\gamma_1 l_0 + \gamma_2 g_0 + \gamma_3 g'_0),$$

A depending on $\Lambda_0$, $\Lambda'_0$, $H_0$, $H'_0$.

If therefore we assume

$$l_0 = g_0 = g'_0 = 0,$$

we will have at the same time

$$\frac{\partial R}{\partial l_0} = \frac{\partial R}{\partial g_0} = \frac{\partial R}{\partial g'_0} = 0.$$

Let us imagine that we change variables by taking for new variables the eccentricities e and e', and the inclinations i and i', i.e., by setting /148

$$\frac{H_0}{\beta} = G_0 = L_0\sqrt{1-e^2}, \qquad G'_0 = L'_0\sqrt{1-e'^2} = \frac{H'_0}{\beta'},$$
$$\Theta_0 = G_0 \cos i, \qquad \Theta'_0 = G'_0 \cos i',$$

such that one of the areal equations becomes

$$\beta L_0\sqrt{1-e^2}\sin i + \beta' L'_0\sqrt{1-e'^2}\sin i' = 0, \tag{6}$$

and the other

$$\beta L_0\sqrt{1-e^2}\cos i + \beta' L'_0\sqrt{1-e'^2}\cos i' = c. \tag{7}$$

It is now a question of seeking the maxima of R considered as a function of e, e', i and i', these four variables being assumed connected by the areal equations (6) and (7). We can therefore write the equations to which we will be led and which, joined to (7), must determine e, e', i and i' in the following form (where k designates an auxiliary quantity):

$$\left.\begin{aligned}
\frac{\partial R}{\partial e} &= k\beta L_0 \frac{e\cos i}{\sqrt{1-e^2}},\\
\frac{\partial R}{\partial i} &= k\beta L_0 \sin\sqrt{1-e^2},\\
\frac{\partial R}{\partial e'} &= k\beta' L'_0 \frac{e'\cos i'}{\sqrt{1-e'^2}},\\
\frac{\partial R}{\partial i'} &= k\beta' L'_0 \sin i'\sqrt{1-e'^2}.
\end{aligned}\right\} \tag{8}$$

Is it possible to satisfy these equations? To determine this, let us examine the form of function R. I observe at the outset that this function depends on i and i' only by their difference i-i', such that we have

$$\frac{\partial R}{\partial i} + \frac{\partial R}{\partial i'} = 0.$$

Then R will present itself, in the form of a series developed in terms of the increasing powers of e, e', i and i', such that the general term of the development will be in the following form (to within a coefficient depending only on $L_0$ and $L'_0$)

$$e^{\alpha_1} e'^{\alpha_2} i^{\alpha_3} i'^{\alpha_4} \cos(\gamma_1 l_0 + \gamma_2 l'_0 + \gamma_3 g_0 + \gamma_4 g'_0).$$

In addition, we will have to have, as I said above, /149

$$n\gamma_1 + n'\gamma_2 = 0$$

and, on the other hand,

$$|\gamma_1 + \gamma_2| < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$
$$\alpha_1 > |\gamma_1 + \gamma_3|, \quad \alpha_2 > |\gamma_2 + \gamma_4|.$$

I say that the terms of R for which $\gamma_1$ and $\gamma_2$ will not be zero at the same time will be of the third degree at least with respect to the eccentricities and to the inclinations, unless n is a multiple of $\frac{n-n'}{2}$.

In fact, let there be two integers $\gamma_1$ and $\gamma_2$ which can be positive or negative, but which are not zero at the same time and which satisfy the equalities

$$n\gamma_1 + n'\gamma_2 = 0, \quad |\gamma_1 + \gamma_2| = 0, 1 \text{ or } 2.$$

If we set

$$\gamma_1 + \gamma_2 = \varepsilon, \quad \varepsilon = 0 \pm 1 \text{ or } \pm 2,$$

we will have

$$\gamma_1 = \varepsilon \frac{n'}{n' - n}, \quad \gamma_2 = \varepsilon \frac{n}{n - n'}.$$

I see first that $\epsilon$ cannot be zero without $\gamma_1$ and $\gamma_2$ being zero at the same time. Because, on the other hand, $\gamma_1$ and $\gamma_2$ must be integers, and since $\epsilon$ is equal to ±1 or to ±2, the number $\frac{2n}{n-n'}$ would have to be integral, which means that n would have to be a multiple of $\frac{n-n'}{2}$. This is what we will not assume.

Therefore, in order to calculate R up to terms of the second order inclusively, it suffices to make in $F_1$, $\gamma_1=\gamma_2=0$, i.e., to retain in $F_1$ only the so-called secular terms.

Now the calculation of these terms was done long ago by the founders of Celestial Mechanics. I will therefore restrict myself to referring, for example, to Tisserand's Mécanique Céleste (Vol. 1, p. 406). We then find

$$R = \tfrac{1}{2}A^{(0)} + \tfrac{1}{8}B^{(1)}[e^2 + e'^2 - (\iota - \iota')^2] - \tfrac{1}{4}B^{(2)}ee'\cos(g_0 - g'_0) + \Omega.$$

The coefficients $A^{(0)}$, $B^{(1)}$ and $B^{(2)}$ which depend only on $L_0$ and $L'_0$ are defined in the cited work of Tisserand and $\Omega$ designates a combination of terms of the third degree at least with respect to e, e', i and i'. /15

The question is therefore to render this function R a maximum or minimum, when assuming that e, e', i and i' are joined by the relation

$$\beta L_0\sqrt{1-e^2}\cos i + \beta' L'_0\sqrt{1-e'^2}\cos i' = C. \qquad (7)$$

Equations (8) can then be written (assuming, as above $\iota'_0 = g_0 = g'_0 = 0$),

$$\left.\begin{aligned} \tfrac{1}{4}B^{(1)}e - \tfrac{1}{4}B^{(2)}e' + D_1 &= k(\beta L_0 e + D_2), \\ \tfrac{1}{4}B^{(1)}(i' - i) + D_3 &= k(\beta L_0 i + D_4), \\ \beta L_0\sqrt{1-e^2}\sin i + \beta' L'_0\sqrt{1-e'^2}\sin i' &= 0, \\ \tfrac{1}{4}B^{(1)}e' - \tfrac{1}{4}B^{(2)}e + D_7 &= k(\beta' L'_0 e' + D_8), \end{aligned}\right\} \qquad (9)$$

the $D_i$ designating a group of terms of at least the second degree with respect to e, e', i and i'.

As for equation (7), it will be written

$$\beta L_0(e^2 + i^2) + \beta' L'_0(e'^2 + i'^2) + D_9 = \rho^2, \qquad (10)$$

$\rho^2$ designating a positive constant equal to

$$2\beta L_0 + 2\beta' L'_0 - 2C$$

and $D_9$ designating a group of terms of at least the third degree with respect to e, e', i and i'.

From equations (9) and (10) we can determine e, e', i and i' as series developed in terms of the increasing powers of $\rho$, and this in six different ways.

Let us, in fact, set

$$e = \varepsilon\rho, \qquad e' = \varepsilon'\rho, \qquad i = \iota\rho, \qquad i' = \iota'\rho;$$

let us substitute in equations (9), which we will divide by $\rho$, and in equations (10), which will divide by $\rho^2$. The two members of these equations will therefore be developed in terms of the increasing powers of k, $\varepsilon$, $\varepsilon'$, $\iota$, $\iota'$ and $\rho$.

I will add even that the two members of these equations may be developed in terms of the powers of $\rho$, $\varepsilon - \varepsilon_0$, $\varepsilon' - \varepsilon'_0$, $\iota - \iota_0$, $\iota' - \iota'_0$, $k - k_0$ (if these quantitites are sufficiently small in absolute value), whatever the constants $\varepsilon_0$, $\varepsilon'_0$, $\iota_0$, $\iota'_0$, $k_0$ may be. /15

For $\rho=0$, these equations reduce to

$$\left.\begin{aligned} &\tfrac{1}{2}B^{(1)}\epsilon - \tfrac{1}{4}B^{(2)}\epsilon' = k\beta L_0\epsilon,\\ &\tfrac{1}{2}B^{(1)}\epsilon' - \tfrac{1}{4}B^{(2)}\epsilon = k\beta' L'_0\epsilon',\\ &\tfrac{1}{4}B^{(1)}(\iota'-\iota) = k\beta L_0\iota,\\ &\beta L_0\iota + \beta' L'_0\iota' = 0.\\ &\beta L_0(\epsilon^2+\iota^2) + \beta' L'_0(\epsilon'^2+\iota'^2) = 1. \end{aligned}\right\} \tag{11}$$

Equations (11) contain six solutions, namely:

$$\left.\begin{aligned} &k = \frac{-1}{+4}B^{(1)}\left(\frac{1}{\beta L_0} \div \frac{1}{\beta' L'_0}\right), \quad \epsilon=\epsilon'=0, \quad \iota = \frac{\beta' L'_0}{h},\\ &\iota' = \frac{-\beta L_0}{h}, \quad h = +\sqrt{\beta\beta' L_0 L'_0(\beta L_0 + \beta' L'_0)},\\ &k = \frac{-1}{+4}B^{(1)}\left(\frac{1}{\beta L_0} + \frac{1}{\beta' L'_0}\right), \quad \epsilon=\epsilon'=0, \quad \iota = \frac{\beta' L'_0}{h},\\ &\iota' = \frac{-\beta L_0}{h}, \quad h = -\sqrt{\beta\beta' L_0 L'_0(\beta L_0 + \beta' L'_0)},\\ &k = k_1, \quad \iota=\iota'=0, \quad \epsilon = \epsilon_1, \quad \epsilon' = \epsilon'_1,\\ &k = -k_1, \quad \iota=\iota'=0, \quad \epsilon = -\epsilon_1, \quad \epsilon' = -\epsilon'_1,\\ &k = k_2, \quad \iota=\iota'=0, \quad \epsilon = \epsilon_2, \quad \epsilon' = \epsilon'_2,\\ &k = k_2, \quad \iota=\iota'=0, \quad \epsilon = -\epsilon_2, \quad \epsilon' = -\epsilon'_2. \end{aligned}\right\} \tag{12}$$

Each of these six solutions is simple, from which we can conclude, according to what we have seen in article 30, that we can develop $\epsilon$, $\epsilon'$, $\iota$, and $\iota'$, and consequently e, e', i and i', in six different ways in terms of the increasing powers of $\rho$.

We therefore will be able to write

$$e = f_{1,\lambda}(\rho), \quad e' = f_{2,\lambda}(\rho), \quad i = f_{3,\lambda}(\rho), \quad i' = f_{4,\lambda}(\rho), \tag{13}$$

where $\lambda$ will be able to take on the values 1, 2, 3, 4, 5 and 6; we will take $\lambda=1$, when we take for a starting point the first of solutions (12); we will take $\lambda=2$, when we choose as a starting point the second of solutions (12), and so forth.

From the six developments (13), the last four must be rejected, for they give

$$i = i' = 0,$$

and the periodic solutions to which they would lead do not differ from solutions of the second kind studied in the preceding article. However, the first two can be retained and lead to new periodic solutions for which the inclinations are not zero, and which we can call solutions of the third kind. /152

The two developments do not lead, however, to two truly distinct periodic solutions.

We have studied more specially the solutions for which we have

$$l'_0 = g_0 = g'_0;$$

these equations express that there is a symmetrical conjunction at the beginning of the period in the case of $\mu=0$.

One argument, quite similar to that of article 40, would show that for all values of $\mu$ there is still symmetrical conjunction at the beginning and in the middle of each period.

This does not mean that there do not exist just as well periodic solutions of the third kind for which there is no symmetrical conjunction; it might occur, in fact, that function R admits other maxima or minima than those which correspond to the case of $l'_0=g_0=g'_0=0$. There will therefore be room to return to this question.

Applications of Periodic Solutions

49. It is, as we have said, of infinitely small probability that in any practical application the initial conditions of motion would be found to be precisely those which correspond to a periodic solution. It seems then that the considerations set forth in this chapter must necessarily remain fruitless. Nothing of the kind; they have already been useful to Astronomy and I do not doubt that astronomers will often have recourse to them in the future.

I will show in the following chapter how we can take a periodic solution as a starting point of a series of successive approximations, and thus study the solutions which differ very little from them. The usefulness of periodic solutions will then appear obvious.

However, I want, for the moment, to place myself at a slightly different point of view. /15

Let us assume that in the motion of an arbitrary heavenly body a very considerable irregularity appears. It may happen that the true motion of this heavenly body differs extremely little from that of an ideal heavenly body for which the orbit corresponds to a periodic solution.

It will then occur rather frequently that for the considerable irregularity of which we have just spoken we will have practically the same coefficient for the real heavenly body as for this ideal heavenly body whose motion is simpler and whose orbit is periodic, but this coefficient will be much more easily calculable for the ideal heavenly body.

It is to Hill that we owe the first application of this principle. In his Lunar Theory, he replaces this satellite in a first approximation by an ideal Moon, of which the orbit is periodic. The motion of this ideal Moon is then that which was described in article 41, where we spoke of this particular case of periodic solutions of the first kind, the knowledge of which we owe to Hill.

It then occurs that the motion of this ideal Moon, like that of the real Moon, is affected by a considerable irregularity well known under the name of variation; the coefficient is approximately the same for the two Moons.

Hill calculates its value for his ideal Moon with a large number of decimals. It would be necessary, to pass to the case in nature, to correct the coefficient thus obtained by taking account of the eccentricities, inclination and parallax. This is doubtless what Hill would have done, if he had completed publication of his admirable memoir.

Here is another case which will present itself often and to which I would like to direct attention. We have seen above that the periodic solutions of the first kind cease to exist when the ratio of the mean motions n and n' is equal to

$$\frac{n}{n'} = \frac{j-1}{j},$$

j being an integer; that is to say, when $\frac{n'}{n'-n}$ is equal to an integer j.

However, if the relationship $\frac{n'}{n'-n}$, without being integral, is very close to an integer, the periodic solution exists, and it then presents a very considerable irregularity. If the actual initial conditions of the motion differ but little /154 from those which correspond to a similar periodic solution, this large irregularity will still exist and its coefficient will be appreciably the same; we will therefore be able to calculate its value advantageously by consideration of the periodic solutions.

This is what Tisserand did (Bulletin astronomique, Vol. III, p. 425) in the study of the motion of Hyperion (satellite of Saturn). The relationship of the mean motion of this satellite to that of Titan is, in fact, very close to 3/4.

The same considerations are applicable to those of the small planets whose mean motion is approximately double that of Jupiter, and which have been the object of a noteworthy work by Harzer, and to the minor planet Hilda, whose mean motion is approximately equal to 3/2 times that of Jupiter.

Tisserand reports, in addition, in the work which we cite the case of Uranus and Neptune where the relationship of the motions is near 1/2. In all these cases there exists an important irregularity and the study of this inequality can be facilitated by consideration of periodic solutions of the first kind.

On the contrary, periodic solutions of the second and third kind have still not received practical applications; everything indicates, however, that this will one day come about, and this is what would happen if the conjectures of Gauss on the subject of Pallas came to be confirmed.

## Satellites of Jupiter

50. However, the most striking example is furnished us by Laplace himself and by his admirable theory of the satellites of Jupiter.

There exist, in fact, true periodic solutions of the first kind when, instead of three bodies, we consider four or a larger number. Let us consider, in fact, a central body of large mass and three other small bodies of zero mass circulating about the first in conformity with Kepler's laws. Let us imagine that the eccentricities and inclinations are zero, such that the motions are circular. Let us assume that there is, among the three mean motions n, n' and n", a linear relation with integral coefficients /155

$$\alpha n + \beta n' + \gamma n'' = 0,$$

$\alpha$, $\beta$, $\gamma$ being three integers, first among them, such that

$$\alpha + \beta + \gamma = 0 \quad ;$$

we will then be able to find three integers $\lambda$, $\lambda'$ and $\lambda''$, such that

$$\alpha\lambda + \beta\lambda' + \gamma\lambda'' = 0$$

and we will have

$$n = \lambda A + B, \quad n' = \lambda' A + B, \quad n'' = \lambda'' A + B,$$

A and B being arbitrary quantities.

At the end of time T, the longitudes of the three bodies will have increased by

$$\lambda AT + BT, \quad \lambda' AT + BT, \quad \lambda'' AT + BT,$$

and the difference of longitudes of the second and third satellites with the first will have increased by

$$(\lambda - \lambda')AT, \quad (\lambda - \lambda'')AT.$$

If, therefore, we chose T such that AT is a multiple of $2\pi$, the angles formed by the radius vectors led from the central body to the three satellites will have resumed their primitive value. Thus the solution considered for $\mu=0$ is periodic of period T.

Will the problem still allow a periodic solution of period T when we take account of the mutual actions of the three small bodies, and their motion will no longer be Keplerian, or in other words, when we no longer assign to the parameter $\mu$ the value 0, but a small finite value instead?

An analysis very similar to that of article 40 proves that it is effectively true; there is a periodic solution of period T analogous to the solutions of the first kind and where the orbits are almost circular. The three small bodies are as much at the beginning as in the middle of each period in symmetrical conjunction or in opposition.

Laplace has demonstrated that the orbits of the three satellites of Jupiter differ very little from those which they would follow in a similar periodic solution, and the positions of these three small bodies oscillate constantly about positions which they would have in this periodic solution. /156

## Periodic Solutions in the Neighborhood of a Position of Equilibrium

51. The periodic solutions which we have discussed until now are not the only ones whose existence it is possible to demonstrate. Thus the Problem of Three Bodies contains periodic solutions of the following nature: the two small bodies describe around the large body orbits very little different from the two Keplerian ellipses E and E'; at a certain moment, these two small bodies pass very close to each other and exercise on each other considerable perturbations; they then separate again and then describe orbits which closely approach two new Keplerian ellipses $E_1$ and $E_1'$, very different from E and from E'. The two small bodies diverge very little from the ellipses $E_1$ and $E_1'$, until they again find themselves very near each other. Thus, the motion is almost Keplerian, except at certain moments where the distance of the two bodies becomes very small and where very considerable but very short-lived perturbations are produced. It can occur that these types of collisions are reproduced periodically and such that at the end of a certain time the two bodies are again found on the ellipses E and E'. The solution is then periodic. I will later return to this type of periodic solution which differs completely from those which we have studied in this chapter.

I will equally reserve for another volume the periodic solutions which I have called those of the second kind and which I have defined in my memoir of Volume XIII of Acta mathematica, but whose study cannot precede that of integral invariants.

It is, nevertheless, a category of periodic solutions the theory of which resembles that for the solutions of the second kind, but of which I want to say several words here, free to return with more detail at the proper time and place.

Let /157

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n \qquad (1)$$

be a system of differential equations. I assume that $X_i$ can be developed in terms of the increasing powers of $x_1, x_2, \dots, x_n$ and of a parameter $\mu$.

I assume in addition that for

$$x_1 = x_2 = \dots = x_n = 0$$

we have at the same time (and no matter what $\mu$ may be)

$$X_1 = X_2 = \dots = X_n = 0.$$

Then system (1) will admit as particular solution

$$x_1 = 0, \quad x_2 = 0, \quad \dots, \quad x_n = 0,$$

and as the values of $x_1, x_2, \ldots, x_n$ are constants, this solution may be regarded as an arbitrary periodic solution.

I propose to study the periodic solutions which differ very little from them.

Let $\beta_1, \beta_2, \ldots, \beta_n$ be the initial values of $x_1, x_2, \ldots, x_n$ ; let $\psi_1+\beta_1, \psi_2+\beta_2, \ldots, \psi_n+\beta_n$ be the values of these same variables for t=T.

We can develop $\psi_1, \psi_2, \ldots, \psi_n$, in terms of the powers of $\beta_1, \beta_2, \ldots, \beta_n$ and $\mu$.

Let us consider the following equation in S

$$\begin{vmatrix} \frac{dX_1}{dx_1} - S & \frac{dX_1}{dx_2} & \cdots & \frac{dX_1}{dx_n} \\ \frac{dX_2}{dx_1} & \frac{dX_2}{dx_2} - S & \cdots & \frac{dX_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dX_n}{dx_1} & \frac{dX_n}{dx_2} & \cdots & \frac{dX_n}{dx_n} - S \end{vmatrix} = 0,$$

where we assume that we have made

$$\mu = x_1 = x_2 = \ldots = x_n = 0.$$

If this equation has no multiple root, I will call $S_1, S_2, \ldots, S_n$ its n roots. /15

We then verify the functional determinant of $\psi$ with respect to $\beta$, when we here make

$$\mu = \beta_1 = \beta_2 = \ldots = \beta_n = 0,$$

become equal to

$$\Delta = (e^{S_1 T} - 1)(e^{S_2 T} - 1)\ldots(e^{S_n T} - 1).$$

In order for the considered solution to be periodic of period T, it is necessary and sufficient that we have

$$\psi_1 = \psi_2 = \ldots = \psi_n = 0. \qquad (2)$$

This system contains a solution which is obvious and which is the following:

$$\beta_1 = \beta_2 = \ldots = \beta_n = 0. \qquad (3)$$

This teaches us nothing new, since we already know this can be regarded as a periodic solution of equations (1). Beyond this obvious periodic solution, do these equations admit others which are distinct from, but differing from this one by only very little? In other words, can equations (2) be satisfied when we substitute in them functions of $\alpha$ instead of $\beta$, which, without being identically zero, vanish for $\mu=0$?

If the determinant $\Delta$ is not zero, solution (3) is for $\mu=0$ a simple solution of system (2); therefore, aside from solution (3), system (2) will not be able to be satisfied by functions $\beta$ vanishing with $\mu$.

If, on the contrary, the determinant $\Delta$ vanishes, we will be able to find in one or several manners convergent series ordered in terms of the fractional powers of $\mu$, vanishing with this variable and which, substituted in place of $\beta_i$, will satisfy equations (2). This is what a special discussion, to which I will return when I treat periodic solutions of the second kind, alone would teach us; if these series have their real coefficients, they define a new category of periodic solutions which exists for small values of $\mu$ and for which $x_1$, $x_2$, ..., and $x_n$ never take anything more than very small values.

In order for $\Delta$ to vanish, it is necessary and sufficient that one of its factors vanish, i.e., that we have /159

$$e^{S_i T} = 1,$$

$S_i$ being one of the roots of the equation in S. In order for this to be possible, it is necessary that $S_i$ be imaginary; the equation in S will then admit the imaginary conjugate root $S_j$ and we will still have

$$e^{S_j T} = 1,$$

which shows that two of the factors of $\Delta$ will cancel at the same time.

## Moons Without Quadrature

52. As application, let us again take the equations

$$\left\{\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + \left(\frac{\mu}{r^3} - 3n^2\right)\xi &= 0 \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \frac{\mu\eta}{r^3} &= 0 \end{aligned}\right\} \qquad r^2 = \xi^2 + \eta^2. \tag{1}$$

These equations are satisfied if we make

$$\eta = 0, \qquad \xi = \sqrt[3]{\frac{\mu}{3n^2}}. \tag{2}$$

We see that $\xi$ and $\eta$ are then constants; equations (2) can be regarded as defining a periodic solution of equations (1).

It is easy to perceive the astronomical significance of this solution. The equation $\eta=0$ signifies that the Moon is constantly in conjunction or opposition, and the second of equations (2) signifies that the distance from the Moon to the Earth is constant. This periodic solution is therefore nothing other than that which Laplace defined in his Mécanique céleste, Book VI, Chapter X.

However, we propose to determine the periodic solutions which differ very little from it by applying the principles of the preceding article.

To do so, let us begin by assuming that the unit of length has been chosen such that /16

$$\frac{\mu}{3n^2} = 1, \qquad \mu = 3n^2,$$

and that the unit of time has been chosen such that

$$n = 1 + \alpha,$$

$\alpha$ being a very small parameter.

If we set $\xi = 1 + x$, system (1) can be replaced by the following, which is analogous to system (1) from the preceding article

$$\frac{dx}{dt} = x' \qquad \frac{dx'}{dt} = 2(1+\alpha)\eta' + 3(1+\alpha)^2(x+1)\left(\frac{1}{r^3} - 1\right),$$
$$\frac{d\eta}{dt} = \eta' \qquad \frac{d\eta'}{dt} = -2(1+\alpha)x' + 3(1+\alpha)^2\frac{\eta}{r^3}.$$

If we then form the equation in S from the preceding article, we have

$$S^4 - 2S^2 - 27 = 0.$$

This equation admits two real roots and two imaginary roots

$$S_1 = \sqrt{-1}\sqrt{\sqrt{28} - 1},$$
$$S_2 = -\sqrt{-1}\sqrt{\sqrt{28} - 1}.$$

If we then take

$$T = \frac{2\pi}{\sqrt{\sqrt{28} - 1}},$$

we will have

$$e^{S_1 T} = e^{S_2 T} = 1.$$

The determinant $\Delta$ from the preceding article is therefore zero.

We can therefore form series ordered in terms of the fractional powers of $\mu$ (here these series would be ordered in terms of the integral powers of $\sqrt{\mu}$) and which, substituted in place of the $\beta_i$, satisfy equations (2) from the preceding article.

We would verify (and I will return to this later) that the coefficients of these series are real.

Equations (1) of Hill admit therefore periodic solutions differing but little from solution (2). In the solutions $\eta$ remains very small and the Moon, conse- /161 quently, is always almost in opposition (or in conjunction). Hill therefore had reason to state that we can imagine a class of satellites which will never be able to be in quadrature; only the process by which he had thought he could arrive at a result, which he had, shall we say, guessed, was in no way capable of leading him to it; for this class of satellites is not, as he had believed, the analytical continuation of that which he had first studied in such an exhaustive and brilliant manner.

I will add that, in this category of periodic solutions, the Moon is found in symmetrical opposition at the beginning and in the middle of each period.

## CHAPTER 4. CHARACTERISTIC EXPONENTS

### Equations of Variation

53. It is very improbable that, in any application, the initial conditions of motion are exactly those which correspond to a periodic solution; but it may occur that they differ thereby very little. If then we consider the coordinates of the three bodies in their actual motion, and, on the other hand, the coordinates which these three same bodies would have in the periodic solution, the difference will remain very small for at least a certain time and we can, in a first approximation, neglect the square of this difference. /162

Let

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of differential equations where $X_i$ are known functions of $x_1$, $x_2$, ..., $x_n$.

Let

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t) \tag{1a}$$

be any solution of these equations which we will call a generating solution.

Let

$$x_1 = \varphi_1(t) + \xi_1, \quad x_2 = \varphi_2(t) + \xi_2, \quad \ldots, \quad x_n = \varphi_n(t) + \xi_n \tag{1b}$$

be a solution differing little from the first.

If we neglect the squares of $\xi$, we will be able to write

$$\frac{d\xi_i}{dt} = \frac{dX_i}{dx_1}\xi_1 + \frac{dX_i}{dx_2}\xi_2 + \ldots + \frac{dX_i}{dx_n}\xi_n \quad (i = 1, 2, \ldots, n). \tag{2}$$

Equations (2) will be what we will call the equations of variation of equations (1). We imagine that we may in a first approximation use these equations of variation to determine $\xi$. /163

The preceding suffices to make the importance of these equations of variation understood.

We are therefore going to make a detailed study of them, insisting especially on those of the equations of Dynamics.

54. Let us again take equations (1) from the preceding article and equations (2) which are their equations of variation.

When we know a solution of equations (1) containing a certain number of arbitrary constants, we can deduce from them particular solutions of equations (2).

Let us assume, in fact, that equations (1) are satisfied when we set

$$\left.\begin{aligned} x_1 &= \varphi_1(t, h_1, h_2, \ldots, h_p), \\ x_2 &= \varphi_2(t, h_1, h_2, \ldots, h_p), \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_n &= \varphi_n(t, h_1, h_2, \ldots, h_p). \end{aligned}\right\} \tag{3}$$

I assume that the generating solution is obtained by setting in equations (3)

$$h_1 = h_2 = \ldots = h_p = 0,$$

where $h_1$, $h_2$, ..., $h_p$ are p arbitrary constants.

It is clear that equations (2) admit the p particular solutions

$$\begin{array}{llll} \xi_1 = \dfrac{d\varphi_1}{dh_1}, & \xi_2 = \dfrac{d\varphi_2}{dh_1}, & \ldots, & \xi_n = \dfrac{d\varphi_n}{dh_1}, \\ \xi_1 = \dfrac{d\varphi_1}{dh_2}, & \xi_2 = \dfrac{d\varphi_2}{dh_2}, & \ldots, & \xi_n = \dfrac{d\varphi_n}{dh_2}, \\ \ldots\ldots, & \ldots\ldots, & \ldots, & \ldots\ldots, \\ \xi_1 = \dfrac{d\varphi_1}{dh_p}, & \xi_2 = \dfrac{d\varphi_2}{dh_p}, & \ldots, & \xi_n = \dfrac{d\varphi_n}{dh_p}. \end{array}$$

It is understood that in these derivatives $\frac{d\varphi_i}{dh_k}$ after the differentiation /164
we make the substitution

$$h_1 = h_2 = \ldots = h_p = 0.$$

Let us assume now that we know an integral of equations (1), and let

$$F(x_1, x_2, \ldots, x_n) = \text{const.}$$

be this integral.

As solution (1a) we will have

$$F[\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)] = c,$$

and for solution (1b)

$$F[\varphi_1(t) + \xi_1, \varphi_2(t) + \xi_2, \ldots, \varphi_n(t) + \xi_n] = c',$$

c and c' being two numerical constants.

If we assume that values $\xi$ are very small, it will be the same for c'-c and, if we neglect the squares of these quantities, it will follow that

$$\frac{dF}{dx_1}\xi_1 + \frac{dF}{dx_2}\xi_2 + \ldots + \frac{dF}{dx_n}\xi_n = c' - c = \text{const.} \tag{4}$$

In the partial derivatives $\frac{dF}{dx_i}$ it is, of course, necessary after differentiation to make the substitution

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t) \quad \ldots, \quad x_n = \varphi_n(t).$$

Equation (4) then gives us an integral of equations (2); it is important to observe that this integral will generally contain the time explicitly.

Thus, if we know an integral of equations (1), we can deduce from it an integral of equations (2).

Application to Lunar Theory

55. I spoke above, in article 53, of the possible applications of equations of variation and of their usefulness for Astronomy. A striking example is furnished us by Hill's admirable Lunar Theory. /16

I said in article 41 how this astronomer and scientist, after having formed the equations of motion for the Moon, studied in detail a particular solution of these equations which differs but little from the solution corresponding to the true initial conditions of motion. This solution is periodic and belongs to those I designated in the preceding chapter under the name of solutions of the first kind.

Holding to this solution, this comes back to neglecting at the same time not only parallax and eccentricity of the Sun, but inclinations in the orbits and eccentricity of the Moon.

Nevertheless, this first approximation permits us to know rather exactly, as I said in article 49, the coefficient of one of the most important inequalities of the Moon known under the name of variations.

Now let

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad x_3 = \varphi_3(t) = 0$$

be the coordinates of the Moon in this particular periodic solution.

Let

$$x_i = \varphi_i(t) + \xi_i$$

be the true coordinates of the Moon.

In a second approximation, Hill neglects the squares of $\xi$ and therefore arrives at a system of linear differential equations. In other words, he forms equations of variation taking as a generating solution the periodic solution which he had first studied.

Nevertheless, this second approximation gives him some of the most important elements of lunar motion namely the motion of the perigee, that of the node and the coefficient of the evection.

Actually the results published concern only the motion of the perigee (Cambridge, U.S.A. 1877, and Acta mathematica, Vol. VIII), but the number obtained is extremely satisfactory.

## Equations of Variation of Dynamics

56. Let F be a function of a double series of variables /166

$$x_1, \; x_2, \; \dots, \; x_n, \\ y_1, \; y_2, \; \dots, \; y_n$$

and of time t.

Let us assume that we have the differential equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}. \tag{1}$$

Let us consider two infinitely close solutions of these equations, the first,

$$x_1, \; x_2, \; \dots, \; x_n, \; y_1, \; y_2, \; \dots, \; y_n,$$

which will serve as generating solution and the second,

$$x_1+\xi_1, \; x_2+\xi_2, \; \dots, \; x_n+\xi_n, \; y_1+\eta_1, \; y_2+\eta_2, \; \dots, \; y_n+\eta_n,$$

values $\xi$ and $\eta$ being sufficiently small for us to be able to neglect their squares.

Values $\xi$ and $\eta$ will then satisfy the linear differential equations

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= \sum_k \frac{d^2F}{dy_i\,dx_k}\xi_k + \sum_k \frac{d^2F}{dy_i\,dy_k}\eta_k, \\ \frac{d\eta_i}{dt} &= -\sum_k \frac{d^2F}{dx_i\,dx_k}\xi_k - \sum_k \frac{d^2F}{dx_i\,dy_k}\eta_k, \end{aligned}\right\} \tag{2}$$

which are the variational equations of equations (1).

Let $\xi'_i$, $\eta'_i$ be another solution of these linear equations, such that

$$\left.\begin{aligned} \frac{d\xi'_i}{dt} &= \sum_k \frac{d^2F}{dy_i\,dx_k}\xi'_k + \sum_k \frac{d^2F}{dy_i\,dy_k}\eta'_k, \\ \frac{d\eta'_i}{dt} &= -\sum_k \frac{d^2F}{dx_i\,dx_k}\xi'_k - \sum_k \frac{d^2F}{dx_i\,dy_k}\eta'_k. \end{aligned}\right\} \tag{2'}$$

Let us multiply equations (2) and (2'), respectively, by $\eta'_i$, $-\xi'_i$, $-\eta_i$, $\xi_i$ and from the sum of all these equations, it will follow that /167

$$\sum_i \left( \eta'_i \frac{d\xi_i}{dt} - \xi'_i \frac{d\eta_i}{dt} - \eta_i \frac{d\xi'_i}{dt} + \xi_i \frac{d\eta'_i}{dt} \right)$$
$$= \sum_i \sum_k \left( \xi_k \eta'_i \frac{d^2F}{dy_i dx_k} + \eta_k \eta'_i \frac{d^2F}{dy_i dy_k} + \xi_k \xi'_i \frac{d^2F}{dx_i dx_k} + \eta_k \xi'_i \frac{d^2F}{dx_i dy_k} \right)$$
$$- \sum_i \sum_k \left( \eta_i \xi'_k \frac{d^2F}{dy_i dx_k} + \eta_i \eta'_k \frac{d^2F}{dy_i dy_k} + \xi_i \xi'_k \frac{d^2F}{dx_i dx_k} + \xi_i \eta'_k \frac{d^2F}{dx_i dy_k} \right)$$

or

$$\sum \frac{d}{dt} [\eta'_i \xi_i - \xi'_i \eta_i] = 0$$

or finally

$$\eta'_1 \xi_1 - \xi'_1 \eta_1 + \eta'_2 \xi_2 - \xi'_2 \eta_2 + \dots + \eta'_n \xi_n - \xi'_n \eta_n = \text{const.} \tag{3}$$

Here is a relation which connects two arbitrary solutions of linear equations (2).

It is easy to find other analogous relations.

Let us consider four solutions of equations (2)

$$\begin{matrix} \xi_i, & \xi'_i, & \xi''_i, & \xi'''_i, \\ \eta_i, & \eta'_i, & \eta''_i, & \eta'''_i. \end{matrix}$$

Let us then consider the sum of the determinants

$$\sum_i \sum_k \begin{vmatrix} \xi_i & \xi'_i & \xi''_i & \xi'''_i \\ \eta_i & \eta'_i & \eta''_i & \eta'''_i \\ \xi_k & \xi'_k & \xi''_k & \xi'''_k \\ \eta_k & \eta'_k & \eta''_k & \eta'''_k \end{vmatrix},$$

where the indices i and k vary from 1 to n. We can verify without difficulty that this sum is still a constant.

More generally, if we form the sum of the determinants with the aid of 2p solutions of equations (2)

$$\Sigma_{a_1 a_2 \dots a_p} |\xi_{a_1} \eta_{a_1} \xi_{a_2} \eta_{a_2} \dots \xi_{a_p} \eta_{a_p}|$$
$$(a_1, a_2, \dots, a_p = 1, 2, \dots, n),$$

this sum will be a constant.

In particular, the determinant formed by the values of the 2n quantities $\xi$ and $\eta$ in 2n solutions of equations (2) will be a constant.

These considerations permit finding a solution of equations (2) when we know one of its integrals and vice versa.

Let us assume, in fact, that

$$\xi_i = \alpha_i, \qquad \eta_i = \beta_i$$

is a particular solution of equations (2) and let us designate by $\xi_i$ and $\eta_i$ an arbitrary solution of these same equations. We must have

$$\Sigma(\xi_i \beta_i - \eta_i \alpha_i) = \text{const.},$$

which will be an integral of equations (2).

Reciprocally, letting

$$\Sigma A_i \xi_i + \Sigma B_i \eta_i = \text{const.}$$

be an integral of equations (2), we must have

$$\sum_i \frac{dA_i}{dt}\xi_i + \sum_i \frac{dB_i}{dt}\eta_i + \sum_i A_i\left(\sum_k \frac{d^2F}{dy_i\,dx_k}\xi_k + \sum_k \frac{d^2F}{dy_i\,dy_k}\eta_k\right) - \sum_i B_i\left(\sum_k \frac{d^2F}{dx_i\,dx_k}\xi_k + \sum_k \frac{d^2F}{dx_i\,dy_k}\eta_k\right) = 0,$$

from which, identifying

$$\frac{dA_i}{dt} = -\sum_k \frac{d^2F}{dy_k\,dx_i}A_k + \sum_k \frac{d^2F}{dx_k\,dx_i}B_k,$$
$$\frac{dB_i}{dt} = -\sum_k \frac{d^2F}{dy_k\,dy_i}A_k + \sum_k \frac{d^2F}{dx_k\,dy_i}B_k,$$

which shows that

$$\xi_i = B_i, \qquad \eta_i = -A_i$$

is a particular solution of equations (2).

If now

$$\Phi(x_i, y_i, t) = \text{const.}$$

is an integral of equations (1),

$$\sum \frac{d\Phi}{dx_i}\xi_i + \sum \frac{d\Phi}{dy_i}\eta_i = \text{const.}$$

will be an integral of equations (2), and consequently

$$\xi_i = \frac{d\Phi}{dy_i}, \qquad \eta_i = -\frac{d\Phi}{dx_i}$$

will be a particular solution of these equations.

If $\Phi$=const., $\Phi_1$=const. are two integrals of equations (1), we will have /1

$$\sum\left(\frac{d\Phi}{dx_i}\frac{d\Phi_1}{dy_i}-\frac{d\Phi}{dy_i}\frac{d\Phi_1}{dx_i}\right)=\text{const.}$$

This is the theorem of Poisson.

Let us consider the particular case where values represent the rectangular coordinates of n points in space; we designate them by the notation with double indices

$$x_{1i},\quad x_{2i},\quad x_{3i},$$

the first index relating to the three rectangular axes of coordinates and the second index with n material points. Let $m_i$ be the mass of the i-th material point. We will then have

$$m_i\frac{d^2x_{ki}}{dt^2}=\frac{dV}{dx_{ki}},$$

V being the force function.

We will then have for the vis viva equation

$$F=\sum\frac{m_i}{2}\left(\frac{dx_{ki}}{dt}\right)^2-V=\text{const.}$$

Let us then set

$$y_{ki}=m_i\frac{dx_{ki}}{dt},$$

from which

$$F=\sum\frac{y_{ki}^2}{2m_i}-V=\text{const.}\tag{3}$$

and

$$\frac{dx_{ki}}{dt}=\frac{dF}{dy_{ki}},\qquad\frac{dy_{ki}}{dt}=-\frac{dF}{dx_{ki}}.\tag{1'}$$

Let

$$x_{ki}=\varphi_{ki}(t),\qquad y_{ki}=m_i\varphi'_{ki}(t)\tag{4}$$

be a solution of these equations (1'); another solution will be

$$x_{ki}=\varphi_{ki}(t+h),\qquad y_{ki}=m_i\varphi'_{ki}(t+h),$$

h being an arbitrary constant.

Regarding h as infinitely small, we will obtain a solution of equations (2') which corresponds to (1') as equations (2) correspond to (1) /170

$$\xi_{ki} = h\varphi'_{ki}(t) = h\frac{y_{ki}}{m_i}, \qquad \eta_{ki} = hm_i\varphi''_{ki}(t) = h\frac{dV}{dx_{ki}},$$

h designating a very small constant factor which we can suppress when we consider only the linear equations (2').

Knowing a solution

$$\xi = \frac{y}{m}, \qquad \eta = \frac{dV}{dx}$$

of these equations, we can deduce an integral

$$\sum \frac{y\eta}{m} - \sum \frac{dV}{dx}\xi = \text{const.}$$

However, this same integral is obtained very easily by differentiating the vis viva equation (3).

If the material points are isolated from all exterior action, we can deduce another solution from solution (4)

$$\begin{aligned} x_{1i} &= \varphi_{1i}(t) + h + kt, & y_{1i} &= m_i\varphi'_{1i}(t) + m_i k, \\ x_{2i} &= \varphi_{2i}(t), & y_{2i} &= m_i\varphi'_{2i}(t), \\ x_{3i} &= \varphi_{3i}(t), & y_{3i} &= m_i\varphi'_{3i}(t), \end{aligned}$$

h and k being arbitrary constants. Regarding these constants as infinitely small, we obtain two solutions of equations (2')

$$\xi_{1i} = 1, \qquad \xi_{2i} = \xi_{3i} = \eta_{1i} = \eta_{2i} = \eta_{3i} = 0,$$

$$\xi_{1i} = t, \qquad \xi_{2i} = \xi_{3i} = \eta_{2i} = \eta_{3i} = 0, \qquad \eta_{1i} = m_i.$$

We thus obtain two integrals of (2')

$$\Sigma_i \eta_{1i} = \text{const.},$$

$$\Sigma \eta_{1i} t - \Sigma m_i \xi_{1i} = \text{const.}$$

We can obtain these integrals by differentiating the equations of motion of the center of gravity /171

$$\Sigma m_i x_{1i} = t\Sigma y_{1i} + \text{const.},$$

$$\Sigma y_{1i} = \text{const.}$$

If we substitute in the solution (4) the rotation through an angle $\omega$ about the z axis, we obtain another solution

$$x_{1i} = \varphi_{1i}\cos\omega - \varphi_{2i}\sin\omega, \qquad \frac{y_{1i}}{m_i} = \varphi'_{1i}\cos\omega - \varphi'_{2i}\sin\omega,$$
$$x_{2i} = \varphi_{1i}\sin\omega + \varphi_{2i}\cos\omega, \qquad \frac{y_{2i}}{m_i} = \varphi'_{1i}\sin\omega + \varphi'_{2i}\cos\omega,$$
$$x_{3i} = \varphi_{3i}, \qquad \frac{y_{3i}}{m_i} = \varphi'_{3i}.$$

Regarding $\varpi$ as infinitely small, we find a solution of (2')

$$\xi_{1i} = -x_{2i}, \qquad \eta_{1i} = -y_{2i},$$
$$\xi_{2i} = x_{1i}, \qquad \eta_{2i} = y_{1i},$$
$$\xi_{3i} = 0, \qquad \eta_{3i} = 0,$$

from which the integral of (2')

$$\Sigma_i(x_{1i}\eta_{2i} - y_{1i}\xi_{2i} - x_{2i}\eta_{1i} + y_{2i}\xi_{1i}) = \text{const.},$$

which we could also obtain by differentiating the integral of the areas of (1')

$$\Sigma(x_{1i}y_{2i} - x_{2i}y_{1i}) = \text{const.}$$

Let us now assume that function V is homogeneous and of degree -1 with respect to x, which is the case in nature.

Equations (1') will not change when we multiply t by $\lambda^3$, x by $\lambda^2$ and y by $\lambda^{-1}$, $\lambda$ being an arbitrary constant. From solution (4) we will therefore deduce the following solution

$$x_{ki} = \lambda^2 \varphi_{ki}\left(\frac{t}{\lambda^3}\right), \qquad y_{ki} = \lambda^{-1} m_i \varphi'_{ki}\left(\frac{t}{\lambda^3}\right).$$

If we regard $\lambda$ as very close to unity, we will obtain as solution of equations (2')

$$\xi_{ki} = 2\varphi_{ki} - 3t\varphi'_{ki}, \qquad \eta_{ki} = -m_i\varphi'_{ki} - 3m_i t\varphi'_{ki}$$

or

$$\xi_{ki} = 2x_{ki} - 3t\frac{y_{ki}}{m_i}, \qquad \eta_{ki} = -y_{ki} - 3t\frac{dV}{dx_{ki}}, \tag{5}$$

from which the following integral of equations (2'), which, unlike those which we have considered until now, cannot be obtained by differentiating a known integral of equations (1') /17

$$\Sigma(2x_{ki}\eta_{ki} + y_{ki}\xi_{ki}) = 3t\left[\sum\left(\frac{y_{ki}\eta_{ki}}{m_i} - \frac{dV}{dx_{ki}}\xi_{ki}\right)\right] + \text{const.}$$

## Application of the Theory of Linear Substitutions

57. Before continuing, I am obliged to recall some of the properties of the linear transformations which will later be useful to us.

Let

$$\left.\begin{aligned}\gamma_1 &= a_1\beta_1 + a_2\beta_2 + a_3\beta_3,\\ \gamma_2 &= b_1\beta_1 + b_2\beta_2 + b_3\beta_3,\\ \gamma_3 &= c_1\beta_1 + c_2\beta_2 + c_3\beta_3\end{aligned}\right\} \qquad (1)$$

be a linear substitution which unites the variables β to the variables γ. The determinant of this substitution is

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

and the equation

$$\begin{vmatrix} a_1 - S & a_2 & a_3 \\ b_1 & b_2 - S & b_3 \\ c_1 & c_2 & c_3 - S \end{vmatrix} = 0 \qquad (2)$$

is called the equation in S of substitution (1). If we make the same linear substitution for β and for γ, that is to say, if we set

$$\beta'_i = \lambda_{i,1}\beta_1 + \lambda_{i,2}\beta_2 + \lambda_{i,3}\beta_3,$$
$$\gamma'_i = \lambda_{i,1}\gamma_1 + \lambda_{i,2}\gamma_2 + \lambda_{i,3}\gamma_3,$$

λ being constant, γ' and β' will be connected by linear relations of the same form as (1), and we will have

$$\left.\begin{aligned}\gamma'_1 &= a'_1\beta'_1 + a'_2\beta'_2 + a'_3\beta'_3,\\ \gamma'_2 &= b'_1\beta'_1 + b'_2\beta'_2 + b'_3\beta'_3,\\ \gamma'_3 &= c'_1\beta'_1 + c'_2\beta'_2 + c'_3\beta'_3.\end{aligned}\right\} \qquad (3)$$

The linear substitution (3) will then be called the transform of substitution (1). /173

The theory of linear substitutions teaches us:

(1) that the new equation in S

$$\begin{vmatrix} a'_1 - S & a'_2 & a'_3 \\ b'_1 & b'_2 - S & b'_3 \\ c'_1 & c'_2 & c'_3 - S \end{vmatrix} = 0$$

does not differ from the old equation in S (2);

(2) that if the determinant $\Delta$ is zero as well as all its minors up to order p inclusively, it will be the same of the determinant

$$\Delta' = \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ c'_1 & c'_2 & c'_3 \end{vmatrix}.$$

The minors of order p and $\Delta'$are, in fact, linear combinations of minors of order p of $\Delta$;

(3) that we can choose $\lambda$ so as to reduce substitution (2) to as simple a form as possible, called the canonical form. Here is what this form consists of:

If the equation in S has all its roots simple, we can cancel at the same time $a'_2$, $a'_3$, $b'_1$, $b'_3$, $c'_1$, $c'_2$.

If the equation in S has a double root, we can cancel at the same time $a'_2$, $a'_3$, $b'_3$, $b'_1$, $c'_1$; we then have $b'_2=c'_2$.

If the equation in S has a triple root, we can cancel at the same time $a'_2$, $a'_3$ and $b'_3$; we then have $a'_1=b'_2=c'_3$.

In all cases, we can always assume that values $\lambda$ were chosen such that

$$a'_2 = a'_3 = b'_3 = 0.$$

If the equation in S has a zero root, $\Delta$ is zero and vice versa.

Let us now assume that $\Delta$ has all its first-order minors zero; then the same is true for $\Delta'$. However, since

$$a'_2 = a'_3 = b'_3 = 0 \quad ,$$

there are three of the minors of $\Delta'$ which reduce to

$$a'_1 b'_2, \quad b'_2 c'_3, \quad a'_1 c'_3;$$

they can vanish only if two of the three quantities $a'_1$, $b'_2$ and $c'_3$ are zero. /171

However, these three quantities are the three roots of the equation in S. Therefore, if the minors of $\Delta$ are still zero, the equation in S has two zero roots.

The converse is not true.

In fact, the equation in S

$$\begin{vmatrix} 1-S & 0 & 0 \\ 0 & -S & 0 \\ 0 & 1 & -S \end{vmatrix} = 0$$

has two zero roots and all its minors are not zero.

We have assumed, in order to establish our ideas, that we were dealing with a linear substitution in only three variables; but the same reasoning is applicable whatever the number of variables may be.

If the determinant of a linear substitution is zero, as well as all its minors of the first, second, etc., of the $(p-1)^{th}$ order, the equation in S will have p zero roots.

58. Let, as in the preceding chapter,

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n)$$

be a system of differential equations. Let

$$x_i = \varphi_i(t)$$

be a periodic solution of these equations of period T.

Let, in a neighboring solution of this periodic solution, $\varphi_i(0)+\beta_i$ be the value of $x_i$ for t=0, and $\varphi_i(0)+\beta_i+\psi_i$ be the value of $x_i$ for $t=T+\tau$.

Let us imagine the functional determinant of $\psi$ with respect to $\beta$

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n} \\ \frac{d\psi_2}{d\beta_1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_n} & \cdots & \cdots & \frac{d\psi_n}{d\beta_n} \end{vmatrix}.$$

We can regard it as the table of coefficients of a linear substitution T. /175

If we make x undergo a linear change of variables, values $\beta$ and $\psi$ will undergo this same linear change, and the linear substitution T will change into the transformed substitution in the sense of the preceding article.

We will therefore be able to choose the linear change of variables undergone by x, $\beta$ and $\psi$ so as to simplify as much as possible the table of coefficients of T, as was explained above. We can therefore always assume that we have made a linear change of variables, such that

$$\frac{d\psi_i}{d\beta_k} = 0 \qquad (1)$$

each time that i<k.

In this case the roots of the equation in S relative to the substitution T are

$$\frac{d\psi_1}{d\beta_1}, \frac{d\psi_2}{d\beta_2}, \ldots, \frac{d\psi_n}{d\beta_n}.$$

We can in addition choose the change of variables which values x, β and ψ undergo such that these roots of the equation in S present themselves in any order that we wish. If, for example, the equation in S has two zero roots, we can choose this change of variables such that

$$\frac{d\psi_{n-1}}{d\beta_{n-1}} = \frac{d\psi_n}{d\beta_n} = 0.$$

If the equation in S has only one root equal to $\frac{d\psi_1}{d\beta_1}$, we will still be able to choose the change of variables, such that one has, in addition,

$$\frac{d\psi_2}{d\beta_1} = \frac{d\psi_3}{d\beta_1} = \ldots = \frac{d\psi_n}{d\beta_1} = 0. \qquad (2)$$

Let us therefore assume that the equation in S has one and only one zero root; according to the preceding, we will be able to assume that this zero root is $\frac{d\psi_1}{d\beta_1}$, so that

$$\frac{d\psi_1}{d\beta_1} = 0,$$

and choose at the same time the change of variables, so as to satisfy conditions (1) and (2). /27

If the equation in S therefore has one and only one zero root, it is always permissible to assume that

$$\frac{d\psi_1}{d\beta_1} = \frac{d\psi_1}{d\beta_2} = \ldots = \frac{d\psi_1}{d\beta_n} = 0, \quad \frac{d\psi_1}{d\beta_1} = \frac{d\psi_2}{d\beta_1} = \ldots = \frac{d\psi_n}{d\beta_1} = 0.$$

## Definition of Characteristic Exponents

99. Let

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

be a system of differential equations where $X_1$, $X_2$, ..., $X_n$ will be given functions of $x_1$, $x_2$, ..., $x_n$. We will be able to assume that either time t does not enter explicitly into these functions $X_i$ or, on the contrary, these functions $X_i$ depend not only on $x_1$, $x_2$, ..., $x_n$, but also on time t; but in this last case the $X_i$ will have to be periodic functions of t.

Let us imagine that these equations (1) admit a periodic solution

$$x_i = \varphi_i(t).$$

Let us take this solution as generating solution and let us form the equations of variation (cf. article 53) of equations (1), setting

$$x_i = \varphi_i(t) + \xi_i$$

and neglecting the squares of ξ.

These equations of variation will be written

$$\frac{d\xi_i}{dt} = \frac{dX_i}{dx_1}\xi_1 + \frac{dX_i}{dx_2}\xi_2 + \ldots + \frac{dX_i}{dx_n}\xi_n. \tag{2}$$

These equations are linear with respect to ξ, and their coefficients $\frac{dX_i}{dx_k}$ (when $x_i$ in them is replaced by $\varphi_i(t)$) are periodic functions of t. /177

We therefore have to integrate linear equations with periodic coefficients.

The general form of the integrals of these equations has been seen in article 29; we obtain n particular solutions of the following form

$$\begin{array}{lll} \xi_1 = e^{\alpha_1 t}S_{11}, & \xi_2 = e^{\alpha_1 t}S_{21}, & \xi_n = e^{\alpha_1 t}S_{n1}, \\ \xi_1 = e^{\alpha_2 t}S_{12}, & \xi_2 = e^{\alpha_2 t}S_{22}, & \xi_n = e^{\alpha_2 t}S_{n2}, \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, & \ldots\ldots\ldots, \\ \xi_1 = e^{\alpha_n t}S_{1n}, & \xi_2 = e^{\alpha_n t}S_{2n}, & \xi_n = e^{\alpha_n t}S_{nn}, \end{array} \tag{3}$$

values α being constants and $S_{ik}$ being periodic functions of t of the same period as $\varphi_i(t)$.

The constants α are called the characteristic exponents of the periodic solution.

If α is purely imaginary such that its square is negative, the modulus of $e^{\alpha t}$ is constant and equal to 1. If on the contrary α is real, or if α is complex such that its square is not real, the modulus $e^{\alpha t}$ tends toward infinity for $t=+\infty$ or for $t=\pm\infty$. If therefore all α have their real and negative squares, the quantities $\xi_1$, $\xi_2$, ..., $\xi_n$ remain finite; I will say then that the periodic solution $x_i=\varphi_i(t)$ is stable; in the contrary case, I will say that this solution is unstable.

An interesting particular case is that case where two or several of the characteristic exponents $\alpha$ are equal. In this case the integrals of equations (2) can no longer be placed in form (3). If, for example,

$$\alpha_1 = \alpha_2,$$

equations (2) would admit two particular solutions which would be written

$$\xi_i = e^{\alpha_1 t} S_{i,1}$$

and

$$\xi_i = t e^{\alpha_1 t} S_{i,1} + e^{\alpha_1 t} S_{i,2},$$

$S_{i,1}$ and $S_{i,2}$ being periodic functions of t (cf. article 29).

If three of the characteristic exponents were equal to each other, not only t but $t^2$ as well would appear outside the trigonometric and exponential signs.

## The Equation Which Defines These Exponents

60. Returning to equations (1) of the preceding article, let us consider /1
an arbitrary solution

$$x_i = \varphi_i(t) + \xi_i.$$

Let T be the period of the periodic generating solution $x_i = \varphi_i(t)$; let $\varphi_i(0) + \beta_i$ be the value of $x_i$ for t=0 and $\varphi_i(T) + \beta_i + \psi_i$ be the value of $x_i$ for t=T.

As values $\psi_i$ vanish with $\beta_i$ and can be developed in terms of the increasing powers of $\beta_i$, we can write, by the formula of Taylor,

$$\psi_i = \frac{d\psi_i}{d\beta_1}\beta_1 + \frac{d\psi_i}{d\beta_2}\beta_2 + \ldots + \frac{d\psi_i}{d\beta_n}\beta_n + \ldots.$$

If the solution under consideration differs sufficiently little from the periodic solution so that we can neglect the squares of $\xi$, we will also be able to neglect the squares of $\beta$ and the following is obtained

$$\psi_i = \frac{d\psi_i}{d\beta_1}\beta_1 + \frac{d\psi_i}{d\beta_2}\beta_2 + \ldots + \frac{d\psi_i}{d\beta_n}\beta_n \qquad (i = 1, 2, \ldots, n).$$

In considering one of the particular solutions of the equations of variation (2), we will have for t=0

$$\xi_i = \beta_i$$

and for t=T

$$\xi_i = \beta_i + \psi_i.$$

Among these particular solutions, we saw in article 59 that there are n of them which are of a special form: they are solutions (3); let

$$\xi_1 = e^{\alpha_k t} S_{1,k}, \quad \xi_2 = e^{\alpha_k t} S_{2,k}, \quad \ldots, \quad \xi_n = e^{\alpha_k t} S_{n,k}$$

be one of these solutions (3) where, by suppressing the index k for brevity in writing,

$$\xi_i = e^{\alpha t} S_i(t).$$

The functions $S_i(t)$ are periodic functions of t, of period T; we therefore have, for t=0, /179

$$\beta_i = S_i(o)$$

and, for t=T,

$$\beta_i + \psi_i = e^{\alpha T} S_i(T) = e^{\alpha T} S_i(o) = e^{\alpha T} \beta_i$$

or, replacing $\psi_i$ by its value,

$$\beta_i(e^{\alpha T} - 1) = \frac{d\psi_i}{d\beta_1}\beta_1 + \frac{d\psi_i}{d\beta_2}\beta_2 + \ldots + \frac{d\psi_i}{d\beta_n}\beta_n \qquad (i = 1, 2, \ldots, n).$$

Eliminating $\beta_1, \beta_2, \ldots, \beta_n$ among these n equations, it follows that

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1} + 1 - e^{\alpha T} & \frac{d\psi_1}{d\beta_2} & \frac{d\psi_1}{d\beta_n} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} + 1 - e^{\alpha T} & \frac{d\psi_2}{d\beta_n} \\ \cdots & \cdots\cdots\cdots\cdots & \cdots\cdot \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \frac{d\psi_n}{d\beta_n} + 1 - e^{\alpha T} \end{vmatrix} = 0$$

gives us the following rule:

In order to obtain the characteristic exponents $\alpha$, we form the functional determinant of $\psi$ with respect to $\beta$; we form the corresponding equation in S: the roots of this equation are equal to $e^{\alpha T}-1$.

In the partial derivatives $\frac{d\psi_i}{d\beta_k}$ it goes without saying that it is necessary, according to the differentiations, to set $\beta_i=0$.

Case Where Time Does Not Enter Explicitly

61. When time t does not enter explicitly into equations (1) of article 59,

at least one of the characteristic exponents is zero. In fact, let

$$x_i = \varphi_i(t)$$

be the generating function; if we set

$$x_i = \varphi_i(t+h),$$

h being an arbitrary constant, we will still have a solution of equations (1); /15
then, according to article 51, we will have a solution of the equations of variation, by setting

$$\xi_i = \frac{d\varphi_i}{dh} = \frac{d\varphi_i}{dt}. \tag{4}$$

However, $\varphi_i$ being a periodic function of t, it will be the same for its derivative $\frac{d\varphi_i}{dt}$.

Solution (4) is definitely of form (3) (i.e., equal to an exponential multiplied by a periodic function of t). Only here the exponential reduces to unity and the characteristic exponent is equal to 0.

Q. E. D.

In addition we have already seen in the preceding chapter that in this case the functional determinant of the $\Psi$ with respect to the $\beta$ is zero.

New Statement of the Theorem of Articles 37 and 38

62. In article 37 we first saw the case where equations (1) depend on time t and on a parameter $\mu$, and admit for $\mu$=0 one and only one periodic solution. We have seen that if the functional determinant

$$\Delta = \frac{\partial(\psi_1, \psi_2, \ldots, \psi_n)}{\partial(\beta_1, \beta_2, \ldots, \beta_n)} \lessgtr 0,$$

the equations will admit in addition a periodic solution for small values of $\mu$.

This determinant can be written

$$\Delta = \begin{vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_2}{d\beta_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n} \end{vmatrix}.$$

Now the characteristic exponents $\alpha$ are given by the equation /1

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1}+1-e^{\alpha T} & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2}+1-e^{\alpha T} & \cdots & \frac{d\psi_2}{d\beta_n} \\ \cdots & \cdots\cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n}+1-e^{\alpha T} \end{vmatrix} = 0.$$

To say that $\Delta$ is zero is therefore to say that one of the characteristic exponents is zero, such that we can state thus the first of the theorems demonstrated in the preceding section:

If equations (1), which depend on a parameter $\mu$, admit for $\mu=0$ a periodic solution of which none of the characteristic exponents are zero, they will admit in addition a periodic solution for small values of $\mu$.

63. We can arrive at an analogous result when we assume, as in article 38, that time does not enter explicitly into the differential equations.

We saw in article 38 that the sufficient condition for there to be periodic solutions for small values of $\mu$ is that for $\mu=0$ all determinants contained in the matrix

$$\begin{Vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} & \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_2}{d\beta_n} & \frac{d\psi_2}{d\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n} & \frac{d\psi_n}{d\tau} \end{Vmatrix}$$

not be zero at the same time.

This granted, let us consider the equation in S

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1}-S & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2}-S & \cdots & \frac{d\psi_2}{d\beta_n} \\ \cdots & \cdots\cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n}-S \end{vmatrix} = 0.$$

As we saw in article 60, its roots are equal to $e^{\alpha T}-1$, T being the period and $\alpha$ a characteristic exponent. Time not entering explicitly into the equations, one of these exponents must be zero according to what we saw in article 61. /182

The equation in S has therefore at least one zero root; I say that if it has only one, there will still be periodic solutions for small values of $\mu$.

In fact, according to what we saw in article 58, it is always permissible to assume that

$$\frac{d\psi_1}{d\beta_1} = \frac{d\psi_1}{d\beta_2} = \frac{d\psi_1}{d\beta_3} = \ldots = \frac{d\psi_1}{d\beta_n} = 0,$$
$$\frac{d\psi_1}{d\beta_1} = \frac{d\psi_2}{d\beta_1} = \frac{d\psi_3}{d\beta_1} = \ldots = \frac{d\psi_n}{d\beta_1} = 0.$$

The first member of the equation in S is written

$$-S \begin{vmatrix} \frac{d\psi_2}{d\beta_2} - S & \frac{d\psi_2}{d\beta_3} & \cdots & \frac{d\psi_2}{d\beta_n} \\ \frac{d\psi_3}{d\beta_2} & \frac{d\psi_3}{d\beta_3} - S & \cdots & \frac{d\psi_3}{d\beta_n} \\ \cdots & \cdots\cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_2} & \frac{d\psi_n}{d\beta_3} & \cdots & \frac{d\psi_n}{d\beta_n} - S \end{vmatrix}.$$

If, therefore, the equation in S has only one zero root, the functional determinant $\delta$ of $\psi_2, \psi_3, \ldots, \psi_n$ with respect to $\beta_2, \beta_3, \ldots, \beta_n$ will not be zero.

Then the determinant obtained by suppressing the first column in the matrix will be reduced to

$$\delta \frac{d\psi_1}{d\tau}.$$

I say that it is not zero; in fact, $\frac{d\psi_1}{d\tau}$ cannot be cancelled for the following reason:

We cannot have at the same time

$$\frac{d\psi_1}{d\tau} = \frac{d\psi_2}{d\tau} = \ldots = \frac{d\psi_n}{d\tau} = 0.$$

If it were so, that would mean that if we consider the periodic solution

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t),$$

which corresponds to $\mu=0$ and which serves us as a starting point, we have for t=T (and consequently still for all values of t) /1

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \ldots = \frac{dx_n}{dt} = 0,$$

such that $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ would be constants, which we will not assume.

On the other hand, I say that

$$\frac{d\psi_2}{d\tau} = \frac{d\psi_3}{d\tau} = \ldots = \frac{d\psi_n}{d\tau} = 0.$$

We have, in fact, as we saw above, p. 72,

$$\frac{d\psi_1}{d\tau}\frac{d\psi_i}{d\beta_1} + \frac{d\psi_2}{d\tau}\frac{d\psi_i}{d\beta_2} + \ldots + \frac{d\psi_n}{d\tau}\frac{d\psi_i}{d\beta_n} = 0 \qquad (i = 1, 2, \ldots, n).$$

Now $\frac{d\psi_i}{d\beta_1} = 0$; we therefore have a series of linear equations

$$\frac{d\psi_2}{d\tau}\frac{d\psi_i}{d\beta_2} + \ldots + \frac{d\psi_n}{d\tau}\frac{d\psi_i}{d\beta_n} = 0 \qquad (i = 2, 3, \ldots, n),$$

and, as the determinant of these equations (i.e., $\delta$) is not zero, it follows that

$$\frac{d\psi_2}{d\tau} = \frac{d\psi_3}{d\tau} = \ldots = \frac{d\psi_n}{d\tau} = 0.$$

As we have excluded the case where $\varphi_1(t)$, $\varphi_2(t)$, ..., $\varphi_n(t)$ are constants, a case which will be examined separately, in article 68, we conclude from this that

$$\frac{d\psi_1}{d\tau} \gtrless 0.$$

Q. E. D.

Thus, if the differential equations do not contain time explicitly, if they admit a periodic solution for $\mu=0$, one of the characteristic exponents of this solution will always be zero; if, in addition, no other of these exponents is zero, there will still be a periodic solution for small values of $\mu$.

## Case Where the Equations Admit Uniform Integrals

64. Let us assume that equations /184

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where $X_i$ are uniform functions of $x_1$, $x_2$, ..., $x_n$ and of t, periodic of period $2\pi$ with respect to t, admit a periodic solution of period $2\pi$,

$$x_1 = \varphi_1(t), \qquad x_2 = \varphi_2(t), \quad \ldots, \qquad x_n = \varphi_n(t),$$

such that $\varphi_i(2\pi) = \varphi_i(0)$ is an independent integral of time

$$F(x_1, x_2, \ldots, x_n) = \text{const.},$$

uniform with respect to $x_1, x_2, \ldots, x_n$. I say that one of the characteristic exponents is zero, save in an exceptional case of which I will speak later.

Let us in fact define the quantities $\psi$ and $\beta$ as in article 37, and let us consider the functional determinant of the $\psi$ with respect to $\beta$. I say that this determinant is zero.

In fact, we have identically

$$F[\varphi_i(0)+\beta_i+\psi_i) = F[\varphi_i(0)+\beta_i],$$

writing, for brevity, $F(x_i)$ in place of

$$F(x_1, x_2, \ldots, x_n).$$

Differentiating this identity with respect to $\beta_i$, we find

$$\frac{dF}{dx_1}\frac{d\psi_1}{d\beta_i}+\frac{dF}{dx_2}\frac{d\psi_2}{d\beta_i}+\ldots+\frac{dF}{dx_n}\frac{d\psi_n}{d\beta_i}=0. \qquad (2)$$

It is necessary, in $\frac{dF}{dx_1}, \frac{dF}{dx_2}, \ldots, \frac{dF}{dx_n}$, to replace $x_1, x_2, \ldots, x_n$ by $\varphi_1(0), \varphi_2(0), \ldots, \varphi_n(0)$.

In equations (2) we can make $i=1, 2, \ldots, n$; we therefore have n linear equations with respect to the n quantities /18

$$\frac{dF}{dx_1}, \frac{dF}{dx_2}, \ldots, \frac{dF}{dx_n}.$$

Then one of two things is so: either the determinant of these equations (2), i.e., the functional determinant of $\psi$ with respect to $\beta$, will be zero, or else, according to what we have seen in article 62, one of the characteristic exponents will be zero.

Or we have at the same time

$$\frac{dF}{dx_1}=\frac{dF}{dx_2}=\ldots=\frac{dF}{dx_n}=0. \qquad (3)$$

These equations will have to be satisfied for

$$x_1=\varphi_1(2\pi), \quad x_2=\varphi_2(2\pi), \quad \ldots, \quad x_n=\varphi_n(2\pi)$$

or, what amounts to the same thing, for

$$x_1=\varphi_1(0), \quad x_2=\varphi_2(0), \quad \ldots, \quad x_n=\varphi_n(0).$$

But the origin of time has remained entirely arbitrary; we must therfore conclude that equations (3) will be satisfied whatever t may be, for

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t).$$

We can again make this clear in the following manner:

Let us assume that equations (3) are satisfied for a system of values of $x_1$, $x_2$, ..., $x_n$; I say that they will still be satisfied for an infinitely near system $x_1+dx_1$, $x_2+dx_2$, ..., $x_n+dx_n$, provided that we have, in conformity with the differential equations,

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}.$$

In other words, I say that equations (3) entail the following,

$$\frac{d^2F}{dx_i\,dx_1} X_1 + \frac{d^2F}{dx_i\,dx_2} X_2 + \ldots + \frac{d^2F}{dx_i\,dx_n} X_n = 0$$

$$(i = 1, 2, \ldots, n).$$

In fact, we have identically (since F is an integral of the differential equations) /186

$$\frac{dF}{dx_1} X_1 + \frac{dF}{dx_2} X_2 + \ldots + \frac{dF}{dx_n} X_n = 0.$$

Differentiating this identity with respect to $x_i$, it follows that

$$\sum_{k=1}^{k=n} \left( \frac{d^2F}{dx_i\,dx_k} X_k + \frac{dF}{dx_k} \frac{dX_k}{dx_i} \right) = 0$$

where, by virtue of equations (3),

$$\sum_k \frac{d^2F}{dx_i\,dx_k} X_k = 0.$$

Q. E. D.

Thus, if the differential equations admit a uniform integral, one of the characteristic exponents of an arbitrary periodic solution will be zero, at least all the partial derivatives of the integral do not vanish identically for all points of this periodic solution. This last circumstance will occur only in exceptional cases.

65. Let us still assume that differential equations (1) contain time explicitly and are, with respect to this variable, periodic functions of period $2\pi$.

I say that if the differential equations admit two uniform integrals, F and $F_1$, two of the characteristic exponents will be zero.

We will find, in fact, as in the preceding article,

$$\left.\begin{aligned} &\frac{dF}{dx_1}\frac{d\psi_1}{d\beta_i}+\frac{dF}{dx_2}\frac{d\psi_2}{d\beta_i}+\ldots+\frac{dF}{dx_n}\frac{d\psi_n}{d\beta_i}=0, \\ &\frac{dF_1}{dx_1}\frac{d\psi_1}{d\beta_i}+\frac{dF_1}{dx_2}\frac{d\psi_2}{d\beta_i}+\ldots+\frac{dF_1}{dx_n}\frac{d\psi_n}{d\beta_i}=0 \end{aligned}\right\} \tag{2}$$

$$(i=1,2,\ldots,n)$$

$$[x_1=\varphi_1(0),\quad x_2=\varphi_2(0),\quad \ldots\quad x_n=\varphi_n(0)].$$

We can conclude from this that not only the functional determinant of $\psi$ with respect to $\beta$ is zero, but the same is true of all its first-order minors, unless we have at the same time /1

$$\frac{\frac{dF}{dx_1}}{\frac{dF_1}{dx_1}}=\frac{\frac{dF}{dx_2}}{\frac{dF_1}{dx_2}}=\ldots=\frac{\frac{dF}{dx_n}}{\frac{dF_1}{dx_n}}\quad . \tag{3}$$

However, according to article 57, this can take place only if the equation in S, formed with the aid of the functional determinant of $\psi$, has two zero roots, i.e., (since these roots are equal to $e^{\alpha T}-1$) if there are two zero characteristic exponents.

If, therefore, there are two uniform integrals, there will be two zero characteristic exponents, unless equations (3) are satisfied at all points of the periodic solution, which can only happen in exceptional cases.

It may be demonstrated also that if there are p uniform integrals $F_1$, $F_2$, ..., $F_p$, p of the characteristic exponents will be zero unless all determinants contained in the matrix

$$\left\| \begin{matrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \ldots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \ldots & \frac{dF_2}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dF_p}{dx_1} & \frac{dF_p}{dx_2} & \ldots & \frac{dF_p}{dx_n} \end{matrix} \right\|$$

vanish in all points of the periodic solution under consideration.

66. Let us imagine now that time does not enter our differential equations explicitly, and in addition, that these equations admit a uniform integral

$$F(x_1, x_2, \ldots, x_n) = \text{const.},$$

independent of time t.

I say that two characteristic exponents will be zero.

We first saw that one of these exponents is always zero when time does not enter explicitly. If, in addition, there is an integral F, we will have, as in article 64, /188

$$F[\varphi_i(0) + \beta_i + \psi_i] = F[\varphi_i(0) + \beta_i],$$

and, differentiating this relation with respect to $\beta_i$ and $\tau$, it follows that

$$\frac{dF}{dx_1}\frac{d\psi_1}{d\beta_i} + \frac{dF}{dx_2}\frac{d\psi_2}{d\beta_i} + \ldots + \frac{dF}{dx_n}\frac{d\psi_n}{d\beta_i} = 0$$
$$(i = 1, 2, \ldots, n),$$

$$\frac{dF}{dx_1}\frac{d\psi_1}{d\tau} + \frac{dF}{dx_2}\frac{d\psi_2}{d\tau} + \ldots + \frac{dF}{dx_n}\frac{d\psi_n}{d\tau} = 0.$$

From this we conclude either that we have at the same time

$$\frac{dF}{dx_1} = \frac{dF}{dx_2} = \ldots = \frac{dF}{dx_n} = 0$$

for all points of the periodic solution, or that all determinants contained in the matrix

$$\begin{Vmatrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_2} & \cdots & \frac{d\psi_1}{d\beta_n} & \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \cdots & \frac{d\psi_2}{d\beta_n} & \frac{d\psi_2}{d\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d\psi_n}{d\beta_1} & \frac{d\psi_n}{d\beta_2} & \cdots & \frac{d\psi_n}{d\beta_n} & \frac{d\psi_n}{d\tau} \end{Vmatrix}$$

are zero at the same time.

Now we saw in article 63 that this can occur only if two characteristic exponents vanish.

67. I now propose to establish the following:

Let us still assume that time does not enter explicitly into our differential equations; let us assume that these equations admit p analytic and uniform integrals and where time no longer enters explicitly. Let $F_1$, $F_2$, ..., $F_p$ be these p integrals.

Then, either p+1 characteristic exponents will be zero, or all determinants contained in the matrix /189

$$\left\| \frac{dF_i}{dx_k} \right\| \qquad (i = 1, 2, \ldots, p; \quad k = 1, \ldots, n)$$

will be zero for all points of the periodic generating solution.

Let us assume, in fact, to fix the ideas,

$$n = 4, \qquad p = 2.$$

We will then have the following equations

$$\left.\begin{aligned} \frac{dF_1}{dx_1}\frac{d\psi_1}{d\beta_i} + \frac{dF_1}{dx_2}\frac{d\psi_2}{d\beta_i} + \frac{dF_1}{dx_3}\frac{d\psi_3}{d\beta_i} + \frac{dF_1}{dx_4}\frac{d\psi_4}{d\beta_i} = 0, \\ \frac{dF_2}{dx_1}\frac{d\psi_1}{d\beta_i} + \frac{dF_2}{dx_2}\frac{d\psi_2}{d\beta_i} + \frac{dF_2}{dx_3}\frac{d\psi_3}{d\beta_i} + \frac{dF_2}{dx_4}\frac{d\psi_4}{d\beta_i} = 0 \end{aligned}\right\} \quad (i = 1, 2, 3, 4),$$

$$\begin{aligned} \frac{dF_1}{dx_1}\frac{d\psi_1}{d\tau} + \frac{dF_1}{dx_2}\frac{d\psi_2}{d\tau} + \frac{dF_1}{dx_3}\frac{d\psi_3}{d\tau} + \frac{dF_1}{dx_4}\frac{d\psi_4}{d\tau} = 0, \\ \frac{dF_2}{dx_1}\frac{d\psi_1}{d\tau} + \frac{dF_2}{dx_2}\frac{d\psi_2}{d\tau} + \frac{dF_2}{dx_3}\frac{d\psi_3}{d\tau} + \frac{dF_2}{dx_4}\frac{d\psi_4}{d\tau} = 0. \end{aligned}$$

From these equations it is permissible to conclude:

either that all determinants contained in the matrix

$$\left\| \begin{matrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \frac{dF_1}{dx_3} & \frac{dF_1}{dx_4} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \frac{dF_2}{dx_3} & \frac{dF_2}{dx_4} \end{matrix} \right\|$$

are zero at the same time; or that all determinants contained in the matrix

$$\left\| \begin{matrix} \frac{d\psi_1}{d\beta_1} & \frac{d\psi_1}{d\beta_2} & \frac{d\psi_1}{d\beta_3} & \frac{d\psi_1}{d\beta_4} & \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} & \frac{d\psi_2}{d\beta_3} & \frac{d\psi_2}{d\beta_4} & \frac{d\psi_2}{d\tau} \\ \frac{d\psi_3}{d\beta_1} & \frac{d\psi_3}{d\beta_2} & \frac{d\psi_3}{d\beta_3} & \frac{d\psi_3}{d\beta_4} & \frac{d\psi_3}{d\tau} \\ \frac{d\psi_4}{d\beta_1} & \frac{d\psi_4}{d\beta_2} & \frac{d\psi_4}{d\beta_3} & \frac{d\psi_4}{d\beta_4} & \frac{d\psi_4}{d\tau} \end{matrix} \right\| \qquad (1)$$

are zero at the same time, as well as their first-order minors.

According to what we saw in article 58, we can always assume that /1

$$\frac{d\psi_i}{d\beta_k} = 0$$

for

$$i < k.$$

On the other hand, all of the minors of the determinant obtained by suppressing the last column of the matrix (1) having to be zero, the corresponding equation in S will have two zero roots: I may therefore assume

$$\frac{d\psi_3}{d\beta_3} = \frac{d\psi_4}{d\beta_4} = 0.$$

I propose to demonstrate that the equation in S has a third zero root and consequently that we have

$$\frac{d\psi_1}{d\beta_1} = 0$$

or

$$\frac{d\psi_2}{d\beta_2} = 0.$$

In fact, according to the same definition of $\psi_i$, we have $\psi_i=0$ if we make

$$\beta_k = \varphi_k(h) - \varphi_k(0),$$

h being an arbitrary constant; from which by differentiating with respect to h and then making h=0,

$$\sum \frac{d\psi_i}{d\beta_k} \varphi'_k(0) = 0.$$

However,

$$\varphi'_k(0) = \frac{d\psi_k}{d\tau};$$

therefore we have

$$\frac{d\psi_i}{d\beta_1}\frac{d\psi_1}{d\tau} + \frac{d\psi_i}{d\beta_2}\frac{d\psi_2}{d\tau} + \frac{d\psi_i}{d\beta_3}\frac{d\psi_3}{d\tau} + \frac{d\psi_i}{d\beta_4}\frac{d\psi_4}{d\tau} = 0 \qquad (i = 1, 2, 3, 4). \tag{2}$$

Setting i=1, it follows that

$$\frac{d\psi_1}{d\beta_1}\frac{d\psi_1}{d\tau} = 0,$$

whence /191

$$\frac{d\psi_1}{d\beta_1} = 0$$

or

$$\frac{d\psi_1}{d\tau} = 0.$$

In the first case, the theorem is demonstrated; in the second case we write equation (2) by setting i=2; it follows that

$$\frac{d\psi_2}{d\beta_2}\frac{d\psi_2}{d\tau} = 0$$

whence

$$\frac{d\psi_2}{d\beta_2} = 0$$

or

$$\frac{d\psi_2}{d\tau} = 0.$$

In the first case, the theorem is demonstrated; in the second case, we have

$$\frac{d\psi_1}{d\tau} = \frac{d\psi_2}{d\tau} = 0,$$

from which we can conclude (since we exclude the case where all the $\frac{d\psi_3}{d\tau}$ are zero at the same time) that $\frac{d\psi_3}{d\tau}$ and $\frac{d\psi_4}{d\tau}$ are both zero. Let us form the minors which we obtain by eliminating in the matrix (1) the third and fourth columns and the third row (or the third and fourth columns and the fourth row).

These two minors will have to be zero, which gives us

$$\frac{d\psi_1}{d\beta_1}\frac{d\psi_2}{d\beta_2}\frac{d\psi_3}{d\tau} = \frac{d\psi_1}{d\beta_1}\frac{d\psi_2}{d\beta_2}\frac{d\psi_4}{d\tau} = 0,$$

from which we obtain this conclusion (since $\frac{d\psi_3}{d\tau}$ and $\frac{d\psi_4}{d\tau}$ are not both zero) that we have

$$\frac{d\psi_1}{d\beta_1} = 0$$

or

$$\frac{d\psi_2}{d\beta_2} = 0.$$

Q. E. D.

68. In the preceding articles we excluded the case where /192

$$\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$$

are constants, i.e., the case where we have at the same time

$$\frac{d\psi_1}{d\tau} = \frac{d\psi_2}{d\tau} = \ldots = \frac{d\psi_n}{d\tau} = 0.$$

If we always assume that time does not enter explicitly into the differential equations, we then have the equations

$$\frac{d\psi_i}{d\beta_1}\frac{d\psi_1}{d\tau} + \frac{d\psi_i}{d\beta_2}\frac{d\psi_2}{d\tau} + \ldots + \frac{d\psi_i}{d\beta_n}\frac{d\psi_n}{d\tau} = 0.$$

However, these equations no longer imply the consequence that the functional determinant of $\psi$ with respect to $\beta$ be zero, or that one of the characteristic exponents be always zero.

If the differential equations admit p integrals, we will therefore be able to conclude only that there are at least p zero characteristic exponents (and not p+1), as in the case where time enters explicitly into the equations.

## Case of the Equations of Dynamics

69. Let us now proceed to the equations of Dynamics

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots n), \tag{1}$$

where I assume that time does not enter explicitly. They will admit the vis viva integral

$$F = \text{const.}$$

Let us assume that equations (1) admit a periodic solution of period $2\pi$

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t),$$

and let us form the equations of variation by setting

$$x_i = \varphi_i(t) + \xi_i, \quad y_i = \psi_i(t) + \eta_i.$$

We saw in article 56 that if $\xi_i$, $\eta_i$ and $\xi''_i$, $\eta'_i$ are two arbitrary particular /193 solutions of the equations of variation, we have

$$\sum_{i=1}^{n}(\xi_i\eta'_i - \xi'_i\eta_i) = \text{const.}$$

I say that the result of this is that the characteristic exponents are equal in pairs and of opposite sign.

Let in fact $\xi_i^0$ and $\eta_i^0$ be the initial values of $\xi_i$ and $\eta_i$ for t=0 in one of the solutions of the equations of variation; let $\xi_i^1$ and $\eta_i^1$ be the corresponding values of $\xi_i$ and of $\eta_i$ for $t=2\pi$. It is clear that $\xi_i^1$ and $\eta_i^1$ will be linear functions of $\xi_i^0$ and of $\eta_i^0$, such that the substitution T which changes $\xi_i^0$ and $\eta_i^0$ into $\xi_i^1$ and $\eta_i^1$ will be a linear substitution.

Let

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,2n} \\ a_{21} & a_{22} & \dots & a_{2,2n} \\ \dots & \dots & \dots & \dots \\ a_{2n,1} & a_{2n,2} & \dots & a_{2n,2n} \end{vmatrix}$$

be the matrix of coefficients of this linear substitution.

Let us form the equation in $\lambda$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1,2n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2,2n} \\ \dots & \dots & \dots & \dots \\ a_{2n,1} & a_{2n,2} & \dots & a_{2n,2n}-\lambda \end{vmatrix} = 0.$$

The 2n roots of this equation will be what are called the 2n multipliers of the linear substitution T. However, this linear substitution T cannot be arbitrary: it is necessary that it should not alter the bilinear form

$$\Sigma_i(\xi_i \eta'_i - \xi'_i \eta_i).$$

For this to be true, the equation in $\lambda$ must be reciprocal. In fact, the theory of linear substitutions teaches us that if one linear substitution does not alter a quadratic form, its equation in S must be reciprocal. If therefore we set

$$\lambda = e^{2\alpha\pi},$$

the quantities $\alpha$ must be equal in pairs and of opposite sign. /19

Q. E. D.

We will return to this point in article 70.

70. Equations (1) from the preceding article always admit the said vis viva integral

$$F = \text{const.}$$

I assume that they admit, in addition, p uniform integrals

$$F_1 = \text{const.}, \quad F_2 = \text{const.}, \quad \ldots, \quad F_p = \text{const.}$$

I assume, in addition, that the brackets formed for each pair of these integrals are zero, i.e.,

$$[F_i, F_k] = 0 \quad (i, k, = 1, 2, \ldots, p).$$

We know, in addition, that for an arbitrary integral $F_i$ we have

$$[F, F_i] = 0.$$

I propose to demonstrate that in this case either all functional determinants of $F, F_1, F_2, \ldots, F_p$, with respect to arbitrary p+1 of the variables $x_i$ and $y_i$, are zero at the same time at all points of the periodic solution, or 2p+2 characteristic exponents are zero.

In fact, let us again take equations (2) from article 56, i.e., the equations of variation (1). Let

$$\xi_i, \quad \eta_i$$

be a particular solution of these equations (2); let us call this solution S; let $\xi'_i$, $\eta'_i$ be another solution of these same equations; let us call this solution S'.

We know that we have

$$\Sigma(\xi_i \eta'_i - \xi'_i \eta_i) = \text{const.}$$

I will call (S, S') the first member of this relation.

We have seen in article 59 that among the solutions of the proposed equations there are some of which the form is notable. For these, each of the quantities $\xi_i$ and $\eta_i$ is equal to an exponential $e^{\alpha t}$ multiplied by a periodic function of t. /195

I will call them solutions of the first kind.

For others, each of quantities $\xi_i$ and $\eta_i$ is equal to an exponential $e^{\alpha t}$, multiplied by an integral polynomial in t, of which the coefficients are periodic functions of t. I will call them solutions of the second kind.

Equations (2) can admit only 2n linearly independent solutions. An arbitrary solution may therefore always be regarded as a linear combination of 2n solutions which may be called fundamental.

If of 2n characteristic exponents p are distinct, we will be able to choose as fundamental solutions p solutions of the first type and 2n-p solutions of the second type.

Let

$$S_1, \; S_2, \; \ldots, \; S_q$$

be q linearly independent particular solutions of equations (2) and let us designate an arbitrary solution by S'.

There may not be more than 2n-q linearly independent solutions S' which satisfy conditions

$$(S_1, S') = (S_2, S') = \ldots = (S_q, S') = 0. \tag{3}$$

In fact, let

$$\xi_i = \xi_i^k, \qquad \eta_i = \eta_i^k$$

be the solution $S_k$; let us retain the letters $\xi_i$ and $\eta_i$ to designate solution S'; then relations (3) give us q linear relations between $\xi_i$ and $\eta_i$; these relations are distinct if the particular solutions $S_1$, $S_2$, ..., $S_q$ are linearly independent. They will therefore serve to lower by q units the order of the linear differential equations (2). After this reduction, these equations will retain only 2n-q linearly independent solutions.

Q. E. D.

This granted, let us assume that S is a solution of the first or second kind, admitting as characteristic exponent $\alpha$, and that S' is a solution of first or second kind admitting as characteristic exponent $\beta$. Let us form the expression

$$(S, S').$$

This expression is of the following form: an exponential $e^{(\alpha+\beta)t}$, multiplied by an integral polynomial in t whose coefficients are periodic functions of t. /19

However, this expression must reduce to a constant. It is clear that this can occur only in two ways:

either if this constant is zero;

or if $\alpha+\beta=0$.

From this we can conclude that if there are q characteristic exponents equal to $+\alpha$, there will be q equal to $-\alpha$, which confirms the result obtained in article 69. If, in fact, there are q exponents equal to $+\alpha$ there will be q solutions of the first kind or of the second kind linearly independent and admitting $\alpha$ as exponent.

Let $S_1$, $S_2$, ..., $S_q$ be these q solutions.

There can be no more than 2n-q linearly independent solutions S' which satisfy the relations

$$(S_1, S') = (S_2, S') = \dots = (S_q, S') = 0.$$

Consequently, among the fundamental solutions (which are all of the first or second kind), there will be q for which one of the constants $(S_i, S')$ at least will not be zero, and consequently for which the exponent $\beta$ will be equal to $-\alpha$.

71. Let us now assume that equations (1) admit an integral

$$F_1 = \text{const.}$$

According to what we have seen in article 54, equations (2) will admit as a particular solution

$$\xi_i = \frac{dF_1}{dy_i}, \qquad \eta_i = -\frac{dF_1}{dx_i}.$$

Let us call this solution $S_1$, the functions $\frac{dF_1}{dx_i}$ and $\frac{dF_1}{dy_i}$ (where we must replace $x_i$ and $y_i$ by their values corresponding to the periodic generating solution) will be periodic functions of t. Therefore solution $S_1$ is of the first kind and its characteristic exponent is zero.

If $F_2$=const. is another integral and if we call $S_2$ the solution /197

$$\xi_i = \frac{dF_2}{dy_i}, \qquad \eta_i = -\frac{dF_2}{dx_i},$$

it will follow that

$$(S_1, S_2) = [F_1, F_2].$$

Let us assume therefore that our equations (1) admit p+1 integrals

$$F = \text{const.}, \qquad F_1 = \text{const.}, \qquad F_2 = \text{const.}, \qquad \dots, \qquad F_p = \text{const.},$$

and let

$$S, \ S_1, \ S_2, \ \dots, \ S_p$$

be the p+1 solutions of equations (2) which correspond to these p+1 integrals.

One of two things is true:

either these p+1 solutions will be independent;

or all functional determinants of $F, F_1, F_2, \ldots, F_p$ with respect to p+l variables chosen among $x_i$ and $y_i$ will be zero at the same time in all points of the periodic solution.

Let us assume that this is not the case and that the solution $S, S_1, \ldots, S_p$ are independent.

We will in each case have

$$[F, F_i] = 0 \qquad (i = 1, 2, \ldots, p),$$

from which

$$(S, S_i) = 0.$$

I assume that we have in addition

$$[F_i, F_k] = 0 \qquad (i, k = 1, 2, \ldots, p).$$

We equally have

$$(S_i, S_k) = 0.$$

I will choose for the 2n fundamental solutions the p+l solutions $S, S_1, S_2, \ldots, S_p$ and 2n-p-l other solutions of the first or second kind.

Among the fundamental solutions, there will certainly be p+l, which (if I call them S') will not at the same time satisfy the relations $(S, S') = (S_1, S') = \ldots = (S_p, S') = 0$, and which consequently will have a zero characteristic exponent.

However, these p+l solutions will not coincide with the p+l solutions /1

$$S, \quad S_1, \quad \ldots, \quad S_p.$$

I say that we cannot have, for example,

$$S' = S_k,$$

for we have, by hypothesis,

$$(S, S_k) = (S_1, S_k) = \ldots = (S_p, S_k) = 0,$$

and, according to the definition of S', S' does not enjoy this property.

There are therefore altogether 2p+2 fundamental solutions of which the exponent is zero; there are therefore at least 2p+2 characteristic exponents which are zero.

Q. E. D.

72. Let us now assume that there exist p integrals (in addition to F=const.). namely,

$$F_1 = \text{const.}, \quad F_2 = \text{const.}, \quad F_p = \text{const.},$$

but that the brackets formed from the pairs of these p integrals are not zero. All that we can affirm then is that p+2 characteristic exponents will be zero. However, we will know that p+1 fundamental solutions at least (which are those which we have called S, $S_1$, $S_2$, ..., $S_p$) will be of the first kind with a zero exponent.

If, therefore, we established that equations (2) only admit p linearly independent solutions which are of the first kind with a zero exponent, we would be certain that equations (1) do not permit p+1 integrals (including F=const.), or at least that, if these p+1 integrals exist, all their functional determinants with respect to p+1 of the 2n variables x and y are zero simultaneously in all points of the periodic solution.

## Changes of Variables

73. Let us see what changes take place in the characteristic exponents when we change variables.

Let /199

$$\frac{dx_i}{dt} = X_i$$

be our differential equations, where I will assume that time does not enter explicitly. Let

$$x_i = \varphi_i(t)$$

be a periodic solution of period T. Let

$$x_i = \varphi_i(t) + \xi_i,$$

whence the equations of variation

$$\frac{d\xi_i}{dt} = \sum \frac{dX_i}{dx_k} \xi_k.$$

Let

$$\xi_i = e^{\alpha t} \psi_i(t)$$

be a solution of these equations of variation, $\varphi_i$ being periodic in t.

Let us change variables by replacing time t by a new variable τ defined by the relation

$$\frac{dt}{d\tau} = \Phi,$$

$\Phi$ being a known function of $x_1$, $x_2$, ..., $x_n$; whence follow the differential equations

$$\frac{dx_i}{d\tau} = X_i \Phi \tag{1a}$$

and the equations of variation

$$\frac{d\xi_i}{d\tau} = \Phi \sum \frac{dX_i}{dx_k} \xi_k + X_i \sum \frac{d\Phi}{dx_k} \xi_k. \tag{2a}$$

Equations (1a) admit a periodic solution

$$x_i = \varphi_i'(\tau)$$

corresponding to

$$x_i = \varphi_i(t)$$

and whose period is equal to /20

$$T' = \int_0^T \frac{dt}{\Phi}.$$

We must replace $x_i$ by $\varphi_i(t)$ in $\Phi$ before integration.

In order to solve equations (2a), we will take account of the value of $\frac{dt}{d\tau}$ and we will write them as

$$\frac{d\xi_i}{dt} = \sum \frac{dX_i}{dx_k} \xi_k + X_i \sum \frac{\frac{d\Phi}{dx_k} \xi_k}{\Phi}.$$

Let us then set

$$\xi_i = \eta_i + X_i \lambda,$$

it follows that

$$\frac{d\eta_i}{dt} + X_i \frac{d\lambda}{dt} + \sum \lambda \frac{dX_i}{dx_k} X_k = \sum \frac{dX_i}{dx_k} \eta_k + \sum \lambda \frac{dX_i}{dx_k} X_k + \frac{X_i}{\Phi} \sum \frac{d\Phi}{dx_k} \eta_k + \frac{X_i \lambda}{\Phi} \sum \frac{d\Phi}{dx_k} X_k,$$

which shows that we can satisfy equations (2b) by taking

$$\eta_i = e^{\alpha t} \psi_i(t) \quad \text{and} \quad \Phi \frac{d\lambda}{dt} = \lambda \sum \frac{d\Phi}{dx_k} X_k + \sum \frac{d\Phi}{dx_k} e^{\alpha t} \psi_k(t).$$

From this we can conclude that

$$\lambda = e^{\alpha t} \theta(t)$$

and that

$$\xi_i = e^{\alpha t}\theta_i(t),$$

$\theta(t)$ and $\theta_i(t)$ being periodic in t. It will then be necessary to replace it by its values obtained from the equation

$$\frac{dt}{d\tau} = \Phi[\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)].$$

We thus find

$$t = \frac{T}{T'}\tau + f(\tau),$$

$f(\tau)$ being a periodic function of $\tau$; we therefore have

$$\xi_i = e^{\frac{\alpha T}{T'}\tau} e^{\alpha f(\tau)}\theta_i\left[\frac{T}{T'}\tau + f(\tau)\right],$$

which shows that after the change of variables the new characteristic exponents /201
are equal to the old, multiplied by $\frac{T}{T'}$.

Development of Exponents. Calculation of the First Terms

74. Let us return to the equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots, n) \tag{1}$$

of article 13 with the hypotheses of that article.

Let us set

$$n_i = -\frac{dF_0}{dx_i}.$$

For $\mu=0$, $x_i$ are constants and we have, on the other hand,

$$y_i = n_i t + \varpi_i,$$

values $\varpi_i$ being constants.

Let $n_1^0, n_2^0, \ldots, n_n^0$ be values of $n_i$ such that the quantities $n_i^0 T$ are multiples of $2\pi$. Let $x_i^0$ be values of $x_i$ such that

$$n_i = n_i^0.$$

We have seen in articles 42 and 44 that equations (1) will admit a periodic solution of period T which will be developable in terms of the powers of $\mu$, and which for $\mu=0$ will reduce to

$$x_i = x_i^0, \qquad y_i = n_i^0 t + \varpi_i^0,$$

$\varpi_i^0$ being certain particular values of the constants $\varpi_i$. This granted, let us imagine an arbitrary solution.

Let $x_i^0 + \beta_i$ be the initial value of $x_i$ and $\varpi_i$ that of $y_i$ for t=0. Let $\Delta x_i$ be the increase which $x_i$ undergoes and $n_i^0 T + \Delta y_i$ the increase which $y_i$ undergoes when t passes from the value 0 to the value T.

This is how we will form the equation which gives the characteristic exponents. We will construct a determinant of which the elements will be given by the following table. In this table, the first column indicates the row number, the second indicates the column number and the third introduces the corresponding element of the determinant. /20

| Row No. | | Column No. | | Expression of element |
|---|---|---|---|---|
| $i$ | $(i \leqq n)$ | $k$ | $(k \leqq n,\ k \lessgtr i)$ | $\dfrac{d\Delta x_k}{d\beta_i}$ |
| $i$ | $(i \leqq n)$ | $k = i$ | | $\dfrac{d\Delta x_i}{d\beta_i} - S$ |
| $i+n$ | $(i > 0)$ | $k$ | $(k \leqq n)$ | $\dfrac{d\Delta x_k}{d\varpi_i}$ |
| $i$ | $(i \leqq n)$ | $k+n$ | $(k > 0)$ | $\dfrac{d\Delta y_k}{d\beta_i}$ |
| $i+n$ | $(i > 0)$ | $k+n$ | $(k > 0,\ k \lessgtr i)$ | $\dfrac{d\Delta y_k}{d\varpi_i}$ |
| $i+n$ | $(i > 0)$ | $k+n = i+n$ | | $\dfrac{d\Delta y_i}{d\varpi_i} - S$[1]. |

(2)

Equating the determinant thus formed to zero, we have an equation in S whose roots are

$$e^{\alpha T} - 1,$$

$\alpha$ being one of the characteristic exponents.

Values $\Delta x_i$ and $\Delta y_i$ can be developed as power series in $\mu$, $\beta_i$ and $\varpi_i - \varpi_i^0$. The same is true for the quantities

$$\frac{d\Delta x_k}{d\beta_i}, \ \frac{d\Delta x_k}{d\varpi_i}, \ \frac{d\Delta y_k}{d\beta_i}, \ \frac{d\Delta y_i}{d\varpi_i} \ . \qquad (3)$$

[1]It is thus, for example, that the first element of the k-th column will be equal to $\dfrac{d\Delta x_k}{d\beta_i}$ provided that $i \leqq n,\ k \leqq n,\ k \gtrless i$.

We must here replace $\beta_i$ and $\varpi_i$ by the values which correspond to the periodic solution and which can be developed as a power series in $\mu$, such that after this substitution the quantities (3) will be developed as a power series of $\mu$.

Since, on the other hand, we have

$$S = e^{\alpha T} - 1,$$

we see that our determinant is an integral function of $\alpha$, developable in addition as a power series of $\mu$. I will call this function $G(\alpha, \mu)$ and I will have, in order to determine $\alpha$ as a function of $\mu$, the equation /203

$$G(\alpha, \mu) = 0. \quad (4)$$

This granted, let us set

$$\alpha = \varepsilon\sqrt{\mu}.$$

Let us divide the n first lines of the determinant, as well as the last n columns by $\sqrt{\mu}$. The elements of the determinant will become, by writing them in the same order as in table (2),

$$\frac{d\Delta x_k}{\sqrt{\mu}\,d\beta_i},\quad \frac{d\Delta x_i}{\sqrt{\mu}\,d\beta_i} - \frac{S}{\sqrt{\mu}},\quad \frac{d\Delta x_k}{\mu\,d\varpi_i},\quad \frac{d\Delta y_k}{d\beta_i},\quad \frac{d\Delta y_k}{\sqrt{\mu}\,d\varpi_i},\quad \frac{d\Delta y_i}{\sqrt{\mu}\,d\varpi_i} - \frac{S}{\sqrt{\mu}},$$

and equation (4) becomes

$$\mu^{-n}G(\varepsilon\sqrt{\mu}, \mu) = G_1(\varepsilon, \sqrt{\mu}) = 0.$$

Let us determine what this equation becomes for $\mu=0$ or, in other words, let us form the determinant $G_1(\varepsilon, 0)$.

For $\mu=0$, $\Delta x_k$ is zero, and $\Delta y_k$ depends only on $\beta_i$. Therefore $\frac{d\Delta x_k}{d\beta_i}$, $\frac{d\Delta x_k}{d\varpi_i}$ and $\frac{d\Delta y_k}{d\varpi_i}$ are divisible by $\mu$. We therefore have

$$\lim \frac{d\Delta x_k}{\sqrt{\mu}\,d\beta_i} = \lim \frac{d\Delta y_k}{\sqrt{\mu}\,d\varpi_i} = 0 \quad \text{for} \quad \mu = 0.$$

On the other hand,

$$\lim \frac{S}{\sqrt{\mu}} = \lim \frac{e^{T\varepsilon\sqrt{\mu}} - 1}{\sqrt{\mu}} = \varepsilon T.$$

It then follows that (for $\mu=0$)

$$\Delta y_k = -\int_0^T \frac{dF_0}{dx_k}\,dt = -T\frac{dF_0}{dx_k}.$$

In $\frac{dF_0}{dx_k}$, $x_i$ must be replaced by $x_i^0+\beta_i$. We therefore have

$$\frac{d\Delta y_k}{d\beta_i} = -T\frac{d^2F_0}{dx_i dx_k}.$$

In $\frac{d^2F_0}{dx_i dx_k}$ one must after differentiation make $\beta_i=0$, i.e., $x_i=x_i^0$. /20

We have (still for $\mu=0$)

$$\frac{1}{\mu}\Delta x_k = \int_0^T \frac{dF_1}{dy_k}\,dt.$$

In $\frac{dF_1}{dy_k}$, we replace $x_i$ by $x_i^0$ and $y_i$ by $n_i t+\varpi_i$, which shows first that

$$\frac{dF_1}{dy_k} = \frac{dF_1}{d\varpi_k}.$$

As we propose to differentiate $\Delta x_k$ with respect to the $\varpi_i$, but not with respect to $\beta_i$, we can immediately give $\beta_i$ their final values and make

$$\beta_i = 0, \text{ whence } n = n_i^0.$$

Then $F_1$ becomes a periodic function of period T with respect to t and of period $2\pi$ with respect to the $\varpi_i$. Let

$$[F_1] = R$$

be the mean value of $F_1$, considered a periodic function of t; it follows that

$$\frac{\Delta x_k}{\mu} = T\frac{dR}{d\varpi_k},$$

from which

$$\frac{d\Delta x_k}{\mu\, d\varpi_i} = T\frac{d^2R}{d\varpi_i d\varpi_k}.$$

Thus the elements of the determinant $G_1(\epsilon, 0)$ will be, writing them in the same order as in table (2),

$$0,\quad -\epsilon T,\quad T\frac{d^2R}{d\varpi_i d\varpi_k},\quad -T\frac{d^2F_0}{dx_i dx_k},\quad 0,\quad -\epsilon T.$$

We thus have an algebraic equation in $\epsilon$; in general, this equation will have two zero roots and all others will be distinct and different from 0. Applying the theorem from article 30, we will see that we can conclude from the equation /2

$$G_1(\epsilon, \sqrt{\mu}) = 0$$

the $\epsilon$ (and consequently $\alpha$) are power series in $\sqrt{\mu}$.

We are therefore led to discuss the equation

$$G_1(\epsilon, 0) = 0.$$

If we change $\epsilon$ to $-\epsilon$, this equation does not change.

In fact, if we multiply the n first rows by -1, as well as the last n columns, the determinant will not change, and none of the elements of the determinant will change either, with the exception of those elements on the principal diagonal which were equal to $-\epsilon T$ and which became equal to $+\epsilon T$.

I say that the equation has two zero roots. If, in fact, we make $\epsilon=0$, the determinant becomes equal to the product of two others, namely:

(1) the Hessian of $-TF_0$ with respect to $x_i$;

(2) the Hessian of TR with respect to the $\varpi_i$.

This last Hessian is zero, for we have, according to the defintion of R,

$$n_1^0 \frac{d^2 R}{d\varpi_i d\varpi_1} + n_2^0 \frac{d^2 R}{d\varpi_i d\varpi_2} + \dots + n_n^0 \frac{d^2 R}{d\varpi_i d\varpi_n} = 0.$$

Therefore the equation is satisfied for $\epsilon=0$ and, as its roots are equal in pairs and of opposite sign, it must have two zero roots.

In order for there to be more than two zero roots, it would be necessary for the coefficient of $\epsilon^2$ in $G_1$ to be zero. This coefficient can now be calculated as follows.

Let us multiply the first row of $G_1$ by $n_1^0$ and let us add to it the second multiplied by $n_2^0$, the third by $n_3^0$, ..., the n-th by $n_n^0$. All elements of $G_1$ remain unaltered with the exception of those of the first row, which become

$$-n_1^0 \epsilon T, \ -n_2^0 \epsilon T, \ -n_3^0 \epsilon T, \ \dots, \ -n_n^0 \epsilon T, \ 0, \ 0, \ \dots, \ 0.$$

Let us now multiply the (n+1)-th column by $n_1^0$ and let us add to it the (n+2)-th multipled by $n_2^0$, the (n+3)-th by $n_3^0$, ..., the 2n-th by $n_n^0$. /206

All elements remain unaltered except those of the (n+1)-th column, which become

$$0, \ 0, \ \dots, \ 0, \ -n_1^0 \epsilon T, \ -n_2^0 \epsilon T, \ \dots, \ -n_n^0 \epsilon T.$$

The determinant $G_1$, by this double operation, has been multipled by $(n_1^0)^2$. Let us divide it now by $\epsilon^2$, dividing by $\epsilon$ the first line on one hand and the (n+1)-th column on the other.

Let us then make $\epsilon=0$ and we will have the desired coefficient.

The determinant thus obtained has its elements conforming to the following table:

| Column No. | Row No. | Value of element |
|---|---|---|
| $i \ (i<n)$ | $1$ | $-n_i^0 T$ |
| $n+1$ | $k \ (k<n)$ | $0$ |
| $i+n \ (i>1)$ | $k \ (k>1, k<n)$ | $T \dfrac{d^2 R}{d\varpi_i d\varpi_k}$ |
| $i \ (i<n)$ | $k \ (k>1, k<n)$ | $0$ |
| $i+n \ (i>1)$ | $1$ | $0$ |
| $i \ (i<n)$ | $k+n \ (k>0)$ | $-T \dfrac{d^2 F_0}{dx_i dx_k}$ |
| $n+1$ | $k+n \ (k>0)$ | $-n_k^0 T$ |
| $i+n \ (i>1)$ | $k+n \ (k>0)$ | $0$ |

We see that this determinant is equal to within its sign

$$T^{2n} H_1 H_2.$$

Values $H_i$ and $H_2$ being the two following determinants

$$H_1 = \begin{vmatrix} n_1^0 & n_2^0 & \cdots & n_n^0 & 0 \\ \dfrac{d^2 F_0}{dx_1^2} & \dfrac{d^2 F_0}{dx_1 dx_2} & \cdots & \dfrac{d^2 F_0}{dx_1 dx_n} & n_1^0 \\ \dfrac{d^2 F_0}{dx_1 dx_2} & \dfrac{d^2 F_0}{dx_2^2} & \cdots & \dfrac{d^2 F_0}{dx_2 dx_n} & n_2^0 \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots \\ \dfrac{d^2 F_0}{dx_1 dx_n} & \cdots\cdots & \cdots & \dfrac{d^2 F_0}{dx_n^2} & n_n^0 \end{vmatrix}$$

and $H_2$ being the Hessian of R with respect to

$$\varpi_2, \ \varpi_3, \ \ldots, \ \varpi_n.$$

If I observe $n_i^0$ is equal, to within its sign, to $\dfrac{df_0}{dx_i}$, I can see that we do not change $H_1$ by replacing $n_i$ by $\dfrac{dF_0}{dx_i}$ in the first line and the last column.

The determinant thus formed will be called the bordered Hessian of $F_0$ with respect to $x_1, x_2, \ldots, x_n$.

Thus the equation $G_1(\epsilon, 0)=0$ can have no more than two zero roots, and consequently, there cannot be any more than two characteristic zero exponents only if $H_1$ or $H_2$ vanishes.

In the particular case of the Problem of Three Bodies which we dealt with in article 9, there are only 2 degrees of freedom, and we have

$$F_0 = \frac{1}{2x_1^2} + x_2.$$

It then follows that

$$H_1 = \begin{vmatrix} \frac{dF_0}{dx_1} & \frac{dF_0}{dx_2} & 0 \\ \frac{d^2F_0}{dx_1^2} & \frac{d^2F_0}{dx_1 dx_2} & \frac{dF_0}{dx_1} \\ \frac{d^2F_0}{dx_1 dx_2} & \frac{d^2F_0}{dx_2^2} & \frac{dF_0}{dx_2} \end{vmatrix} = \begin{vmatrix} -x_1^{-3} & 1 & 0 \\ 3x_1^{-4} & 0 & -x_1^{-3} \\ 0 & 0 & 1 \end{vmatrix} = -3x_1^{-4};$$

therefore $H_1$ is not zero; on the other hand, it is verified that $H_2 = \frac{d_2R}{d\omega_2^2}$ is not zero either.

Therefore, in this particular case of the Problem of Three Bodies, there are two characteristic zero exponents and the other two are different from 0.

75. The determinant $G_1$ can be somewhat simplified by a proper choice of variables. I say that one can always assume

$$n_2^0 = n_3^0 = \ldots = n_n^0 = 0. \tag{1}$$

In fact, if this were not so, we would change variables by taking $x_i'$ and $y_i'$ as new variables and setting

$$y_i' = \alpha_{1,i} y_1 + \alpha_{2,i} y_2 + \ldots + \alpha_{n,i} y_n,$$
$$x_i = \alpha_{i,1} x_1' + \alpha_{i,2} x_2' + \ldots + \alpha_{i,n} x_n',$$

values $\alpha_{i,k}$ being constant coefficients. After this linear change of variables, the equations will be in canonical form.

After this change of variables, the quantities which will correspond to $n_1^0$, $n_2^0$, ..., $n_n^0$, and which we will call $n_i'^0$, $n_2'^0$, ..., $n_n'^0$, will be given by the relations /208

$$n_i'^0 = \alpha_{1,i} n_1^0 + \alpha_{2,i} n_2^0 + \ldots + \alpha_{n,i} n_n^0,$$

for

$$n_i'^0 = -\frac{dF_0}{dx_i'}, \quad n_i^0 = -\frac{dF_0}{dx_i}, \quad \frac{dF_0}{dx_i'} = \sum_k \frac{dF_0}{dx_k}\frac{dx_k}{dx_i'} = \sum \frac{dF_0}{dx_k}\alpha_{k,i}.$$

As the numbers $n_1^0, n_2^0, \ldots, n_n^0$ are commensurable to each other, we will always be able to choose values $\alpha_{i,k}$ in such a manner:

(1) that $\alpha_{i,k}$ are integrals;

(2) that their determinant is equal to 1. These two conditions are necessary in order for F to remain periodic with respect to y', as it was with respect to the y;

(3) that

$$n_2'^0 = n_3'^0 = \ldots = n_n'^0 = 0.$$

Thus we can always assume that conditions (1) are fulfilled and from this we deduce the following equations

$$\frac{d^2 R}{d\varpi_1 d\varpi_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{2}$$

76. An interesting particular case is that where one or several of the variables $x_i$ does not enter into $F_0$. Let us assume, for example, that $F_0$ does not depend on $x_n$. Then all elements of the n-th column (and those of the 2n-th row) are all zero, except the one among them which belongs to the principal diagonal and which remains equal to $\epsilon T$.

I will, in addition, assume that the variables have been chosen such that conditions (1) and (2) from the preceding article are fulfilled. The result is that the elements of the first row (and those of the (n+i)-th column) are all zero, with the exception of the one among them which belongs to the principal diagonal and which remains equal to $-\epsilon T$.

Thus all elements of rows 1 and 2n and all those from columns n and n+1 are divisible by $\epsilon$ (I will add that each element which belongs at the same time to one of these two rows and to one of these two columns is zero and consequently divisible by $\epsilon^2$); the result is that the determinant is divisible by $\epsilon^4$, and, consequently, that the equation $G_1=0$ has four zero roots. /209

In what case can it have more than four?

In order for us to ascertain this, let us divide rows 1 and 2n and columns n and n+1 by $\epsilon$, and let us then make $\epsilon=0$. In what case will the determinant thus obtained and which will be equal to

$$\lim \frac{G_1}{\epsilon^4} \text{ for } \epsilon = 0$$

be zero?

We can just as well divide the determinant $G_1$ by $\epsilon^4 T^4$, suppressing lines 1, n, n+1 and 2n and the columns from the same article. If we then set $\epsilon=0$, we see that all elements are zero except those which belong to one of the n-2 last remaining columns, and to one of the n-2 first rows, or inversely to one of the n-2 first columns and to one of the last n-2 rows.

Thus the determinant is equal, within a power of T, to the product of two Hessians, namely:

(1) the Hessian of $F_0$ with respect to $x_2, x_3, \ldots, x_{n-1}$;

(2) and the Hessian of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-1}$.

If neither of these two Hessians is zero, the equation $G_1=0$ will not have more than four zero roots and there will certainly be not more than four characteristic exponents which are zero.

What does this condition become when one assumes that the variables are arbitrary and that conditions (1) and (2) from the preceding article are not fulfilled?

In this case, we will submit the determinant to the same transformation as at the end of article 74; we will then see, as at the end of this article, that after this transformation, the elements of the first row become equal to

$$-n_1^0\varepsilon T, \quad -n_2^0\varepsilon T, \quad \ldots, \quad -n_n^0\varepsilon T, \quad 0, \quad 0, \quad \ldots, \quad 0$$

and those of the (n+1)-th column to

$$0, \quad 0, \quad \ldots, \quad 0, \quad -n_1^0\varepsilon T, \quad -n_2^0\varepsilon T, \quad \ldots, \quad -n_n^0\varepsilon T.$$

It is only important to observe here that $n_n^0$ is zero, since

$$\frac{dF_0}{dx_n} = 0.$$

We will be able to divide this determinant by $\epsilon^4 T^4$, suppressing rows n and 2n and the columns from the same numbers, and dividing the elements of the first row and the (n+1)-th column by $\epsilon T$. /210

If we then set $\epsilon=0$, we see that the determinant reduces to the product of two others, namely:

(1) the bordered Hessian of $F_0$ with respect to $x_1, x_2, \ldots, x_{n-1}$;

(2) the Hessian of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-1}$.

In order for there to be more than four characteristic exponents, it is necessary (but not sufficient) that one of these two **Hessians be zero.**

Let us assume that $F_0$ not only does not contain $x_n$, but does not contain $x_{n-1}$ either; reasoning in this manner, we would arrive at the following result:

The equation $G_1(\epsilon, 0)$ has still six zero roots; in order for it to have more, it is necessary and sufficient that the bordered Hessian of $F_0$ with respect to $\varpi_1, \varpi_3, \ldots, \varpi_{n-2}$ be zero. This condition is therefore necessary (but not sufficient) for there to be more than six characteristic zero exponents.

77. Let us again take the hypotheses made at the beginning of article 76, namely that $F_0$ does not depend on $x_n$ and that conditions (1) and (2) from article 75 are fulfilled.

We have seen that the equation

$$G_1(\varepsilon, 0) = 0$$

then admits four and only four zero roots, and from this we have concluded that there can be no more than four zero exponents. On the contrary, it is not permitted to conclude from this that there are four characteristic zero exponents; this proves only that, when one develops these exponents as power series of $\mu$, the first term of the development is zero for four of them.

It remains for us to see if the following terms of the development are also zero.

I know that two exponents are zero, since time does not enter explicitly into the differential equations, and that F=const. is an integral. I propose to find out what happens to the other two, and to do so, I am going to calculate /21
the coefficient of $\mu$ in their development.

I am going to set

$$\alpha = \eta\mu, \text{ from which } \varepsilon = \eta\sqrt{\mu}.$$

I will divide the equation

$$G(\alpha, \mu) = G(\eta\mu, \mu) = 0$$

by a suitable power of $\mu$ and I will then make $\mu$=0, and I will have an equation which will give me the values of $\eta$.

From the fact that $F_0$ does not depend on $x_n$ we can conclude that the quantities which we called $n_i$ and which are equal to $-\frac{dF_0}{dx_i}$ neither depend on $x_n$ nor consequently on $\beta_n$.

We will therefore have $n_i = n_i^0$, not only as in article 74, when all values $\beta$ are zero, but even if $\beta_n$ are not zero, provided that the other $\beta$ are.

If, therefore, we assume

$$\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0, \qquad \beta_n \gtrless 0,$$

we will still have

$$\frac{\Delta x_k}{\mu} = T \frac{dR}{d\varpi_k}.$$

This permits us to differentiate this identity with respect to $\beta_n$ and to write

$$\frac{d\Delta x_k}{\mu\, d\beta_n} = T \frac{d^2 R}{d\varpi_k d\beta_n}.$$

Let us now calculate

$$\frac{d\Delta y_n}{\mu\, d\varpi_k} \text{ and } \frac{d\Delta y_n}{\mu\, d\beta_n}.$$

It follows that

$$\Delta y_n = -\int_0^T \frac{dF}{dx_n}\, dt$$

where, since $\frac{dF_0}{dx_n} = 0$, we will have for $\mu = 0$,

$$\frac{\Delta y_n}{\mu} = -\int_0^T \frac{dF_1}{dx_n}\, dt = -T \frac{dR}{d\beta_n}.$$

This identity holds provided that /212

$$\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0.$$

We can therefore differentiate it with respect to $\varpi_k$ or to $\beta_n$, which gives us

$$\frac{d\Delta y_n}{\mu\, d\varpi_k} = -T \frac{d^2 R}{d\beta_n d\varpi_k}, \qquad \frac{d\Delta y_n}{\mu\, d\beta_n} = -T \frac{d^2 R}{d\beta_n^2}. \tag{3}$$

Concerning the quantities

$$\frac{d\Delta y_n}{d\beta_k}, \quad \frac{d\Delta y_k}{d\beta_n},$$

it is sufficient for us to observe that they are divisible by $\mu$.

We have yet to examine the elements of the first row of our determinant and those of the $(n+1)$-th column.

The elements of the first row are equal to

$$1+\frac{d\Delta x_1}{d\beta_1}-e^{\eta\mu T},\ \frac{d\Delta x_1}{d\beta_2},\ \frac{d\Delta x_1}{d\beta_3},\ \dots,\ \frac{d\Delta x_1}{d\beta_{n-1}},\ \frac{d\Delta x_1}{d\beta_n},\ \frac{d\Delta x_1}{d\varpi_1},\ \dots,\ \frac{d\Delta x_1}{d\varpi_n}.$$

They are all divisible by $\mu$, but I say that the n+1 last elements, i.e.,

$$\frac{d\Delta x_1}{d\beta_n} \text{ and } \frac{d\Delta x_1}{d\varpi_k},$$

are divisible by $\mu^2$. In fact, we have found for $\mu=0$

$$\frac{d\Delta x_1}{\mu\, d\beta_n}=T\frac{d^2 R}{d\varpi_1\, d\beta_n},\qquad \frac{d\Delta x_1}{\mu\, d\varpi_k}=T\frac{d^2 R}{d\varpi_1\, d\varpi_k}.$$

Now, by virtue of the definition of R, we have

$$n_1^0\frac{dR}{d\varpi_1}+n_2^0\frac{dR}{d\varpi_2}+\dots+n_n^0\frac{dR}{d\varpi_n}=0,$$

or, because of relations (1) from article 75,

$$\frac{dR}{d\varpi_1}=0,$$

from which, (differentiating this identity)

$$\frac{d\Delta x_1}{\mu\, d\beta_n}=\frac{d\Delta x_1}{\mu\, d\varpi_k}=0$$

for $\mu=0$. Q. E. D.

The elements of the (n+1)-th column are written /2

$$\frac{d\Delta x_1}{d\varpi_1},\ \frac{d\Delta x_2}{d\varpi_1},\ \dots,\ \frac{d\Delta x_n}{d\varpi_1},\ \frac{d\Delta y_1}{d\varpi_1}+1-e^{\eta\mu T},\ \frac{d\Delta y_2}{d\varpi_1},\ \dots,\ \frac{d\Delta y_n}{d\varpi_1}.$$

All these elements are divisible by $\mu$; but I say that the first n and the last are divisible by $\mu^2$, or, what goes back to the same thing, that

$$\frac{d\Delta x_k}{\mu\, d\varpi_1}=\frac{d\Delta y_n}{\mu\, d\varpi_1}=0 \text{ for } \mu=0.$$

In fact, we have found

$$\frac{d\Delta x_k}{\mu\, d\varpi_1}=T\frac{d^2 R}{d\varpi_1\, d\varpi_k},\qquad \frac{d\Delta y_n}{\mu\, d\varpi_1}=-T\frac{d^2 R}{d\varpi_1\, d\beta_n},$$

and

$$\frac{dR}{d\varpi_1}=0,$$

from which, by differentiation,

$$\frac{d^2 R}{d\varpi_i \, d\varpi_k} = \frac{d^2 R}{d\varpi_i \, d\beta_n} = 0.$$

Q. E. D.

This granted, in our determinant $G(\eta, \mu, \mu)$ I divide each element by T; I then divide:

the first row by $\mu$, lines 2, 3, 4, ..., n , 2n by $\sqrt{\mu}$; and

the (n+1)-th column by $\mu$, columns n, n+2, n+3, ..., 2n by $\sqrt{\mu}$.

The determinant is finally divided by $T^{2n}\mu^{n+2}$.

I then make $\mu=0$.

I observe that the following elements are zero:

| Row to which the element belongs | Column to which the element belongs | Power of $\mu$ by which the element was divisible | Power of $\mu$ by which the element was divided | |
|---|---|---|---|---|
| 2 to n incl. and 2n | 1 to n-1 incl. | $\mu$ | $\sqrt{\mu}$ | (4) |
| 1 | n and n+2 to 2n incl. | $\mu^2$ | $\mu\sqrt{\mu}$ | |
| 2 to n incl. and 2n | n+1 | $\mu^2$ | $\mu\sqrt{\mu}$ | |
| n+1 to 2n-1 incl. | n and n+2 to 2n incl. | $\mu$ | $\sqrt{\mu}$ | |

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and that the following elements are finite:

| | | | | |
|---|---|---|---|---|
| n+1 to 2n-1 incl. | 1 to n-1 incl. | power 0 | power 0 | |
| 1 | 1 to n-1 incl. | $\mu$ | $\mu$ | (4a) |
| 2 to n incl. and 2n | n and n+2 to 2n incl. | $\mu$ | $\mu$ | |
| 1 | n+1 | $\mu^2$ | $\mu^2$ | |
| n+1 to 2n-1 incl. | n+1 | $\mu$ | $\mu$ | |

The only elements which are finite therefore belong to rows 1 and n+1 to 2n-1 incl. and to columns 1 to n-1 incl. and n+1 or to rows 1 to n incl. and 2n and to columns n and n+2 incl.

Our determinant therefore becomes equal to the product of two others which I will call $D_1$ and $D_2$.

The determinant $D_1$ will be obtained by suppressing the rows

$$1, \; n+1, \; n+2, \; \ldots, \; 2n-1,$$

and the columns

$$1, \; 2, \; 3, \; \ldots, \; n-1, \; n+1.$$

The determinant $D_2$ will be obtained by suppressing the rows

$$2, \; 3, \; 4, \; \ldots, \; n, \; 2n,$$

and the columns

$$n, \; n+2, \; n+3, \; \ldots, \; 2n.$$

Let us see how these determinants will depend on $\eta$. To do so I will remark that

$$\eta = \lim \frac{S}{\mu T} \text{ (for } \mu=0)$$

only enters into the terms of the principal diagonal; now the determinant $D_1$ contains two of these terms one belonging to the column and to row n, the other belonging to the column and to row 2n.

The determinant $D_2$ also contains two of these terms, one belonging to the column and to row 1, and the other belonging to the column and row n+1.

The result of this is that $D_1$ and $D_2$ are polynomials of the second degree in $\eta$. Thus our equation in $\eta$ breaks down into two second-degree equations,

$$D_1 = 0, \qquad D_2 = 0.$$

Let us first examine the equation $D_1=0$. /2

In order to form the determinant $D_1$, we can apply the following rule:

Write the Hessian of R with respect to

$$\varpi_2, \; \varpi_3, \; \ldots, \; \varpi_n, \; \beta_n;$$

change the signs of the last row which contains the derivatives of $\frac{dR}{d\beta_n}$; then add $-\eta$ to the two elements which are equal to $\frac{d^2R}{d\varpi_n d\beta_n}$ and to $-\frac{d^2R}{d\varpi_n d\beta_n}$.

We obtain the same equation more simply (the first member being only changed in sign) by taking the Hessian of R and adding $-\eta$ to one of the two elements which are equal to $\frac{d^2R}{d\varpi_n d\beta_n}$ and $+\eta$ to the other. Let us write the equation $D_1=0$ assuming

$n=4$ in order to fix the ideas:

$$
\begin{vmatrix}
\frac{d^2R}{d\varpi_2^2} & \frac{d^2R}{d\varpi_2 d\varpi_3} & \frac{d^2R}{d\varpi_2 d\varpi_4} & \frac{d^2R}{d\varpi_2 d\beta_4} \\
\frac{d^2R}{d\varpi_2 d\varpi_4} & \frac{d^2R}{d\varpi_3^2} & \frac{d^2R}{d\varpi_3 d\varpi_4} & \frac{d^2R}{d\varpi_3 d\beta_4} \\
\frac{d^2R}{d\varpi_2 d\beta_4} & \frac{d^2R}{d\varpi_3 d\varpi_4} & \frac{d^2R}{d\varpi_4^2} & \frac{d^2R}{d\varpi_4 d\beta_4} + \eta \\
\frac{d^2R}{d\varpi_2 d\varpi_3} & \frac{d^2R}{d\varpi_3 d\beta_4} & \frac{d^2R}{d\varpi_4 d\beta_4} - \eta & \frac{d^2R}{d\beta_4^2}
\end{vmatrix} = 0.
$$

In this form one sees immediately what could have been anticipated: that this equation in $\eta$ has its two roots equal and of opposite sign.

These two roots will be finite, if the Hessian of R with respect to

$$\varpi_2, \quad \varpi_3, \quad \varpi_4, \quad \ldots, \quad \varpi_{n-1}$$

is not zero.

They will be different from 0 if the Hessian of R with respect to

$$\varpi_2, \quad \varpi_3, \quad \varpi_4, \quad \ldots, \quad \varpi_{n-1}, \quad \varpi_n, \quad \beta_n$$

is not zero.

As for equation $D_2=0$, it will have its two roots zero. In fact, we know that there are always two characteristic zero exponents and, consequently, that two of the values of $\eta$ are zero. We have just seen that the roots of $D_1=0$ are not zero /216 in general; it is therefore necessary to admit that it is the roots of $D_2=0$ which are always zero.

How would these results be modified, if condition (1) from article 75 were not itself fulfilled?

In this case one would multiply (as we have done in article 74) the first line by $n_1^0$, and add to it the 2-nd, 3-rd, ..., n-th lines, multiplied, respectively, by $n_2^0$, $n_3^0$, ..., $n_n^0$ (I recall in addition that $n_n^0$ is zero); we would then multiply the (n+1)-th column by $n_1^0$, and add to it the n+2-th, n+3-th, ..., 2n-th columns, multiplied, respectively, by $n_2^0$, $n_3^0$, ..., $n_n^0$. After this transformation, all elements of the determinant $G(\eta\mu, \mu)$ remain the same except those of the first row and of the (n+1)-th column.

In addition, each element (those of the first row and of the (n+1)-th column as well as the others) is divisible by the power of $\mu$ indicated in the 3-rd column in tables (4) and (4a).

We will then divide each element by T and by the power of $\mu$ indicated in the 4-th column of the same tables.

When we make $\mu=0$, a certain number of elements will be zero and others will remain finite thus conforming with tables (4) and (4a). Our determinant will still be found equal to the product of the other two $D_1$ and $D_2$, which will be obtained as above.

All elements of these two determinants will have the same expression as in the preceding case, except those of the first row and of the (n+1)-th column. Now $D_1$ contains no element of this row and of this column.

Therefore $D_1$ has the same expression as in the preceding case and the same conclusions hold.

The values of $\eta$ are finite if the Hessian of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-1}$ is not zero, and they are different from 0, if the Hessian of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_n, \beta_n$ is not zero.

Summarizing, if $F_0$ does not depend on $x_n$, if the bordered Hessian of $F_0$ with respect to $x_1, x_2, \ldots, x_{n-1}$ is not zero, if the Hessians of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-1}$, and with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-1}, \varpi_n, \beta_n$ are not zero, there will be only two characteristic zero exponents.

Let us proceed to the case where $F_0$ depends neither on $x_{n-1}$ nor on $x_n$.

We would see by reasoning in the same manner that:

If the bordered Hessian of $F_0$ with respect to $x_1, x_2, \ldots, x_{n-2}$ is not zero, if the Hessians of R with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-2}$ and with respect to $\varpi_2, \varpi_3, \ldots, \varpi_{n-2}, \varpi_{n-1}, \varpi_n, \beta_{n-1}$ and $\beta_n$ are not zero, there will be only two zero exponents.

## Application to the Problem of Three Bodies

78. Let us apply the preceding to the Problem of Three Bodies; we have seen in article 15 and 16 how one can reduce the number of degrees of freedom to 3 in the case of the plane problem and to 4 in the general case.

Let us therefore write the equations of motion in the form which we gave them in these articles 15 and 16.

The two series of conjugate variables are then

$$\begin{matrix} \beta L, & \beta' L', & H, \\ l, & l', & h \end{matrix}$$

in the case of the plane problem, and

$$\begin{matrix} \beta L, & \beta' L', & \beta\Gamma, & \beta'\Gamma', \\ l, & l', & g, & g' \end{matrix}$$

in the general case. We have in addition

$$F_0 = AL^{-2} + A'L'^{-2},$$

A and A' being constant coefficients.

We see therefore that $F_0$ does not depend on H in the case of the plane problem, nor on Γ and on Γ' in the general case.

In the first place, the bordered Hessian of $F_0$ with respect to βL and β'L' is equal to

$$BL^{-4}L'^{-6} + B'L^{-6}L'^{-4},$$

B and B' being constant coefficients. The bordered Hessian is therefore not zero.

The Hessians of R will not always be zero in general, as one can be assured /218
of from some examples; in addition, we will return to this point in greater detail in the following chapter.

Therefore the periodic solutions of the Problem of Three Bodies have two characteristic zero exponents, but they have only two.

### Complete Calculation of Characteristic Exponents

79. Let us again take equations (1) from article 74 by making n=3 in order to fix the ideas:

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, 3). \tag{1}$$

Let us assume that one has found a periodic solution of these equations

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t)$$

and let us propose to determine the characteristic exponents of this solution.

To do so, we will set

$$x_i = \varphi_i(t) + \xi_i, \quad y_i = \psi_i(t) + \eta_i,$$

then we will form the equations of variation (1), which we will write

$$\left\{\begin{aligned} \frac{d\xi_i}{dt} &= \sum_k \frac{d^2F}{dy_i\,dx_k}\xi_k - \sum_k \frac{d^2F}{dy_i\,dy_k}\eta_k \\ \frac{d\eta_i}{dt} &= -\sum_k \frac{d^2F}{dx_i\,dx_k}\xi_k - \sum_k \frac{d^2F}{dx_i\,dy_k}\eta_k \end{aligned}\right\} \quad (i,\ k = 1,\ 2,\ 3), \tag{2}$$

and we will attempt to integrate these equations by setting

$$\xi_i = e^{\alpha t} S_i, \qquad \eta_i = e^{\alpha t} T_i, \tag{3}$$

$S_i$ and $T_i$ being periodic functions of t. We know that there exist in general six particular solutions of this form (linear equations (2) being of the sixth order). But it is important to observe that, in the particular case which concerns us, there are only four particular solutions which retain this form, because two of the characteristic solutions are zero, and there are consequently two particular solutions in a degenerate form.

This granted, let us first assume $\mu=0$; then F reduces to $F_0$ and depends only on $x_1^0$, $x_2^0$, $x_3^0$.

Then equations (2) reduce to

$$\frac{d\xi_i}{dt} = 0, \qquad \frac{d\eta_i}{dt} = -\sum_k \frac{d^2F_0}{dx_i^0\,dx_k^0}\xi_k. \tag{2'}$$

The coefficients of $\xi_k$ in the second equation (2') are constants.

We will take as solutions of equations (2')

$$\xi_1 = \xi_2 = \xi_3 = 0, \qquad \eta_1 = \eta_1^0, \qquad \eta_2 = \eta_2^0, \qquad \eta_3 = \eta_3^0,$$

$\eta_1^0$, $\eta_2^0$, $\eta_3^0$ being three integration constants.

This solution is not the most general, because it contains only three arbitrary constants, but it is the most general among those which we can reduce to form (3). We thus see that, for $\mu=0$, the six characteristic exponents are zero.

Let us no longer assume now that $\mu$ is zero. We shall now attempt to develop $\alpha$, $S_i$ and $T_i$, not as an increasing power series in $\mu$, but in powers of $\sqrt{\mu}$ by writing

$$\begin{aligned} \alpha &= \alpha_1\sqrt{\mu} + \alpha_2\mu + \alpha_3\mu\sqrt{\mu} + \dots \\ S_i &= S_i^0 + S_i^1\sqrt{\mu} + S_i^2\mu + S_i^3\mu\sqrt{\mu} + \dots \\ T_i &= T_i^0 + T_i^1\sqrt{\mu} + T_i^2\mu + T_i^3\mu\sqrt{\mu} + \dots \end{aligned}$$

I first propose to establish that this development is possible.

We saw first in article 74 that the characteristic exponents $\alpha$ can be developed as an increasing power series in $\sqrt{\mu}$.

Let us now demonstrate that $S_i$ and $T_i$ can also be developed in powers of $\mu$.

$S_i$ and $T_i$ are given to us, in fact, by the following equations:

$$\left.\begin{aligned}\frac{dS_i}{dt}+\alpha S_i &= \sum \frac{d^2F}{dy_i\,dx_k}S^k+\sum\frac{d^2F}{dy_i\,dy_k}T_k,\\ \frac{dT_i}{dt}+\alpha T_i &= -\sum \frac{d^2F}{dx_i\,dx_k}S_k+\sum\frac{d^2F}{dx_i\,dy_k}T_k.\end{aligned}\right\} \qquad (2'')$$

Let $\beta_i$ be the initial value of $S_i$ and $\beta'_i$ those of $T_i$; the values of $S_i$ and of $T_i$ /220

for an arbitrary value of t may, according to article 27, develop in powers of $\mu$, $\alpha$, $\beta_i$ and $\beta'_i$. In addition, because of the linear form of the equations, these values will be linear and homogeneous functions of $\beta_i$ and $\beta'_i$.

In order to use notations analogous to those from article 37, let $\beta_i+\psi_i$ be the value of $S_i$ and $\beta'_i+\psi'_i$ that of $T_i$ for t=T. The condition for the solution to be periodic is that we have

$$\psi_i=\psi'_i=0.$$

Values $\psi_i$ and $\psi'_i$ are linear functions of $\beta_i$ and $\beta'_i$; these equations are therefore linear with respect to these quantities. In general, these equations admit no other solution than

$$\beta_i=\beta'_i=0,$$

such that (2") have no other periodic solution than

$$S_i=T_i=0.$$

However, we know that, if one chooses $\alpha$ so as to satisfy $G(\alpha,\mu)=0$, equations (2") admit periodic solutions other than $S_i=T_i=0$. Consequently, the determinant of the linear equations $\psi_i=\psi'_i=0$ is zero. We will therefore be able to extract from these equations the relations

$$\frac{\beta_i}{\beta'_1}\text{ and }\frac{\beta'_i}{\beta'_1}$$

in the form of power series developed in $\alpha$ and $\mu$.

As $\beta'_1$ remains arbitrary, we will agree to take $\beta'_1=1$ such that the initial value of $T_1$ is equal to 1. Values $\beta_i$ and $\beta_i'$ are then developed as a power series in $\alpha$ and $\mu$; but $S_i$ and $T_i$ are, as we have seen, developable as power series in $\alpha$, $\mu$, $\beta_i$ and $\beta_i'$ and, on the other hand, $\alpha$ can be developed as power series in $\sqrt{\mu}$.

Therefore $S_i$ and $T_i$ will be developable as power series in $\sqrt{\mu}$.

Q. E. D.

We will have in particular /2

$$T_1 = T_1^0 + T_1^1\sqrt{\mu} + T_1^2\mu + \dots$$

Since, according to our hypothesis, $\beta_1'$, which is the initial value of $T_1$, must be equal to 1, whatever $\mu$ may be, we will have for t=0

$$T_1^0 = 1, \qquad 0 = T_1^1 = T_1^2 = \dots = T_1^m = \dots$$

Having thus demonstrated the existence of our series, we are going to seek to determine its coefficients.

We have

$$S_i^0 = 0, \qquad T_i^0 = \eta_i^0$$

and

$$\left.\begin{aligned}
&\xi_i = e^{\alpha t}(S_i^0 + S_i^1\sqrt{\mu} + \dots), \qquad \eta_i = e^{\alpha t}(T_i^0 + T_i^1\sqrt{\mu} + \dots),\\
&\frac{d\xi_i}{dt} = e^{\alpha t}\left|\begin{array}{l} \dfrac{dS_i^0}{dt} + \sqrt{\mu}\,\dfrac{dS_i^1}{dt} + \dots \\ + \alpha S_i^0 + \alpha\sqrt{\mu}\,S_i^1 + \dots \end{array}\right|,\\
&\frac{d\eta_i}{dt} = e^{\alpha t}\left|\begin{array}{l} \dfrac{dT_i^0}{dt} + \sqrt{\mu}\,\dfrac{dT_i^1}{dt} + \dots \\ + \alpha T_i^0 + \alpha\sqrt{\mu}\,T_i^1 + \dots \end{array}\right|.
\end{aligned}\right\} \tag{4}$$

We will develop, on the other hand, the second derivatives of F which enter as coefficients in equations (2), writing

$$\left.\begin{aligned}
\frac{d^2F}{dy_i\,dx_k} &= A_{ik}^0 + \mu A_{ik}^1 + \mu^2 A_{ik}^2 + \dots,\\
\frac{d^2F}{dy_i\,dy_k} &= B_{ik}^0 + \mu B_{ik}^1 + \mu^2 B_{ik}^2 + \dots,\\
-\frac{d^2F}{dx_i\,dx_k} &= C_{ik}^0 + \mu C_{ik}^1 + \mu^2 C_{ik}^2 + \dots,\\
-\frac{d^2F}{dx_i\,dy_k} &= D_{ik}^0 + \mu D_{ik}^1 + \mu^2 D_{ik}^2 + \dots.
\end{aligned}\right\} \tag{5}$$

These developments contain only integral powers of $\mu$ and do not possess, as developments (4), terms dependent on $\sqrt{\mu}$.

We will observe that /222

$$\left.\begin{aligned} A^0_{ik} = B^0_{ik} = D^0_{ik} = 0, \\ C^m_{ik} = C^m_{ki}, \quad B^m_{ik} = B^m_{ki}, \quad A^m_{ik} = -D^m_{ki}. \end{aligned}\right\} \qquad (6)$$

We will substitute in equations (2) the values (4) and (5) in place of $\xi$, $\eta$ their derivatives and second derivatives of F. In expressions (4) I assume that $\alpha$ is developed in powers of $\sqrt{\mu}$, except when this quantity $\alpha$ enters into an exponential factor $e^{\alpha t}$.

We will then identify by equating the like powers of $\sqrt{\mu}$ and thus obtain a series of equations which permit determining successively

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \ldots, \quad S_i^0, \quad S_i^1, \quad \ldots, \quad T_i^0, \quad T_i^1, \quad \ldots.$$

I will only write the first of these equations, obtained by equating successively the completely known terms, the terms in $\sqrt{\mu}$, the terms in $\mu$, .... In addition I cause the factor $e^{\alpha t}$, which is found everywhere, to vanish.

Let us first equate the terms in $\sqrt{\mu}$; it follows that

$$\left.\begin{aligned} \frac{dS_i^1}{dt} + \alpha_1 S_i^0 = \Sigma_k A^0_{ik} S^1_k + \Sigma_k B^0_{ik} T^1_k, \\ \frac{dT_i^1}{dt} + \alpha_1 T_i^0 = \Sigma_k C^0_{ik} S^1_k + \Sigma_k D^0_{ik} T^1_k. \end{aligned}\right\} \qquad (7)$$

Let us equate the terms in $\mu$; it follows that

$$\left.\begin{aligned} & \frac{dS_i^2}{dt} + \alpha_1 S_i^1 + \alpha_2 S_i^0 \\ & = \Sigma_k (A^0_{ik} S^2_k + A^2_{ik} S^0_k + B^0_{ik} T^2_k + B^2_{ik} T^0_k) \qquad (i = 1, 2, 3), \end{aligned}\right\} \qquad (8)$$

in addition to three analogous equations giving $\frac{dT_i^2}{dt}$.

If one now takes account of relations (6), equations (7) become

$$\frac{dS_i^1}{dt} = 0, \qquad \frac{dT_i^1}{dt} + \sigma_1 \eta_i^0 = \Sigma_k C^0_{ik} S^1_k.$$

The first of these equations shows that $S^1_1$, $S^1_2$ and $S^1_3$ are constants. As for the second, it shows that $\frac{dT^1_i}{dt}$ is a constant; but as $T^1_i$ must be a periodic function, this constant must be zero, such that one has /223

$$\alpha_1 \eta_i^0 = C_{i1}^0 S_1^1 + C_{i2}^0 S_2^1 + C_{i3}^0 S_3^1, \tag{9}$$

which establishes three relations among the three constants $\eta_i^1$, the three constants $S_i^1$ and the unknown quantity $\alpha_1$.

For its part, equation (8) will be written

$$\frac{dS_i^2}{dt} + \alpha_1 S_i^1 = \Sigma_k B_{ik}^2 \eta_k^0.$$

The $B^2_{ik}$ are periodic functions of t; let us develop them according to the formula of Fourier and let $b_{ik}$ be the completely known term of $B^2_{ik}$.

It will follow that

$$\alpha_1 S_i^1 = \Sigma_k b_{ik} \eta_i^0$$

or, considering equations (9),

$$\alpha_1^2 S_i^1 = \sum_{k=1}^{k=3} b_{ik}(C_{k1}^0 S_1^1 + C_{k2}^0 S_2^1 + C_{k3}^0 S_3^1). \tag{10}$$

Setting i=1, 2 and 3 in this equation (10), we will have three linear and homogeneous relations among the three constants $S_i^1$. Eliminating these three constants, we will then have an equation of the third degree which will determine $\alpha_1^2$.

If, for brevity, we set

$$e_{ik} = b_{i1} C_{1k}^0 + b_{i2} C_{2k}^0 + b_{i3} C_{3k}^0,$$

the equation due to this elimination will be written

$$\begin{vmatrix} e_{11} - \alpha_1^2 & e_{12} & e_{13} \\ e_{21} & e_{22} - \alpha_1^2 & e_{23} \\ e_{31} & e_{32} & e_{33} - \alpha_1^2 \end{vmatrix} = 0. \tag{11}$$

It can also be written

$$\begin{vmatrix} -\alpha_1 & 0 & 0 & C_{11}^0 & C_{12}^0 & C_{13}^0 \\ 0 & -\alpha_1 & 0 & C_{21}^0 & C_{22}^0 & C_{23}^0 \\ 0 & 0 & -\alpha_1 & C_{31}^0 & C_{32}^0 & C_{33}^0 \\ b_{11} & b_{12} & b_{13} & -\alpha_1 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & -\alpha_1 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & -\alpha_1 \end{vmatrix} = 0.$$

The determination of $\alpha_1$ is the only part of the calculation which presents some difficulty. /224

The equations analogous to (7) and (8), formed by equating in equations (2) the coefficients of the like powers of $\sqrt{\mu}$, then permits without difficulty the determination of $\alpha_k$, $S_i^m$ and $T_i^m$. We can therefore state the following result:

The characteristic exponents $\alpha$ can be developed as an increasing power series of $\sqrt{\mu}$.

Therefore, concentrating all our attention on the determination of $\alpha_1$, we are going to study especially equation (11). We must first attempt to determine the quantities $C_{ik}^0$ and $b_{ik}$.

We obviously have

$$C_{ki}^0 = -\frac{d^2 F_0}{dx_i^0 \, dx_k^0}$$

and

$$B_{ik}^2 = \frac{d^2 F_1}{dy_i^0 \, dy_k^0},$$

or

$$B_{ik}^2 = -\Sigma A m_i m_k \sin\omega, \qquad (\omega = m_1 y_1^0 + m_2 y_2^0 + m_3 y_3^0 + h)$$

and

$$b_{ik} = -\mathfrak{S} A m_i m_k \sin\omega.$$

The summation represented by the sign $\Sigma$ extends to all terms, whatever the integral values taken on by $m_1$, $m_2$ and $m_3$. The summation represented by the sign $\mathfrak{S}$ extends only to terms such as

$$n_1 m_1 + n_2 m_2 + n_3 m_3 = 0.$$

Under the sign $\mathfrak{S}$ we have consequently

$$\omega = m_2 \varpi_2 + m_3 \varpi_3 + h.$$

This permits us to write

$$b_{ik} = \frac{d^2 R}{d\varpi_i \, d\varpi_k} \text{ (for i and K=2 or 3).}$$

If one or two of the indices i and k are equal to 1, $b_{ik}$ will be defined by the relation /225

$$n_1 b_{i1} + n_2 b_{i2} + n_3 b_{i3} = 0.$$

By means of this last relation, we are going to transform equation (11) so as to place in evidence the existence of two zero roots and to reduce the equation to the fourth degree.

I find, in fact, by a simple transformation of the determinant and by dividing by $\alpha_1^2$, that

$$\begin{vmatrix} n_1 & n_2 & n_3 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & b_{22} & b_{23} & 0 \\ 0 & 0 & -\alpha_1 & b_{32} & b_{33} & 0 \\ C^0_{13} & C^0_{23} & C^0_{33} & -\alpha_1 & 0 & n_3 \\ C^0_{12} & C^0_{22} & C^0_{32} & 0 & -\alpha_1 & n_2 \\ C^0_{11} & C^0_{21} & C^0_{31} & 0 & 0 & n_1 \end{vmatrix} = 0.$$

In the particular case where we have only 2 degrees of freedom, this equation is written

$$\begin{vmatrix} n_1 & n_2 & 0 & 0 \\ 0 & -\alpha_1 & \dfrac{d^2R}{d\varpi_2^2} & 0 \\ C^0_{12} & C^0_{22} & -\alpha_1 & n_2 \\ C^0_{11} & C^0_{21} & 0 & n_1 \end{vmatrix} = 0$$

or

$$n_1^2 \alpha_1^2 = \frac{d^2R}{d\varpi_2^2}\left(n_1^2 C^0_{22} - 2 n_1 n_2 C^0_{12} + n_2^2 C^0_{11}\right).$$

The expression $n_1^2 C^0_{22} - 2 n_1 n_2 C^0_{12} + n_2^2 C^0_{11}$ depends only on $x_1^0$ and $x_2^0$ or, if one wishes, on $n_1$ and $n_2$. When we are given the two numbers $n_1$ and $n_2$, which must be commensurable, we may regard $n_1^2 C^0_{22} - 2 n_1 n_2 C^0_{12} + n_2^2 C^0_{11}$ as given constant. Then the sign of $\alpha_1^2$ depends only on that of $\dfrac{d^2R}{d\varpi_2^2}$.

When we are given $n_1$ and $n_2$, we form the equation

$$\frac{dR}{d\varpi_2} = 0. \tag{12}$$

We have seen in article 42 a periodic solution that corresponds to each root of this equation. /23

Let us consider the general case where equation (12) has only simple roots;

each of these roots then corresponds to a maximum or minimum of R. However, function R, being periodic, presents in each period at least one maximum and one minimum and precisely as many maxima as minima.

Now for values of $\varpi_2$ corresponding to a minimum, $\frac{d^2R}{d\varpi_2^2}$ is positive; for values corresponding to a maximum, this derivative is negative.

Therefore equation (12) will have precisely as many roots for which this derivative will be positive as roots for which this derivative will be negative, and consequently as many roots for which $\alpha_1^2$ will be positive as roots for which $\alpha_1^2$ will be negative.

This is the same as saying that there will be precisely as many stable periodic solutions as unstable solutions, giving this term the same sense as in article 59.

Thus, to each system of values of $n_1$ and $n_2$ there will correspond at least one stable periodic solution and one unstable periodic solution and precisely as many stable solutions as unstable solutions, provided $\mu$ is sufficiently small.

Here I will not examine how these results could be extended in the case where equation (12) would have multiple roots.

The calculation is continued in the following manner.

Let us imagine that we have completely determined the quantities

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$$

and the functions

$$S_i^0,\ S_i^1,\ \ldots,\ S_i^m,$$
$$T_i^0,\ T_i^1,\ \ldots,\ T_i^{m-1}$$

and that we know the functions $S_i^{m+1}$ and $T_i^m$ to within a constant. Let us assume that we propose then to calculate $\alpha_{m+1}$, to determine functions $S_i^{m+1}$ and $T_i^m$ and then to determine functions $S_i^{m+2}$ and $T_i^{m+1}$ to within a constant.

Equating the like powers of $\mu$ in equations (4), we obtain equations of the following form, analogous to equations (7) and (8), /227

$$\left\{\begin{aligned} -\frac{dT_i^{m+1}}{dt} + \Sigma_k C_{ik}^0 S_k^{m+1} - \alpha_1 T_i^m - \alpha_{m+1} T_i^0 &= \text{known quantity} \\ -\frac{dS_i^{m+2}}{dt} + \Sigma_k B_{ik}^0 T_k^m - \alpha_1 S_i^{m+1} - \alpha_{m+1} S_i^1 &= \text{known quantity} \end{aligned}\right\} (i=1,2,3). \qquad (13)$$

The two members of these equations (12) are periodic functions of t. Let us equate the mean value of these two members. If we designate by [U] the mean value of an arbitrary periodic function U and we observe that, if U is periodic, we have

$$\left[\frac{dU}{dt}\right] = 0;$$

if we recall that, $T_k^m$ being known to within a constant, $T_k^m - [T_k^m]$ and

$$[B_{ik}^2(T_k^m - [T_k^m])]$$

are known quantities, we will obtain the following equations:

$$\left\{\begin{array}{l} \Sigma_k C_{ik}^0[S_k^{m+1}] - \alpha_1[T_i^m] - \alpha_{m+1}T_i^0 = \text{known quantity} \\ \Sigma_k b_{ik}[T_k^m] - \alpha_1[S_i^{m+1}] - \alpha_{m+1}S_i^1 = \text{known quantity} \end{array}\right\} (i = 1, 2, 3). \qquad (14)$$

These equations (14) will serve to calculate $\alpha_{m+1}$, $[T_i^m]$ and $[S_i^{m+1}]$ and consequently to achieve the determination of the functions $T_1^m$ and $S_i^{m+1}$, which are still known only to within a constant.

If we add equations (14) after having multiplied them, respectively, by

$$S_1^1, \quad S_2^1, \quad S_3^1, \quad T_1^0, \quad T_2^0, \quad T_3^0,$$

we find

$$2\Sigma S_i^1 T_i^0 \alpha_{m+1} = \text{known quantity},$$

which determines $\alpha_{m+1}$.

If in equations (14) we replace $\alpha_{m+1}$ by the value thus found, we have, in order to determine the six unknowns $[T_i^m]$ and $[S_i^{m+1}]$, six linear equations, of which only five are independent.

This granted, we will determine $[T_1^m]$ by the condition that $[T_1^m]$ be zero for t=0, in conformity with the hypothesis made above, and the five equations (14) remaining independent will permit calculating the five other unknowns.

Equations (13) will then permit us to calculate $\frac{dT_i^{m+1}}{dt}$ and $\frac{dS_i^{m+2}}{dt}$ and consequently to determine the functions $T_i^{m+1}$ and $S_i^{m+2}$ to within a constant: and so forth.

## Degenerate Solutions

80. Let us return to equations (1) from the preceding article

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, 3). \tag{1}$$

We have assumed that there existed a periodic solution of period T

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t);$$

then setting

$$x_i = \varphi_i + \xi_i, \quad y_i = \psi_i + \eta_i,$$

we have formed the equations of variation

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= \sum \frac{d^2F}{dy_i\,dx_k}\xi_k + \sum \frac{d^2F}{dy_i\,dy_k}\eta_k, \\ \frac{d\eta_i}{dt} &= -\sum \frac{d^2F}{dx_i\,dx_k}\xi_k - \sum \frac{d^2F}{dx_i\,dy_k}\eta_k. \end{aligned}\right\} \tag{2}$$

These equations, having generally four characteristic exponents different from 0, will admit four particular solutions of the form

$$\xi_i = e^{\alpha t} S_i, \quad \eta_i = e^{\alpha t} T_i,$$

$S_i$ and $T_i$ being periodic. We have learned to form these integrals.

However, equations (2) will have in addition two characteristic zero exponents: they will therefore admit two particular solutions of the form

$$\left.\begin{aligned} \xi_i &= S''_i, & \eta_i &= T''_i, \\ \xi_i &= S^*_i + tS''_i, & \eta_i &= T^*_i + tT''_i, \end{aligned}\right\} \tag{3}$$

$S''_i$, $T''_i$, $S^*_i$, $T^*_i$ being periodic of the same period as $\varphi_i$, $\psi_i$, $S_i$ and $T_i$. /229

How must one proceed to form these solutions (3)?

We have seen in article 42 that equations (1) admit a periodic solution

$$x_i = \varphi_i(t, \mu, \varepsilon), \quad y = \psi_i(t, \mu, \varepsilon) \tag{4}$$

of period

$$\frac{T}{1+\varepsilon},$$

which reduces to

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t)$$

for $\varepsilon=0$.

The functions $\varphi_i$ and $\psi_i$ can be developed in increasing powers of $\varepsilon$.

Let us now set

$$t = \frac{u}{1+\varepsilon} \text{ whence } \quad u = t(1+\varepsilon).$$

If we substitute this value in place of t in equations (4), it will follow that

$$x_i = \theta_i(u, \mu, \varepsilon). \qquad y_i = \Theta_i(u, \mu, \varepsilon).$$

The functions $\theta_i$ and $\Theta_i$ will still be developable in powers of $\mu$ and $\varepsilon$, but they will be periodic in u and the period will be constant and equal to T; they will therefore be developable in terms of the sine and cosine of the multiples of $\frac{2\pi u}{T}$.

If h is an arbitrary constant,

$$x_i = \varphi_i(t+h, \mu, \varepsilon), \qquad y_i = \psi_i(t+h, \mu, \varepsilon)$$

is still a solution of equations (1), because time does not enter explicitly into these equations. This solution contains two arbitrary constants h and $\varepsilon$.

Article 54 furnishes us the means of deducing two solutions of the equations of variation (2).

These solutions are written /23

$$\xi_i = \frac{d\varphi_i}{dh}, \qquad \eta_i = \frac{d\psi_i}{dh}$$

and

$$\xi_i = \frac{d\varphi_i}{d\varepsilon}, \qquad \eta_i = \frac{d\psi_i}{d\varepsilon}.$$

After differentiation it is necessary to make $h=\varepsilon=0$.

Now it follows that

$$\varphi_i(t, \mu, \varepsilon) = \theta_i[t(1+\varepsilon), \mu, \varepsilon],$$
$$\psi_i(t, \mu, \varepsilon) = \Theta_i[t(1+\varepsilon), \mu, \varepsilon],$$

from which

$$\frac{d\varphi_i}{dh} = \frac{d\varphi_i}{dt} = \frac{d\theta_i}{du}\frac{du}{dt} = \frac{d\theta_i}{du}(1+\varepsilon),$$
$$\frac{d\psi_i}{dh} = \frac{d\psi_i}{dt} = \frac{d\Theta_i}{du}\frac{du}{dt} = \frac{d\Theta_i}{du}(1+\varepsilon),$$

and for $\varepsilon=0$

$$\frac{d\varphi_i}{dh} = \frac{d\theta_i}{du}, \qquad \frac{d\psi_i}{dh} = \frac{d\Theta_i}{du}.$$

On the other hand,

$$\frac{d\varphi_i}{d\varepsilon} = \frac{d\theta_i}{du}\frac{du}{d\varepsilon} + \frac{d\theta_i}{d\varepsilon} = t\frac{d\theta_i}{du} + \frac{d\theta_i}{d\varepsilon},$$
$$\frac{d\psi_i}{d\varepsilon} = \frac{d\Theta_i}{du}\frac{du}{d\varepsilon} + \frac{d\Theta_i}{d\varepsilon} = t\frac{d\Theta_i}{du} + \frac{d\Theta_i}{d\varepsilon}$$

or, for $\varepsilon=0$,

$$\frac{d\varphi_i}{d\varepsilon} = t\frac{d\varphi_i}{dt} + \frac{d\theta_i}{d\varepsilon}, \qquad \frac{d\psi_i}{d\varepsilon} = t\frac{d\psi_i}{dt} + \frac{d\Theta_i}{d\varepsilon}.$$

The sought solutions of equations (2) are therefore

$$\xi_i = S''_i = \frac{d\varphi_i}{dt}, \qquad \eta_i = T''_i = \frac{d\psi_i}{dt}$$

and

$$\xi_i = tS''_i + S^*_i, \qquad \eta_i = tT''_i + T^*_i$$

with

$$S^*_i = \frac{d\theta_i}{d\varepsilon}, \qquad T^*_i = \frac{d\Theta_i}{d\varepsilon}.$$

I say that the functions $S''_i$, $T''_i$, $S^*_i$, $T^*_i$ are periodic in t of period T. In fact, /231
$\theta_i$ and $\Theta_i$ are periodic of period T in u; this period being independent of $\varepsilon$, the derivatives

$$\frac{d\theta_i}{du}, \quad \frac{d\Theta_i}{du}, \quad \frac{d\theta_i}{d\varepsilon}, \quad \frac{d\Theta_i}{d\varepsilon} \tag{5}$$

will be equally periodic in u. However, for $\varepsilon=0$, u=t; if therefore after differentiation we make $\varepsilon=0$, these four derivatives (5), i.e., the four functions $S''_i$, $T''_i$, $S^*_i$, $T^*_i$, will be periodic in t.

Q. E. D.

These four functions will be, as $\theta_i$ and $\Theta_i$ whose derivatives they are, developable in increasing and positive powers of $\mu$ (I recall that $S_i$ and $T_i$ in the preceding article were developable in powers not of $\mu$, but of $\sqrt{\mu}$).

For $\mu=0$, $\varphi_i$ reduces to a constant $x_i^0$; therefore $\frac{d\varphi_i}{dt} = S''_i$ vanishes.

Therefore $S_i''$ is divisible by $\mu$, as in the preceding article $S_i$ was divisible by $\sqrt{\mu}$.

On the contrary, $S_i^*$ is not divisible by $\mu$.

In a memoir I published in Acta mathematica, Vol. XIII, p. 157, I was led to consider equations analogous to equations (2) and two particular solutions of these equations

$$\xi_i = S_i', \qquad \eta_i = T_i',$$
$$\xi_i = S_i'' + \alpha t S_i', \qquad \eta_i = T_i'' + \alpha t T_i'.$$

I call $\alpha$ one of the characteristic exponents, such that $\alpha$ can be developed in odd powers of $\sqrt{\mu}$, and that $\mu$ is itself developable in powers of $\alpha^2$ and is divisible by $\alpha^2$.

I assume that one replaces $\mu$ by this value, such that all our functions are found developed in powers of $\alpha$. I state then that $S_i''$ and $S_i''!$ are divisible by $\alpha$. In fact, $S_i''$, as we have just seen, is divisible by $\mu$ and $\mu$ by $\alpha^2$.

On the other hand, we have manifestly

$$S_i'' = \alpha S_i^*,$$

since it is necessary to multiply by $\alpha$ the solution which I have just studied /233

$$\xi_i = S_i^* + t S_i',$$

so as to obtain the solution considered in Acta mathematica

$$\xi_i = S_i'' + \alpha t S_i'.$$

I thought it necessary to make this remark because an inattentive reader might not take account of this factor $\alpha$ and consequently believe there to be a contradiction between the result stated in Acta and those which I have just demonstrated.

# CHAPTER 5. NON-EXISTENCE OF UNIFORM INTEGRALS

81. Let us return to our canonical equations /233

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{1}$$
$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots,$$

I first assume that $F_0$, which does not depend on $y_i$, depends on n variables $x_i$ and that its Hessian with respect to these n variables is not zero.

I propose to demonstrate that, except in certain exceptional cases which we will study later, equations (1) admit no other analytic and uniform integral than the integral F=const.

This is what I mean to say:

Let $\Phi$ be an analytic and uniform function of values s, y, and $\mu$, which must in addition be periodic with respect to y.

I am not required to assume that this function is analytic and uniform for all values of s, y and $\mu$.

I assume only that this function is analytic and uniform for all real values of y, for sufficiently small values of $\mu$ and for the systems of values of x belonging to a certain domain D; the domain D can in addition be arbitrary and as small as desired. Under these conditions, the function $\Phi$ can be developed in powers of $\mu$ and I may write

$$\Phi = \Phi_0 + \mu\Phi_1 + \mu^2\Phi^2 + \ldots,$$

$\Phi_0$, $\Phi_1$, $\Phi_2$, ... being uniform with respect to x and y and periodic with respect to y.

I say that a function $\Phi$ in this form cannot be an integral of equations (1). /234

The necessary and sufficient condition for a function $\Phi$ to be an integral is written, resuming the notation of article 3,

$$[F, \Phi] = o,$$

or, replacing F and $\Phi$ by their developments,

$$o = [F_0, \Phi_0] + \mu([F_1, \Phi_0] + [F_0, \Phi_1]) + \mu^2([F_2, \Phi_0] + [F_1, \Phi_1] + [F_0, \Phi_2]) + \ldots.$$

We therefore will have separately the following equations, which I will use later

$$[F_0, \Phi_0] = 0 \tag{2}$$

and

$$[F_1, \Phi_0] + [F_0, \Phi_1] = 0. \tag{3}$$

I say that I may always assume that $\Phi_0$ is not a function of $F_0$.

In fact, let us assume that we have

$$\Phi_0 = \psi(F_0).$$

I say that function $\psi$ generally will be a uniform function, when the variables x remain in the domain D.

We have in fact

$$F_0 = F_0(x_1, x_2, \ldots, x_n).$$

We will be able to solve this equation with respect to $x_1$ and write

$$x_1 = \theta(F_0, x_2, \ldots, x_n),$$

and $\theta$ will be a uniform function unless $\frac{dF_0}{dx_1}$ vanishes within domain D.

Replacing $x_1$ by its value $\theta$ in

$$\Phi_0\begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix},$$

it follows that

$$\Phi_0\begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix} = \psi\begin{pmatrix} F_0, x_2, \ldots, x_n \\ y_1, y^2, \ldots, y_n \end{pmatrix}.$$

$\Phi_0$ is a uniform function of x and y; if we here replace $x_1$ by the uniform function $\theta$, we will obtain a uniform function $\psi$ of $F_0$, of $x_2$, ..., $x_n$ and of y, but by hypothesis this function $\Phi=\psi$ is a uniform function of $F_0$.

Therefore, $\Phi=\psi$ is a uniform function of $F_0$.

This result holds provided $\frac{dF_0}{dx_1}$ does not vanish in domain D; this will hold equally well if any one of the derivatives $\frac{dF_0}{dx_1}$ does not vanish in domain D.

This granted, if $\Phi$ is a uniform integral, the same will be true for

$$\Phi - \psi(F).$$

$\Phi - \psi(F)$ can be developed in powers of μ and in addition is divisible by μ, since $\Phi_0 - \psi(F_0)$ is zero. Let us therefore set

$$\Phi - \psi(F) = \mu\Phi':$$

$\Phi'$ will be an analytic and uniform integral and it will follow that

$$\Phi' = \Phi'_0 + \mu\Phi'_1 + \mu^2\Phi'_2 + \ldots,$$

In general, $\Phi'_0$ will not be a function of $F_0$; if this were to happen, we would repeat the previous operation.

I say that in repeating this operation, we will in the end arrive at an integral which will not reduce to a function of $F_0$ for $\mu=0$.

At least this is true whenever Φ is not a function of F, in which case the two integrals F and Φ would not be distinct.

In fact, let J be the Jacobian or functional determinant of Φ and of F with respect to both variables x and y. I may assume that this Jacobian is not identically zero, because if all Jacobians were zero, Φ would be a function of F, which we do not assume.

J will be manifestly developable in powers of μ. In addition, J will vanish with μ, because $\Phi_0$ is a function of $F_0$. Therefore J will be divisible by a cer- /236
tain power of μ, for example, by $\mu^p$.

Now let J' be the functional determinant or Jacobian of $\Phi'$ and of F; we will have

$$J = \mu J',$$

such that J' will only be divisible by $\mu^{p-1}$.

Thus, after p operations at most, we will arrive at a Jacobian which will no longer vanish with μ and which will consequently correspond to an integral which will not reduce to a function of $F_0$ for $\mu=0$.

Consequently, if there exists an analytic and uniform integral distinct from F, but such that $\Phi_0$ is a function of $F_0$, we will always be able to find from it another of the same form and which will not reduce to a function of $F_0$ for $\mu=0$.

We therefore always have the right to assume that $\Phi_0$ is not a function of $F_0$.

82. I now say that $\Phi_0$ cannot depend on y.

If, in fact, $\Phi_0$ depends on y, it will be a periodic function of these variables, such that we will be able to write

$$\Phi_0 = \Sigma A e^{\sqrt{-1}(m_1 y_1 + m_2 y_2 + \ldots + m_n y_n)} = \Sigma A \zeta,$$

$m_i$ being positive or negative integers, A being functions of $x_i$ and the notation $\zeta$ representing for brevity the imaginary exponential which multiplies A.

This granted, we have

$$[F_0, \Phi_0] = \sum \frac{dF_0}{dx_i} \frac{d\Phi_0}{dy_i},$$

since $F_0$ does not depend on y and values $\frac{dF_0}{dy_i}$ are zero.

On the other hand,

$$\frac{d\Phi_0}{dy_i} = \Sigma \sqrt{-1}\, m_i A \zeta,$$

so that equation (2) may be written

$$\sqrt{-1}\,\Sigma A \left(m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} + \ldots + m_n \frac{dF_0}{dx_n}\right) \zeta = 0,$$

and, as this must be an identity, for all systems of integral values of $m_i$ we will have /23

$$A \Sigma m_i \frac{dF_0}{dx_i} = 0,$$

such that we must have identically either

$$A = 0, \tag{4}$$

or

$$\Sigma m_i \frac{dF_0}{dx_i} = 0. \tag{5}$$

From identity (5) we may by differentiation deduce

$$\sum_{i=1}^{i=n} m_i \frac{d^2 F_0}{dx_i dx_k} = 0 \qquad (K = 1, 2, \ldots n).$$

Now this can take place in only two manners:

Either

$$m_1 = m_2 = \ldots = m_n = 0,$$

or the Hessian of $F_0$ is zero.

Now we have assumed at the beginning that the Hessian was not zero.

Therefore A must be identically zero, except for the term where all $m_i$ are zero.

This is the same as saying that $\Phi_0$ reduces to a single term which does not depend on y.

Q. E. D.

Let us now examine equation (3). As $F_0$ and $\Phi_0$ do not depend on y, this equation can be written

$$-\sum \frac{d\Phi_0}{dx_i}\frac{dF_1}{dy_i} + \sum \frac{dF_0}{dx_i}\frac{d\Phi_1}{dy_i} = 0.$$

On the other hand, $F_1$ and $\Phi_1$ are periodic with respect to y and consequently can be developed as exponentials of the form

$$e^{\sqrt{-1}(m_1 y_1 + m_2 y_2 + \ldots + m_n y_n)},$$

$m_i$ being positive of negative integers.

For brevity I will, as above, designate this exponential by $\zeta$ and I will write /238

$$F_1 = \Sigma B\zeta, \qquad \Phi_1 = \Sigma C\zeta,$$

B and the C being coefficients depending only on x.
We will then have

$$\frac{dF_1}{dy_i} = \sqrt{-1}\,\Sigma B\, m_i \zeta, \qquad \frac{d\Phi_1}{dy_i} = \sqrt{-1}\,\Sigma C m_i \zeta,$$

such that equation (3), divided by $\sqrt{-1}$, will be written

$$-\Sigma B\zeta\left(\Sigma_i m_i \frac{d\Phi_0}{dx_i}\right) + \Sigma C\zeta\left(\Sigma m_i \frac{dF_0}{dx_i}\right) = 0.$$

Since this equation is an identity, for all systems of integral values of $m_i$ we must have

$$B\Sigma m_i \frac{d\Phi_1}{dx_i} = C\Sigma m_i \frac{dF_0}{dx_i}. \qquad (6)$$

Relation (6) must hold for all values of x. Let us then give x values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0 : \tag{7}$$

the second member of (6) vanishes. Whenever values x satisfy equation (7), we must therefore have either

$$B = 0 \tag{8}$$

or

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0. \tag{9}$$

Function $F_1$ is one of the given conditions of the problem and the case consequently will be the same for coefficient B. Therefore it is easy to recognize if equality (7) implies equality (8). In general, we will state that this is not true and we must conclude that equality (9) is a necessary consequence of equality (7).

Now let $p_1, p_2, \ldots, p_n$ be a certain number of integers. Let us consider that we give x values such that /25

$$\frac{dF_0}{p_1\, dx_1} = \frac{dF_0}{p_2\, dx_2} = \ldots = \frac{dF_0}{p_n\, dx_n}. \tag{10}$$

We will be able to find an infinity of systems of integers, $m_1, m_2, \ldots, m_n$, such that

$$m_1 p_1 + m_2 p_2 + \ldots + m_n p_n = 0.$$

For each of these systems of integers, we must have

$$\Sigma m_i \frac{dF_0}{dx_i} = 0$$

and, consequently,

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0.$$

Comparison of these two equations shows that we must have

$$\frac{\frac{dF_0}{dx_1}}{\frac{d\Phi_0}{dx_1}} = \frac{\frac{dF_0}{dx_2}}{\frac{d\Phi_0}{dx_2}} = \ldots = \frac{\frac{dF_0}{dx_n}}{\frac{d\Phi_0}{dx_n}},$$

i.e., that the Jacobian of $F_0$ and of $\Phi_0$ with respect to two arbitrary components of the vector x must be zero.

This must hold true for all values of x which satisfy relations of form (10), i.e., for all values such that $\frac{dF_0}{dx_i}$ be commensurable among themselves. In an arbitrary domain, however small it may be, there is therefore an infinity of systems of values x for which this Jacobian vanishes, and as this Jacobian is a continuous function, it must vanish identically.

To say that all Jacobians of $F_0$ and of $\Phi_0$ are zero is therefore to say that $\Phi_0$ is a function of $F_0$. Now this is contrary to the hypothesis which we have assumed at the end of the preceding article.

We must therefore conclude that equations (1) admit no other uniform integral than F=C.

Q. E. D.

## Case Where the B Vanish

83. In the preceding demonstration we assumed that coefficients B were not zero. If one or several of these coefficients vanished (and especially if infinitely many of them vanished), we would have to examine this reasoning. /240

To make possible the statement of the consequence to which I will be led, I will be forced to introduce a new terminology.

To each system of indices $m_1, m_2, \ldots, m_n$ (where $m_i$ are integers) there corresponds a coefficient B. I will say that this coefficient is secular when $x_i$ take on values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0. \qquad (7)$$

The following will justify this definition.

When, in the calculation of perturbations, we assume that the ratios of the mean motions are commensurable, some of the terms of the perturbing function cease being periodic, and we can then say that they become secular; what happens here is completely analogous.

I will say that two systems of indices $(m_1, m_2, \ldots, m_n)$ and $(m'_1, m'_2, \ldots, m'_n)$ belong to the same class when we have

$$\frac{m_1}{m'_1} = \frac{m_2}{m'_2} = \ldots = \frac{m_n}{m'_n}$$

and that two coefficients B belong to the same class when they correspond to two systems of indices belonging to the same class.

In order to demonstrate the theorem of the preceding article, we have assumed that none of the coefficients B vanishes in becoming secular.

In order for the result to be true, it is sufficient that in each of the classes there be at least one coefficient B which does not vanish in becoming secular.

Let us assume in fact that the coefficient B which corresponds to the system $(m_1, m_2, \ldots, m_n)$ vanishes, but that the coefficient B' which corresponds to to the system $(m_1', m_2', \ldots, m_n)$ does not vanish. /2

If we give x values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0,$$

we will have equally

$$\Sigma m_i' \frac{dF_0}{dx_i} = 0,$$

and consequently

$$B \Sigma m_i \frac{d\Phi_0}{dx_i} = 0, \qquad B' \Sigma m_i' \frac{d\Phi_0}{dx_i} = 0.$$

From the first of these equalities we cannot deduce

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0$$

because B is zero; but, as B' is not zero, the second equality gives us

$$\Sigma m_i' \frac{d\Phi_0}{dx_i} = 0$$

and, consequently,

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0.$$

The rest of the reasoning is carried out as in the preceding article.

Before continuing, let us first consider the particular case where there are only two degrees of freedom.

There will then be only two indices $m_1$ and $m_2$ and one class will be entirely defined by the ratio of these two indices. Let $\lambda$ be an arbitrary commensurable number; let C be the class of indices where $\frac{m_1}{m_2} = \lambda$. For brevity I will say that this class C belongs to domain D, or is in this domain if we can give to $x_i$ a system of values belonging to this domain, and such that

$$\lambda \frac{dF_0}{dx_1} + \frac{dF_0}{dx_2} = 0.$$

I will say that a class is singular when all coefficients of this class vanish in becoming secular, and that it is ordinary in the opposite case. /242

I say that the theorem will still be true if we assume that, in any domain $\delta$ being part of D we can find an infinity of ordinary classes.

Let there be, in fact, an arbitrary system of values of $x_1$ and $x_2$, such that we have at this point

$$\lambda \frac{dF_0}{dx_1} + \frac{dF_0}{dx_2} = 0.$$

Let us assume that $\lambda$ is commensurable and that the class which corresponds to this value of $\lambda$ is ordinary; the reasoning of the preceding article can then be applied to this system of values and one must conclude that, for these values of $x_1$ and of $x_2$, the Jacobian of $F_0$ and of $\Phi_0$ with respect to $x_1$ and to $x_2$ vanishes.

However, by hypothesis there exists, in any domain $\delta$ so small that it is part of D, an infinity of such systems of values of $x_1$ and of $x_2$. Consequently our Jacobian must vanish at all points of D which shows that $\Phi_0$ is a function of $F_0$. From this we would conclude, as in the preceding article, that there exists no uniform integral distinct from F.

The case would not be the same if we could find a domain D of which all the classes are singular.

We could then ask if there does not exist an integral which remains uniform not for all values of x, but only for those values which do not leave domain D. We would see, in general, that while this would not be true; it would be sufficient, in order to be certain of it, to consider in the equation

$$[F, \Phi] = 0,$$

not any longer only the term independent of $\mu$, and the terms in $\mu$, but the term in $\mu^2$ and the following terms.

I do not insist, this has no interest, for I do not believe that in any problem of Dynamics occurring naturally it happens that all classes of a domain D are singular without all coefficients B vanishing in becoming secular.

Let us now proceed to the case where there are more than 2 degrees of freedom.

The results will be analogous, although their statement will be more complicated. /243

Let

$$p_1, \ p_2, \ \dots, \ p_n$$

be n arbitrary integral numbers. Let us consider all systems of indices $m_1$, $m_2$, ..., $m_n$ which satisfy the condition

$$m_1 p_1 + m_2 p_2 + \dots + m_n p_n = 0.$$

I will say that all corresponding coefficients belong to the same family.

Let there be q classes defined by the following systems of indices

$$\begin{array}{llll} m_{1,1}, & m_{2,1}, & \dots, & m_{n,1} \\ m_{1,2}, & m_{2,2}, & \dots, & m_{n,2} \\ \dots, & \dots, & \dots, & \dots \\ m_{1,q}, & m_{2,q}, & \dots, & m_{n,q}. \end{array}$$

If one cannot find q integers

$$a_1, \ a_2, \ \dots, \ a_q,$$

such that one has

$$\sum_{i=1}^{i=q} a_i m_{k,i} = 0 \qquad (k = 1, 2, \dots, n),$$

I will say that these 2 classes are independent.

I will say that a family is ordinary, if we can find in it n-1 independent and ordinary classes, and that it is singular in the opposite case. It will be singular of the first order, if we can find in it n-2 independent classes, ordinary and singular classes of the q-th order, if we can find in it n-q -1 independent and ordinary classes and no more.

I will say that a family defined by the integers ($p_1$, $p_2$, ...,$p_n$) belongs to a domain D, if there exist in this domain values of x such that

$$\frac{dF_0}{p_1 dx_1} = \frac{dF_0}{p_2 dx_2} = \dots = \frac{dF_0}{p_n dx_n}.$$

This granted, I say that if we can find in all domains δ which are part of D an infinity of ordinary families, there can exist no uniform distinct integral of F. /244

The reasoning of the preceding article is, in fact, applicable to all systems of values of x which correspond to an ordinary family.

The Jacobians of $F_0$ and $\Phi_0$, with respect to two arbitraries of variables x, must therefore vanish an infinity of times in all domains δ that are part of D, which can occur only if they are identically zero.

I now say that if we can find in any domain δ which is part of D an infinity of singular classes of the q-th order, the number of distinct uniform integrals which equations (1) can have is at most equal to q+1 (including the integral F).

Let us, in fact, assume that there are q+2 distinct integrals; let

$$F, \quad \Phi^1, \quad \Phi^2, \quad \ldots, \quad \Phi^{q+1}$$

be these integrals and let us assume that for μ=0 they reduce to

$$F_0, \quad \Phi_0^1, \quad \Phi_0^2, \quad \ldots, \quad \Phi_0^{q+1}. \tag{11}$$

Let there be a system of values of x corresponding to an irregular family of the q-th order. Let us set

$$n - q - 1 = p.$$

There will exist in this family p ordinary classes. Let

$$m_{1,k}, m_{2,k}, \ldots, m_{n,k} \qquad (k = 1, 2, \ldots, p)$$

be the systems of indices corresponding to these classes.

We will have for the values of x under consideration

$$\sum_{i=n}^{i=1} m_{i,k} \frac{dF_0}{dx_i} = \sum_{i=n}^{i=1} m_{i,k} \frac{d\Phi_0^h}{dx_i} = 0$$

$$(k = 1, 2, \ldots p, \; h = 1, 2, \ldots, q+1).$$

We will deduce from this that the Jacobians of the q+2 functions (11) with respect to q+2 arbitrary coordinates of x must vanish for the considered values of x.

And since this must take place an infinity of times in each domain δ, we will conclude from it that these Jacobians vanish identically and consequently that our q+2 integrals cannot be distinct. /245

These considerations present no additional practical interest, and I have presented them here only to be complete and rigorous. We can obviously construct problems artificially where these various circumstances are encountered; but in problems of Dynamics occurring naturally it will always occur either that all classes will be singular, or that they all will be ordinary, with the exception of a finite number of them.

## Case Where the Hessian is Zero

84. Let us now proceed to the case where $F_0$ does not depend on all variables $x_1, x_2, \ldots, x_n$.

I will assume that $F_0$ depends on $x_1$ and $x_2$ only, and that its Hessian with respect to these two variables is not zero.

In order to note well the difference between these two variables $x_1$ and $x_2$ and their conjugates $y_1$ and $y_2$ on one hand, and the other variables x and y on the other hand, I will agree to designate

$$x_3, \; x_4, \; \ldots, \; x_n,$$
$$y_3, \; y_4, \; \ldots, \; y_n$$

by the notation

$$z_1, \; z_2, \; \ldots, \; z_{n-2},$$
$$u_1, \; u_2, \; \ldots, \; u_{n-2}.$$

We will first observe that the conclusions of article 81 obtain and, if there exists a uniform integral $\Phi$ distinct from F, it is always permissible to assume that $\Phi_0$ is not a function of $F_0$.

This granted, we must first have

$$[F_0, \Phi_0] = \frac{dF_0}{dx_1}\frac{d\Phi_0}{dy_1} + \frac{dF_0}{dx_2}\frac{d\Phi_0}{dy_2} = 0.$$

Let us set

$$\zeta = e^{\sqrt{-1}(m_1 y_1 + m_2 y_2)},$$

we can write /246

$$\Phi_0 = \Sigma A \zeta$$

values A being coefficients depending on $x_1$, $x_2$, z and $\mu$. It then follows that

$$\sqrt{-1}\,\Sigma A \zeta \left(m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2}\right) = 0.$$

This relation must be an identity and, on the other hand, the Hessian of $F_0$ being not zero, we cannot have identically

$$m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} = 0,$$

unless $m_1$ and $m_2$ are both zero.

From this we would conclude, as in article 82, that $\Phi_0$ depends neither on $y_1$ nor on $y_2$.

If we then write equation (3), we will have

$$-\frac{d\Phi_0}{dx_1}\frac{dF_1}{dy_1}-\frac{d\Phi_0}{dx_2}\frac{dF_1}{dy_2}+\frac{dF_0}{dx_1}\frac{d\Phi_1}{dy_1}$$
$$+\frac{dF_0}{dx_2}\frac{d\Phi_1}{dy_2}+\sum\left(\frac{dF_1}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dF_1}{du_i}\frac{d\Phi_0}{dz_i}\right)=0.$$

Let us also set

$$F_1=\Sigma B\zeta,\qquad \Phi_1=\Sigma C\zeta.$$

When it is necessary to indicate the indices, I will write

$$F_1=\Sigma B_{m_1,m_2}e^{\sqrt{-1}(m_1y_1+m_2y_2)}.$$

It will follow that

$$-\Sigma B\zeta\left(\Sigma m_i\frac{d\Phi_0}{dx_i}\right)+\Sigma C\zeta\left(\Sigma m_i\frac{dF_0}{dx_i}\right)+\Sigma\zeta\sum\left(\frac{dB}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dB}{du_i}\frac{d\Phi_0}{dz_i}\right)=0.$$

This relation must be an identity; we can therefore equate to 0 the coefficient of an arbitrary one of exponentials $\xi$. We will in addition give x values such that

$$m_1\frac{dF_0}{dx_1}+m_2\frac{dF_0}{dx_2}=0, \tag{12}$$

so as to make the terms which depend on C vanish.

It will follow that /247

$$-B\left(m_1\frac{d\Phi_0}{dx_1}+m_2\frac{d\Phi_0}{dx_2}\right)+\sum\left(\frac{dB}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dB}{du_i}\frac{d\Phi_0}{dz_i}\right)=0. \tag{13}$$

We will consider two coefficients $B_{m_1,\,m_2}$, $B_{m_1',\,m_2'}$ as belonging to the same class such that

$$m_1m_2'-m_2m_1'=0,$$

and for brevity I will say that the coefficient $B_{m_1,\,m_2}$ belongs to the class $\frac{m_1}{m_2}$.

It follows from this definition that the coefficient $B_{0,0}$ belongs to all classes at the same time.

According to the preceding, if we give x values which satisfy relation (12), relation (13) must hold for the coefficients of B of class $\frac{m_1}{m_2}$.

Then let p and q be the two first integers among those, such that

$$\frac{m_1}{m_2} = \frac{p}{q}.$$

Let us set

$$\zeta = e^{\sqrt{-1}(py_1+qy_2)}$$

and

$$D_\lambda = B_{\lambda p, \lambda q}\zeta^\lambda, \qquad -\zeta H = p\frac{d\Phi_0}{dx_1} + q\frac{d\Phi_0}{dx_2}.$$

If we give x values such that

$$p\frac{dF_0}{dx_1} + q\frac{dF_0}{dx_2} = 0, \tag{12a}$$

we must have

$$H\frac{dD_\lambda}{d\zeta} + \sum\left(\frac{dD_\lambda}{dz_i}\frac{d\Phi_0}{du_i} - \frac{dD_\lambda}{du_i}\frac{d\Phi_0}{dz_i}\right) = 0, \tag{13a}$$

and this for all integral values of λ, positive, negative or zero.

This can only take place in two ways:

(1) Either we have

$$H = 0, \quad \frac{d\Phi_0}{dz_i} = 0, \quad \frac{d\Phi_0}{du_i} = 0 \quad (i = 1, 2, \ldots, n-2),$$

from which /24

$$\frac{dF_0}{dx_1}\frac{d\Phi_0}{dx_2} - \frac{dF_0}{dx_2}\frac{d\Phi_0}{dx_1} = 0.$$

We would deduce from this, by reasoning quite similar to that of article 82, that $\Phi_0$ is a function of $F_0$, which is contrary to the hypothesis made at the beginning.

(2) Or, on the other hand, if the Jacobian of 2n-3 arbitrary choices of function $D\lambda$ with respect to the 2n-3 variables ξ; $z_i$ and $u_i$ is zero.

From this we would conclude that, if we give $x_1$ and $x_2$ constant values satisfying condition (12a), a relation will result between 2n-3 arbitrary choices of functions $D\lambda$, such that all these functions can be expressed by means of 2n-4 from among them.

We can state this result in still another way.

Let us consider the following expressions

$$B^{\lambda'}_{\lambda p, \lambda q} B^{-\lambda}_{\lambda' p, \lambda' q}. \tag{14}$$

If we assume that we give $x_1$ and $x_2$ constant values satisfying equation (12a), these expressions (14) depend on 2n-4 variables only, namely $z_i$ and $u_i$.

If there exists a uniform integral, all these expressions are functions of 2n-5 from among them; or, in other words, we can find a relation among an arbitrary choice 2n-4 from among them.

What is the condition for which there exist three uniform distinct integrals

$$F = \text{const.}, \qquad \Phi = \text{const.}, \qquad \psi = \text{const.?}$$

Let $F_0$, $\Phi_0$ and $\Psi_0$ be the values of these three integrals for $\mu$=0. We could demonstrate, as above, that we can always assume that there is no relation whatever between $F_0$, $\Phi_0$ and $\Psi_0$.

We would then find, by setting

$$-H'\zeta = p\frac{d\psi_0}{dx_1} + q\frac{d\psi_0}{dx_2},$$

that we have

$$H'\frac{dD_\lambda}{d\zeta} + \sum\left(\frac{dD_\lambda}{dz_i}\frac{d\psi_0}{du_i} - \frac{dD_\lambda}{du_i}\frac{d\psi_0}{dz_i}\right) = 0. \tag{13b}$$

Thus equation (12a) implies as a necessary consequence, not only equation (13a) but equation (13b). By reasoning quite similar to the preceding, we would see that this can occur in only two ways: /249

Either there is a relation between $F_0$, $\Phi_0$ and $\Psi_0$, which is contrary to the hypothesis which we have just made;

Or, if the Jacobian of an arbitrary choice 2n-3 of functions $D\lambda$ is zero as well as all its minors of the first order.

From this it would result that, if $x_1$ and $x_2$ satisfy condition (12a), there is among an arbitrary choice 2n-3 of $D\lambda$ not one, but two relations.

In other words, expressions (14) can be calculated by means of 2n-3 from among them.

Expressions (14) which depend on the coefficients of the development of the function $F_1$ are given quantities of the problem, and we will always be able to verify if there are one or two relations among 2n-4 of these expressions.

Generally, we will discover that there is but one, and from this we will conclude that there exists no analytic and uniform integral other than F.

What will happen, however, if this is not so? To be able to state the result in a complete and rigorous manner, I am going to make use of a terminology analogous to that of the preceding article. I will say that a class is ordinary, if there is no relation between 2n-4 of expressions (14) formed with the coefficients of this class, that it is singular of the first order, if there is one, singular of the second order, if there are two, etc. More generally, a class will be singular of order q if there are q relations among an arbitrary choice 2n-3 of quantities $D\lambda$.

Let $\delta$ be an arbitrary domain including an infinity of systems of values of $x_1$, $x_2$ of z and of u.

If we can find in the domain $\delta$ values of $x_1$ and $x_2$ satisfying condition (12a), I will say that the class $\frac{p}{q}$ belongs to this domain. I have said of the values of $x_1$ and of $x_2$ and not of the values of $x_1$, $x_2$, z and of u, because the first member of (12a) depends only on $x_1$ and $x_2$.

I will then be able to state the following result:

I will designate by D a domain including an infinity of values of $x_1$, $x_2$ of z and u. /250

If in every domain $\delta$ that is part of D, we can find an infinity of ordinary classes, we will be certain that there does not exist outside of F any other integral which is analytic and uniform with respect to x, to y, to z and to u, and in addition periodic with respect to $y_1$ and to $y_2$ and which remains such for all real values of $y_1$ and of $y_2$, for sufficiently small values of $\mu$, and for the values of $x_1$, $x_2$ of z and u which belong to the domain D.

If in all domains $\delta$ that are part of D, we can find an infinity of singular classes of the q-th order, it will not be possible for them to exist more than q+1 distinct uniform integrals, including F.

Application to the Problem of Three Bodies

85. I shall now concern myself with applying the preceding ideas to the various cases of the Problem of Three Bodies.

Let us begin by the particular case defined in article 9. In this case, we have 2 degrees of freedom only and four variables

$$x_1 = L, \quad x_2 = G,$$
$$y_1 = l, \quad y_2 = g - t$$

(cf. article 9); we have, in addition,

$$F_0 = \frac{1}{2x_1^2} + x_2.$$

The Hessian of $F_0$ is zero, but we can, by the artifice of article 43, reduce the problem to the case where the Hessian is not zero.

If, therefore, a uniform integral were to exist, it would be necessary that in the development of $F_1$ (which is the perturbing function of the astronomers), in terms of the sines and cosines of the multiples of $y_1$ and $y_2$, all coefficients vanish at the moment when they become secular.

Examination of the well-known development of the perturbing function shows that this is not the case.

We must therefore conclude that in the particular case of the Problem of Three Bodies there is no uniform integral distinct from F. /251

In my memoir in Acta mathematica (Vol. XIII), in order to establish the same point I made use of the existence of periodic solutions and of the fact that the characteristic exponents are not zero. The demonstration which I give here differs from that of Acta only in form, but it lends itself better to the generalization which will follow.

Let us now consider a somewhat more general case of the Problem of Three Bodies, that where motion occurs in a plane, and let us assume that we have reduced the number of degrees of freedom to 3, as we have said in article 15.

We then have six conjugate variables, namely

$$\beta L, \quad \beta' L', \quad \beta \Pi = H,$$
$$l, \quad l', \quad h = \varpi - \varpi'.$$

Let us assume that we develop the perturbing function $F_1$ in the following manner

$$F_1 = \Sigma B_{m_1, m_2} e^{\sqrt{-1}(m_1 l + m_2 l')},$$

the coefficients $B_{m_1, m_2}$ will be functions of $\beta L$, $\beta' L'$, H and h.

Let p and q be two arbitrary integers first among them; let us form the expressions

$$B^{\lambda'}_{\lambda p, \lambda q} B^{-\lambda}_{\lambda' p, \lambda' q} \quad (\lambda, \lambda' = 0, \pm 1, \pm 2, \ldots, \text{ad inf.}). \tag{14}$$

Let us give L and L' values satisfying condition (12a), i.e., such that the relationship of the mean motions is equal to $-\frac{q}{p}$.

In order for the problem to admit a uniform integral other than the vis viva integral, it would be necessary for there to be a relationship between two arbitrarily chosen from among them (n=3, 2n-4=2), i.e., that all these expressions (14) be functions of $B_{0,0}$, i.e., of the secular part of the perturbing function. Now examination of the well-known development of this function shows that this is not the case.

We must conclude that, outside of the vis viva integral, the problem admits no uniform integral of the following form /25

$$\Phi(L, L', H, l, l', h) = \text{const.}$$

periodic in $l$ and $l'$.

But this is not sufficient for us; we must still demonstrate that the problem admits no integral of the following form

$$\Phi(L, L', \Pi, \Pi', l, l', \varpi, \varpi') = \text{const.},$$

where the function $\Phi$ depends in an arbitrary manner on $\varpi$ and on $\varpi'$ instead of depending on the difference $\varpi - \varpi'$.

To do so we must take the problem with 4 degrees of freedom, as we did in article 16.

We will then have eight conjugate variables

$$\begin{matrix} \beta L, & \beta' L', & \beta\Pi, & \beta'\Pi', \\ l, & l', & \varpi, & \varpi'. \end{matrix}$$

The coefficients $B_{m_1, m_2}$ and expressions (14) then depend on L, L', $\Pi$, $\Pi'$, and $\varpi'$. When we have given L and L' constant values such that the relation of mean motions is equal to $\frac{q}{p}$, expressions (14) will only depend on the four variables $\Pi$, $\Pi'$, $\varpi$ and $\varpi'$.

In order for there to be a uniform integral other than that of the vis viva, it is necessary that we have a relation among four arbitrarily chosen (2n-4=4, n=4) of the expressions (14); this is what occurs since all these expressions are only functions of three variables $\Pi$, $\Pi'$ and $\varpi - \varpi'$.

Therefore nothing opposes the fact that there exists an integral other than that of the vis viva, and in fact there does exist one, namely the areal integral.

In order for there to be two integrals, it would be necessary that there be a relationship among an arbitrarily chosen three of these expressions, that is to say, that all these expressions depend only on two of them. This is not the case.

Therefore, outside the vis viva integral and that of the area, the problem admits no other uniform integral.

Let us at last proceed to the most general case of the Problem of Three Bodies, and let us set the problem as in article 11, i.e., with 6 degrees of freedom and with the twelve variables: /253

$$\beta L,\ \beta G,\ \beta\theta,\ \beta' L',\ \beta' G',\ \beta'\theta'.$$
$$l,\ g,\ \theta,\ l',\ g',\ \theta'.$$

Expressions (14), after we have given L and L' proper constant values chosen as above, still depend on the eight variables G, G', $\Theta$, $\Theta'$, g, g', $\theta, \theta'$.

In order for these to be q uniform variables distinct from F, it would be necessary for there to be a relationship among 2n-3-2=9-q arbitrarily chosen of expressions (14).

It is easy to verify that these expressions depend only on five variables, namely on

$$G,\ G',\ g,\ g'$$

and on the angle of the planes of the two osculatory orbits.

There is therefore a relationship between an arbitrary 6=9-3 of the expressions (14).

Nothing is therefore in oppostion to the existence of three new integrals and they exist effectively: they are the integrals of area. But there is no relationship between an arbitrary (5=9-4) of expressions (14).

Therefore, the Problem of Three Bodies admits no other uniform integral than those of the vis viva and of area.

I have limited myself, in order not to interrupt the reasoning, to affirming that there exist no relationships among expressions (14); I will later return to this question.

It is known that Bruns has demonstrated (Acta mathematica, Vol. II) that the Problem of Three Bodies admits no new algebraic integral beyond the integrals already known.

The preceding theorem is more general in a sense than that of Bruns, because I demonstrate not only that there exists no algebraic integral, but that there exists not even a transcendental uniform integral, and not only that an integral cannot be uniform for all values of the variables, but that it cannot remain uniform even in a restrained domain defined above.

However, in another sense the theorem of Bruns is more general than mine; /2 I establish only, in effect, that there can exist no algebraic integral for all rather small values of the masses, and Bruns demonstrates that there exist none for any system of values of the masses.

## Problems of Dynamics Where There Exists A Uniform Integral

86. There are problems where we know the existence of a uniform integral and where we can propose to verify that the conditions stated in the preceding articles are effectively fulfilled.

Let us take as an example the problem of the motion of a moving point M, attracted by two fixed centers A and B.

I will assume, for simplification, that the motion occurs in a plane; I will assume in addition that the mass A is large, while that of B is equal to a very small quantity $\mu$, in such a manner that one may regard the attraction of B as a perturbing force.

We will then define the location of the point M by the osculating elements of its orbit about A, and we will designate these elements as the letters L, $\Pi$, $l$, $\varpi$, as in article 10. We will then have

$$F = \frac{1}{2L^2} + \frac{\mu}{MB}, \text{ whence } F_0 = \frac{1}{2L^2}, \quad F_1 = \frac{1}{MB};$$

$F_1$ can be developed in the following form

$$F_1 = \Sigma B_m e^{\sqrt{-1}\, m l}.$$

The coefficients $B_m$ then depend on L, $\Pi$, and $\varpi$, and in order for an integral to exist, it is necessary that there be a relationship between three arbitrary quantities of the coefficients of the same class (n-2, 2n-2=2; I say 2n-2 instead of 2n-4 because $F_0$ depends no longer on two variables $x_1$ and $x_2$, as in articles 84 and 85, but on one variable only), when we give L a value satisfying relation (12a).

However, here all coefficients $B_m$ (which have only one index) belong to the same class and one relation (12a) is written simply $m\,(dF_0/dL)=0$ where $L=\infty$. There could therefore be difficulty only for infinite values of L. If, therefore, we again take up the abbreviated language of the preceding articles, of L, $\Pi$, and $\omega$, but such that, for all these systems, the value of L is finite, the class of which all these coefficients B are part will not belong to the domain D; therefore nothing will oppose the existence of an integral which remains uniform in this domain D.

Let us proceed to another problem; that of the motion of a heavy body about fixed point.

This problem has been integrated in three different particular cases by uler, by Lagrange and by Mme. de Kowalevski (cf. Acta mathematica, 12). I beieve that Mme. de Kowalevski has discovered other new cases of integrability.

We can therefore ask if, in this problem, the considerations presented in his chapter oppose the existence of a uniform integral other than those of the is viva and of area.

I will assume that the product of the weight of the body by the distance of he center of gravity to the point of suspension is very small, such that we may rite the equations of the problem in the form

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

$$F = F_0 + \mu F_1.$$

alues $x_i$ and $y_i$ form three pairs of conjugate variables; F designates the total nergy of the system; $F_0$ is its semi-vis viva; $\mu$ is a very small quantity and $\mu G_1$ epresents the product of the weight of the body by the distance from the center f gravity to a horizontal plane passing through the point of suspension.

In the case where $\mu$ is zero (i.e., where the center of gravity coincides ith the point of suspension), the motion of the solid body reduces to a Poinsot otion. Since we assume $\mu$ very small, it is this Poinsot motion which will /256 erve us as first approximation, in the manner of Keplerian motion in the study f the Problem of Three Bodies by successive approximations.

I must, before continuing, define two quantities n and n', which I will call he two mean motions and which will play an important role in what follows. In he Poinsot motion, the ellipsoid of inertia rolls on a fixed plane; let P be the oot of the perpendicular lowered from the point of suspension onto this fixed lane and Q the point of contact. This point of contact belongs to a curve fixed ith respect to the ellipsoid and called the polhody. At the end of a certain ime T, the same point of the polhody returns to Q' in contact with the fixed plane. et $\alpha$ be the angle QPQ'. We will set

$$n = \frac{2\pi}{T}, \qquad n' = \frac{\alpha}{T}$$

nd n and n' will be the two mean motions.

This granted, we will be able to write the equations of Poinsot motion in the ollowing manner:

Let x, y and z be the coordinates of an arbitrary point of the solid body by aking the origin of the coordinates at the point of suspension and the axis of he z vertical.

Let us set

$$l = nt + \varepsilon, \qquad l' = n't + \varepsilon',$$

$\epsilon$ and $\epsilon'$ being two constants of integration.

Let $\xi$, $\eta$ and $\psi$ be three functions of n, n' and $\iota$, periodic of period 2r in $\iota$ (these functions, as is known, depend on the elliptic functions); let $\theta$ and $\varphi$ be two new constants of integration; we will have

$$\begin{aligned} x &= \cos\theta(\xi\cos l' - \eta\sin l') - \sin\theta\cos\varphi(\xi\sin l' + \eta\cos l') + \psi\sin\theta\sin\varphi, \\ y &= \sin\theta(\xi\cos l' - \eta\sin l') + \cos\theta\cos\varphi(\xi\sin l' + \eta\cos l') - \psi\cos\theta\sin\varphi, \\ z &= \sin\varphi(\xi\sin l' + \eta\cos l') + \psi\cos\varphi. \end{aligned}$$

If we assume that the point (x, y, z) is the center of gravity of the solid body, $F_1$ reduces within a constant factor to z, so that we will be able to write

$$F_1 = \Sigma B_{m.1} e^{\sqrt{-1}(ml+l')} + \Sigma B_{m.0} e^{\sqrt{-1}(ml)} + \Sigma B_{m.-1} e^{\sqrt{-1}(ml-l')},$$

the coefficients B depending only on n, n' and $\varphi$.

When we give n and n' constant values satisfying condition (12a), B will only depend on $\varphi$ such that there will be a relationship between two arbitrarily chosen from among them.

Values D depend only on $\varphi$ and $\zeta$ in setting, as in the preceding articles,

$$D_\lambda = B_{\lambda p.\lambda q}\zeta^\lambda.$$

There will therefore be a relationship between an arbitrary (2n-3=3) of the $D\lambda$. Every class will therefore be singular of the first order.

Nothing opposes the existence of a uniform integral distinct from that of the vis viva, and we know, in fact, that there exists one, namely that of area.

But the question is to learn whether a third can exist.

For this purpose, let us seek to learn the classes which are singular of the second order. To do this, it is necessary and sufficient that there be among three arbitrarily chosen of $D_\lambda$ two relationships, and consequently that all $D_\lambda$ be functions of only one of them. We will thus be led to distinguish several types of classes:

(1) The class $\frac{1}{0}$ which contains all coefficients $B_{m.0}$. This one is singular of the second order. We have in fact,

$$B_{m.0} = C_{m.0}\cos\varphi,$$

$C_{m.0}$ depending only on n and n' and consequently having to be regarded as constant since we assumed that we gave n and n' constant values. Then we have

$$D\lambda = C_{\lambda.0} \cos \rho \zeta \lambda.$$

In order for $D\lambda$ to be functions of only one of them, all $C_{\lambda.0}$ must vanish with one exception, or the function $\psi$ must reduce to an exponential

$$e^{\sqrt{-1}m.l}.$$

However, in order to satisfy condition (12a), it is necessary to give n the value 0; what is therefore the Poinsot motion for which n=0? A bit of attention shows that it is the one which corresponds to the uniform rotation about one of the axes of inertia. In a similar motion, the function $\psi$ is a constant independent of $l$. This proves that all $C_{\lambda.0}$ are zero for these particular values of n and of n', /258 with the exception of $C_{0.0}$.

The class is therefore singular of the second order.

(2) The classes of the form $\frac{m}{1}$ which contain only three coefficients

$$B_{m.1}, \quad B_{0.0}, \quad B_{-m.-1}.$$

These classes can be singular of the second order only if

$$B_{m.1} = B_{-m.-1} = 0$$

or, what comes back to the same thing, if in the development of $\xi+i\eta$ and of $\xi-i\eta$ in positive and negative powers of $e^{il}$, there are no terms in $e^{+mil}$ (assuming $\xi$ and $\eta$ real).

This will not happen, in general, when the ellipsoid of inertia is not one of revolution; but, if this ellipsoid is one of revolution, we will have

$$\xi = A\cos l + B\sin l + C, \qquad \eta = A'\cos l + B'\sin l + C',$$

A, B, C, A', B', C' being constants. The result of this will be that we have

$$B_{m.1} = -B_{-m.-1} = 0,$$

unless m=1, 0 or -1.

All classes $\frac{m}{1}$ will then be singular of the second order, with the exception of classes $\frac{1}{1}$, $\frac{0}{1}$ and $\frac{-1}{1}$.

(3) All other classes reducing to the single coefficient $B_{0.0}$ will be singular of the second order.

In summary, if the ellipsoid is one of revolution, all classes are singular of the second order, with the exception of classes $\frac{1}{1}$, $\frac{0}{1}$ and $\frac{-1}{1}$.

Therefore nothing opposes the existence of a third uniform integral and even that it be algebraic, provided that the Jacobian of the three integrals vanishes when we make $n'=0$ or $n'=\pm n$. (This last condition is not necessary in the case of Lagrange, that is, if the point of suspension is on the axis of revolution, because then $\xi$ and $\eta$ reduce to constants.)

If, on the contrary, the ellipsoid is not one of revolution, there is an /25
infinity of classes which are not singular of the second order, namely classes $\frac{m}{1}$; but let us consider a domain D containing an infinity of systems of values of n, n', $\varphi$ and $\theta$ and let us assume that for none of these systems is n' a multiple of n; none of the classes $\frac{m}{1}$ will belong to this domain. Therefore still nothing will oppose the existence of a third uniform integral, provided that the Jacobian of the three integrals vanishes when n' is a multiple of n; here the result is that this third integral cannot, in general, be algebraic.

The conditions stated in this chapter being necessary, but not sufficient, nothing proves that this third integral exists; it is advisable, before making a statement, to await the complete publication of Mme. de Kowalevski's results.[1]

## Nonholomorphic Integrals in $\mu$

87. Until now we have assumed that our uniform integral $\Phi$ was developable in integral powers of $\mu$. It is easy to extend the result to the case where we would abandon this hypothesis. Let us assume, for example, that $\Phi$ can be developed in integral powers of $\sqrt{\mu}$; we will be able to write

$$\Phi = \Phi' + \sqrt{\mu}\,\Phi'',$$

$\Phi'$ and $\Phi''$ being developable in integral powers of $\mu$.

If $\Phi$ is an integral, we must have identically

$$[F, \Phi] = [F, \Phi'] + \sqrt{\mu}[F, \Phi''] = 0.$$

Since $(F, \Phi')$ and $(F, \Phi'')$ can be developed in integral powers of $\mu$, we must have separately

$$[F, \Phi'] = [F, \Phi''] = 0.$$

Therefore $\Phi'$ and $\Phi''$ must both be integrals. /25

[1] Since these lines were written, the scientific world has had to mourn the premature death of Mme. de Kowalevski. Those of her notes which were found are unfortunately insufficient to permit reconstructing her demonstrations and calculations.

If, therefore, we have demonstrated that a uniform integral developable in integral powers of $\mu$ cannot exist we will have demonstrated that a uniform integral developable in integral powers of $\sqrt{\mu}$ cannot exist either.

More generally, let

$$\theta_1(\mu), \quad \theta_2(\mu), \quad \ldots, \quad \theta_p(\mu) \tag{1}$$

be p arbitrary functions of $\mu$.

Let us assume that $\Phi$ is of the form

$$\left.\begin{aligned}\Phi = \quad & A_{0.1}\theta_1(\mu) + A_{1.1}\mu\theta_1(\mu) + A_{2.1}\mu^2\theta_1(\mu) + \ldots \\ & + A_{0.2}\theta_2(\mu) + A_{1.2}\mu\theta_2(\mu) + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + A_{0.p}\theta_p(\mu) + A_{1.p}\mu\theta_p(\mu) + \ldots,\end{aligned}\right\} \tag{2}$$

values A being functions of x and y independent of $\mu$.

We can always assume that among the p functions (1) there are no relationships of the form

$$\varphi_1\theta_1 + \varphi_2\theta_2 + \ldots + \varphi_p\theta_p = 0, \tag{3}$$

$\varphi_1$, $\varphi_2$, ..., $\varphi_p$ being developable in powers of $\mu$. If this were in fact so, one of the functions $\varphi_1$, $\varphi_2$, ..., $\varphi_p$ will not contain $\mu$ as a factor; for, if all these functions contained $\mu$ as a factor, the first member of (3) would be divisible by $\mu$ and we would make the division.

Let us assume, for example, that $\varphi_1$ does not vanish with $\mu$; we will be able to solve equation (3) with respect to $\theta_1$ and we will have

$$\theta_1 = -\frac{\varphi_2}{\varphi_1}\theta_2 - \frac{\varphi_3}{\varphi_1}\theta_3 - \ldots - \frac{\varphi_p}{\varphi_1}\theta_p.$$

$\frac{\varphi_2}{\varphi_1}, \frac{\varphi_3}{\varphi_1}$, ... will be developable in powers of $\mu$, and if we replace $\theta_1$ by this value in expression (2), we will have reduced by unity the number of functions (1).

Let us therefore assume that these functions are not connected by a relation of form (3).

We will be able to write

$$\Phi = \Phi_1\theta_1 + \Phi_2\theta_2 + \ldots + \Phi_p\theta_p,$$

$\Phi_1, \Phi_2, \ldots, \Phi_p$ being developable in powers of $\mu$. If $\Phi$ is an integral, we will have /261

$$[F, \Phi] = \theta_1[F, \Phi_1] + \theta_2[F, \Phi_2] + \ldots + \theta_p[F, \Phi_p] = 0. \tag{4}$$

I say that we will have separately

$$[F, \Phi_1] = [F, \Phi_2] = \ldots = [F, \Phi_p] = 0. \tag{5}$$

For, if this were not so, as quantities $(F, \Phi_i)$ $(i=1,2, \ldots, p)$ are developable in powers of $\mu$, relation (4) would be of form (3), which is contrary to the hypothesis we have just made.

Therefore relations (5) hold true.

Therefore $\Phi_1, \Phi_2, \ldots, \Phi_p$ are integrals.

If, therefore, we have demonstrated that there cannot be a uniform integral developable in powers of $\mu$, we will have demonstrated that neither is there a uniform integral of form (2).

I will add that this reasoning applies when functions (1) are finite in number.

## Discussion of Expressions (14)

88. I return to the subject which I had reserved above, namely the demonstration of the fact that there exists no relationship between an arbitrary 2n-4 expressions of (14) in the case of the Problem of Three Bodies.

In order to define expressions (14), we have assumed that the perturbative function $F_1$ had been developed in the following form

$$F_1 = \Sigma B_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')}, \tag{1}$$

the coefficients $B_{m_1 m_2}$ being functions of the other variables

$$L, \quad L', \quad \Pi, \quad \Pi', \quad \varpi, \quad \varpi'$$

or

$$L, \quad L', \quad G, \quad G', \quad g, \quad g', \quad \theta, \quad \theta', \quad \theta, \quad \theta'.$$

It is not in this form that we ordinarily develop the perturbative function in treatises on Celestial Mechanics.

We take as variables: /262

The major axes, the eccentricities, the inclinations, the mean longitudes and the longitudes of the perihelions and of the nodes.

However, it is easy to see that this goes back to saying the same thing.

If we set

$$B_{m_1 m_2} = C_{m_1 m_2} e^{\sqrt{-1}(m_1 g + m_2 g' + m_1 \theta + m_2 \theta')},$$

it will follow that

$$F_1 = \Sigma C_{m_1 m_2} e^{\sqrt{-1}[m_1(l+g+\theta)+m_2(l'+g'+\theta')]}. \qquad (2)$$

the exponential factor depends only on the mean longitudes

$$l+g+\theta, \quad l'+g'+\theta'$$

and the factor $C_{m_1 m_2}$ depends only on the other variables, major axes, eccentricities, inclinations, longitudes of the perihelions and of the nodes. Thus we will in this way fall back to the usual development of the perturbative function.

Expressions (14) can then be written

$$B^{\lambda'}_{\lambda p.\lambda q} B^{-\lambda}_{\lambda' p.\lambda' q} = C^{\lambda'}_{\lambda p.\lambda q} C^{-\lambda}_{\lambda' p.\lambda' q}.$$

In order for there to be a uniform integral, it is therefore necessary that there be a relationship between an arbitrary 2n-4 (n=4 in the plane, n=6 in space) of expressions

$$C^{\lambda'}_{\lambda p.\lambda q} C^{-\lambda}_{\lambda' p.\lambda' q} \quad (\lambda, \lambda' = 0, \pm 1, \pm 2, \pm 3, \ldots, \text{ad inf.}) \qquad (14a)$$

formed by means of the coefficients of development (2).

Thus, in order to apply the principles of the present chapter, it is not necessary to make a new development of the perturbative function by means of new variables, as it would be in development (1). We can make use of the development already used by astronomers, that is, development (2).

The coefficients $C_{m_1 m_2}$ can be developed in increasing powers of the eccentricities and inclinations. Let us therefore consider the development of one of these coefficients in powers of the eccentricities and inclinations. We know (cf. article 12) that all terms of this development will be of the degree $|m_1+m_2|$ at least with respect to these quantities and, if their degree differs from $|m_1+m_2|$ the difference is an even number.

We will therefore be able to write

$$C_{m_1 m_2} = C^0_{m_1 m_2} + C^1_{m_1 m_2} + \ldots + C^p_{m_1 m_2} + \ldots,$$

$C^p_{m_1 m_2}$ representing the total of the terms of the development which are of the degree

$$|m_1 + m_2| + 2p$$

with respect to the eccentricities and inclinations.

We will say that $C^0_{m_1 m_2}$ is the principal term of $C_{m_1 m_2}$ and that the other terms are its secondary terms.

There will be an exception for coefficient $C_{00}$; we have, in this case,

$$C_{00} = C^0_{00} + C^1_{00} + \ldots .$$

$C^0_{00}$ depends only on the major axes, if these major axes are momentarily regarded as constants, as we have done in previous articles (it is, in fact, in assuming the major axes constant that the existence of a uniform integral implies that of a relationship between 2n-4 expressions (14)); if, therefore, the major axes are constants $C^0_{00}$ will also be a constant which will play no role whatsoever in the calculation.

It is therefore $C^1_{00}$ which is of the second degree with respect to the eccentricities and to the inclinations, which we will agree to call the principal term of $C_{00}$.

If, then, we replace development (2) by the following

$$C^0_{00} + C^1_{00} + \Sigma C^0_{m_1 m_2} e^{\sqrt{-1}[m_1(l+g+\theta) + m_2(l'+g'+\theta')]}, \qquad (3)$$

we will say that we have written the development of the perturbative function $F_1$ reduced to its principal terms.

This granted, what is the condition for which there be a relationship between an arbitrary 2n-4 of the expressions

$$C^{\lambda'}_{\lambda p.\lambda q} C^{-\lambda}_{\lambda' p.\lambda' q} \quad (\lambda, \lambda' = 0, \pm 1, \pm 2, \ldots). \qquad (14)$$

Let us form a table composed of an infinity of rows formed as follows:

The various lines will correspond to the various integral values of the index $\lambda$, positive, negative or zero.

The first element of the row with index $\lambda$ will be /26

$$\lambda C_{\lambda p.\lambda q},$$

the others will be the derivatives of $C_{\lambda p.\lambda q}$ with respect to the different variables

$$e,\ e',\ \varpi,\ \varpi',\ i,\ i',\ \theta,\ \theta',$$

that is, with respect to the eccentricities, longitudes of the perihelions, to the inclinations and to the node longitudes.

Thus, the necessary and sufficient condition for there to be a relationship between 2n-4=8 (n=6 in space) of relations (14) is that all determinants formed by taking nine arbitrary lines in this table be zero.

Needless to add, in the simplest cases, for example when the three bodies move in a plane, the number of columns and rows of these determinants is smaller than 9.

We have seen that all terms of the development of $C_{m_1 m_2}$ are of the degree $|m_1+m_2|$ at least. Therefore, among the elements of row of index λ (which I assume developed in powers of the eccentricities and of the inclinations), the first $\lambda C_{\lambda p.\lambda q}$ begin with terms of the degree

$$|\lambda p+\lambda q|.$$

The case is the same for derivatives $C_{\lambda p, \lambda q}$ with respect to $\varpi$ and to θ, while the derivatives of $C_{\lambda p, \lambda q}$ with respect to e and to i will begin with terms of the degree

$$|\lambda p+\lambda q|-1.$$

For the row index 0, the first term reduces to 0; the developments of the derivatives of $C_{00}$ with respect to $\varpi$ and to θ will begin with terms of the second degree, and those of the derivatives of $C_{00}$ with respect to e and to i will begin with terms of the first degree.

Our determinants are in turn capable of being developed in powers of e and i. If a determinant Δ is formed by the rows of indices

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\ \lambda_9,$$

all the terms of its development will then be at least of the degree /265

$$|p+q|\ (|\lambda_1|+|\lambda_2|+\ldots+|\lambda_8|+|\lambda_9|)-4.$$

I set this quantity equal to $\alpha$.

There is an exception in the case where $\lambda_0=0$; all terms are then at least of the degree

$$|p+q| \ (|\lambda_1|+|\lambda_2|+\ldots+|\lambda_8|)-2.$$

I will still set this quantity equal to $\alpha$.

The determinants $\Delta$ having to be identically zero, the total of the terms of degree $\alpha$ will also have to be identically zero. Now we will obtain these terms of degree $\alpha$, in replacing in the determinant $\Delta$ each of the coefficients $C_{\lambda p.\lambda q}$ by its principal $C^0_{\lambda p,\lambda q}$ (or $C^1_{0.0}$ if $\lambda=0$).

The determinant $\Delta_0$ thus obtained will therefore have to be identically zero; now what does this condition

$$\Delta_0=0$$

signify?

Let us form the expressions

$$(C^0_{\lambda p.\lambda q})^{\lambda'}(C^0_{\lambda' p.\lambda' q})^{-\lambda} \quad (\lambda, \lambda'=\pm 1, \pm 2, \ldots), \tag{14a}$$

obtained by replacing, in expressions (14), each coefficient C by its principal term.

If in expression (14) we make $\lambda=0$, this expression reduces to

$$C_{0.0}.$$

We will adjoin to the table of expressions (14a) the expression $C^1_{00}$ which is a polynomial integral of the second degree with respect to e and to i.

Thus, the condition $\Delta_0=0$ signifies that there is a relationship among an arbitrary eight of the expressions (14a) contained in the table thus completed.

Thus, in order for there to be a uniform integral, it is necessary that there be an integral relationship among an arbitrary of these expressions (14a).

The coefficients C were infinite series, and expressions (14) were presented in the form of the quotient of such series.

On the contrary, expressions (14a) are rational with respect to e, i, the sine and cosine of the $\varpi$ and of the $\theta$.

Verification is therefore facilitated by substitution in the coefficients of their principal terms.

It even becomes easy for small values of the two integers p and q.

When we thus have proved that the determinants corresponding to small values of the integers p and q are not zero, it becomes difficult to retain the illusion that the determinants corresponding to the large values of the same integers can vanish and thus permit the existence of a uniform integral.

A doubt might, nevertheless, still remain.

We could assume, however improbable it may seem, that among the classes (to use the language of article 84), there is a finite number of them which are ordinary and it is precisely these on which verification is based; but there are an infinity of them which are singular.

In order to completely erase this final doubt, it would be necessary to have a general expression of functions (14) or (14a) for all values of the integers $\lambda$, $\lambda'$, p and q and this expression could only be extremely complicated.

Happily, Flamme, in a recent thesis,[1] has given the approximate expression of the terms of increasing rank in the development of the perturbative function, and this approximate expression, much simpler than the complete expression, can suffice for our purpose.

Nevertheless, the form which Flamme has given it is not useful for the problem which concerns us; we will be obliged to complete his results and to transform them considerably.

I will, therefore, return to this topic in the next chapter, after having treated the approximate calculus of the various terms of the perturbative function form; although the preceding considerations are of a nature to convince the most skeptical, they do not, nevertheless, constitute a rigorous mathematical demonstration.

89. One last remark can, to a certain measure, facilitate verification.

Let us again take relations (13), from article 84, which is written

$$-B_{m_1 m_2}\left(m_1 \frac{d\Phi_0}{dx_1} + m_2 \frac{d\Phi_0}{dx_2}\right) + \sum\left(\frac{dB_{m_1 m_2}}{dz_i}\frac{d\Phi_0}{du_i} - \frac{dB_{m_1 m_2}}{du_i}\frac{d\Phi_0}{dz_i}\right) = 0.$$

In setting $m_1 = \lambda p$, $m_2 = \lambda q$ in this equation, I will obtain a particular relationship which I will call (13a); in setting $m_1 = \lambda' p$, $m_2 = \lambda' q$ in it, I will obtain another particular relation which I will call (13b).

Then let

$$M_{\lambda.\lambda'} = B^{\lambda'}_{\lambda p.\lambda q} B^{-\lambda}_{\lambda' p.\lambda' q};$$

[1] Paris, Gauthier-Villars, 1887.

$M_{\lambda.\lambda'}$ will be one of expressions (14) which have played such a large role in the preceding articles.

Let us multiply (13a) and (13b), respectively, by

$$\frac{\lambda'}{B_{\lambda p.\lambda q}} \text{ and } \frac{-\lambda}{B_{\lambda' p.\lambda' q}}$$

and add; it will follow that

$$\sum\left(\frac{d\log M_{\lambda.\lambda'}}{dz_i}\frac{d\Phi_0}{du_i}-\frac{d\log M_{\lambda.\lambda'}}{du_i}\frac{d\Phi_0}{dz_i}\right)=0,$$

or, in adopting the notation of the brackets of Jacobi,

$$[\log M_{\lambda.\lambda'}, \Phi_0]=0,$$

or

$$[M_{\lambda.\lambda'}, \Phi_0]=0.$$

If therefore M and M' are two expressions (14) belonging to the same class, we will have to have

$$[M, \Phi_0]=[M', \Phi_0]=0,$$

or, by virtue of the theorem of Poisson,

$$[[M, M'], \Phi_0]=0,$$

from which we can conclude that (M,M') is a function of 2n-4 of expressions (14). /268

It must not be forgotten that the brackets must be calculated while considering $x_1$ and $x_2$ (that is to say in the case of the Problem of Three Bodies, $\beta L$ and $\beta' L'$) as constants.

## CHAPTER 6. APPROXIMATE DEVELOPMENT OF THE PERTURBATIVE FUNCTION

### Statement of the Problem

90. I have said that Flamme had given a remarkable approximate expression /269 of the terms of higher order of the perturbative function. He was able to do so by applying to this problem the method of Darboux, which permits finding coefficients of high order in the Fourier series or in that of Taylor, when we know the analytic properties of the function represented by these series.

However, the method of Darboux is applicable only to the functions of one variable, whereas the perturbative function must be developed in terms of the sines and cosines of the multiples of the two mean anomalies. Thus we have here the procedure used by Flamme: he first obtains, by the usual process, a first development of the perturbative function whose terms are of the form

$$A\rho^{\alpha}e^{i(\beta v+\gamma u)},\quad \rho'^{\alpha'}e^{i(\beta' v'+\gamma' u')},$$

$\rho$ being the radius vector of the first planet, v being true anomaly, u being the eccentric anomaly; $\rho'$, $v'$, and $u'$ are analogous quantities for the second planet.

Then the two factors

$$\rho^{\alpha}e^{i(\beta v+\gamma u)} \text{ and } \rho'^{\alpha'}e^{i(\beta' v'+\gamma' u')}$$

depend only on a single variable, namely: the first on the mean anomaly $\zeta$ of the first planet, the second on the other mean anomaly $\zeta'$. Flamme applies to each of these two factors the method of Darboux.

This artifice cannot suffice for our purpose; we must, on the contrary, apply the method of Darboux directly to the perturbative function and to do so we /270 must extend this method to the case of the functions of two variables.

91. The function which we must develop is the one we have called $F_1$ and of which I am going to recall the expression by returning to the notations of article 11.

We then have

$$F=\frac{y_1^2+y_2^2+y_3^2}{2\beta}+\frac{y_4^2+y_5^2+y_6^2}{2\beta'}-\frac{m_2m_3}{\mu BC}-\frac{m_1m_3}{\mu AC}-\frac{m_1m_2}{\mu AB}$$

The function F thus defined depends on variables (4) from article 11, on $m_1$, $m_2$, $m_3$ and $\mu$. If we assume that $m_1$, $m_2$ and $m_3$ are known functions of the parameter $\mu$ and can be developed in increasing powers of this parameter, F will depend only on variables (4) and on $\mu$, and will be developable in increasing powers of $\mu$.

This can occur in an infinity of ways; we may assume, for example, that $m_1$, $\beta$ and $\beta'$ are constants independent of $\mu$.

The variables (4) are the Keplerian variables relative to two osculatory orbits defined in article 11. The radius vector in the first osculatory orbit is AB, in the second orbit the radius vector is CD. The angle between these two radii (which is nothing other than the difference in true longitudes of the two osculatory orbits if these two orbits are in the same plane) is the angle BDC, which I will call simply D.

The quantities $(y_1^2+y_2^2+y_3^2)$, $(y_4^2+y_5^2+y_6^2)$ and AB depend only on variables (4), not on $\mu$. On the other hand, $\alpha_2$, $\alpha_3$, AC and BC depend not only on the variables (4) but also on $\mu$. We can therefore propose to develop $\alpha_2$, $\alpha_3$, $\frac{1}{AC}$, $\frac{1}{BC}$ in powers of $\mu$. We thus find

$$\alpha_2 = \beta + \frac{\mu\beta^2}{m_1} + \text{terms divisible by } \mu^2,$$

$$\alpha_3 = \beta' + \frac{\mu\beta'^2}{m_1} + \text{terms divisible by } \mu^2,$$

$$\frac{1}{BC} = \frac{1}{\sqrt{AB^2+CD^2-2AB.CD\cos D}} + \text{terms divisible by } \mu^2,$$

$$\frac{1}{AC} = \frac{1}{CD} - \frac{\beta\mu}{m_1}\frac{AB\cos D}{CD^2} + \text{terms divisible by } \mu^2.$$

If we then set

$$F = F_0 + \mu F_1 + \dots,$$

it follows that

$$F_0 = \frac{y_1^2+y_2^2+y_3^2}{2\beta} + \frac{y_4^2+y_5^2+y_6^2}{2\beta'} - \frac{\beta'_3 m_1}{CD} - \frac{\beta_2 m_1}{AB},$$

$$F_1 = -\frac{\beta^2}{AB} - \frac{\beta'^2}{CD} - \frac{\beta\beta'}{\sqrt{AB^2+CD^2-2AB.CD.\cos D}} + \frac{\beta\beta' AB\cos D}{CD^2}.$$

Let us consider successively the various terms of the perturbative function $F_1$.

At the outset the first term

$$-\frac{\beta^2}{AB} = -\frac{\beta^2}{r}$$

depends only on the mean anomaly $l$ and not at all on the mean anomaly $l'$; it therefore cannot in the final development give us the terms in

$$\sin(ml + nl') \quad \text{or} \quad \cos ml + nl'),$$

where $n \gtrless 0$.

In the same way, the second term

$$-\frac{\beta'^2}{CD} = -\frac{\beta'^2}{r'}$$

cannot in the final development give us the terms in

$$\sin(ml + nl') \quad \text{or} \quad \cos(ml + nl'),$$

where $m \gtrless 0$.

We will therefore in general be able to set aside these first two terms.

The last term

$$\frac{\beta\beta' AB \cos D}{CD^2}$$

can be placed in another form. If I designate by i the inclination of the orbits and by v and v' the true longitudes measured from the node, I have

$$\cos D = \cos v \cos v' + \cos i \sin v \sin v',$$

whence /272

$$\frac{AB \cos D}{CD^2} = (AB \cos v)\frac{\cos v'}{CD^2} + \cos i (AB \sin v)\frac{\sin v'}{CD^2}.$$

The method of Flamme is directly applicable to the four factors

$$AB \cos v, \quad \frac{\cos v'}{CD^2}, \quad AB \sin v, \quad \frac{\sin v'}{CD^2}.$$

It therefore remains to develop the third term

$$F_1^0 = \frac{\beta\beta'}{\sqrt{AB^2 + CD^2 - 2AB.CD.\cos D}},$$

which is known under the name of the principal part of the perturbative function. We shall now concern ourselves with the development of this principal part.

Digression on a Property of the Perturbative Function

92. We might be tempted to avoid the necessity of developing the principal part of the perturbative function by employing the following artifice.

We have found

$$F_1 = -\frac{\beta^2}{r} - \frac{\beta'^2}{r'} + \beta\beta' F_1^0 + \frac{\beta\beta' r \cos\omega}{r'^2},$$

through designating by r and r' the two radii vectors and by $\omega$ the angle of these two radii vectors.

In order to arrive at this result, we have taken, as in article 11, for osculatory orbits the orbit of B with respect to A and that of C with respect to D. the center of gravity of A and of B.

However, it is clear that we could equally have chosen as osculatory orbits that of C with respect to A and that of B with respect to E, the center of gravity of A and of C.

This goes back to permuting two planets B and C; we could therefore have found in this way as a new perturbative function,

$$F_1' = -\frac{\beta^2}{r} - \frac{\beta'^2}{r'} + \beta\beta' F_1^0 + \frac{\beta\beta' r' \cos\omega}{r^2},$$

whence /2

$$F_1' - F_1 = \beta\beta'\left(\frac{r'\cos\omega}{r^2} - \frac{r\cos\omega}{r'^2}\right).$$

If there exists an integral

$$\Phi = \text{const.},$$

we will be able to write, taking the osculatory elements of the first two orbits [variables (4) of article 11], as variables and we will thus have

$$\Phi_0 + \mu\Phi_1 + \ldots = \text{const.}$$

We will also be able to write it by taking for variables the osculatory elements of the two new orbits (orbits of C with respect to A and of B with respect to E); we will then have

$$\Phi_0' + \mu\Phi_1' + \ldots = \text{const.}$$

$\Phi_0'$ will be formed of the elements of the two new orbits as $\Phi_0$ was formed from the corresponding elements of the two former orbits, but $\Phi_1'$ will not be formed as $\Phi_1$.

We must then have, as we have seen in article 81,

$$[\Phi_0, F_1] + [\Phi_1, F_0] = 0,$$

and similarly

$$[\Phi_0', F_1'] + [\Phi_1', F_0] = 0;$$

since $\Phi_0'$ is formed in the same way as $\Phi_0$, I can suppress the prime and write

$$[\Phi_0, F_1'] + [\Phi_1', F_0] = 0,$$

whence

$$[\Phi_0, F_1' - F_1] + [\Phi_1' - \Phi_1, F_0] = 0. \quad (1)$$

We have seen that if there exists a uniform integral and if, after having developed $F_1$, we form the expressions (14) of article 84, a certain number of relationships must hold between these expressions.

However, in reasoning from equation (1), as we have done from equation (3) of article 81, we would arrive at an analogous result. Let us develop $F_1'-F_1$ and let us by means of this development form expressions (14); if there exists a uniform integral, a certain number of relationships must exist among these expressions.

If, therefore we could establish that these relationships do not exist we would have demonstrated that there can no longer exist any more uniform integrals. Because the development of $F_1'-F_1$ is incomparably easier than that of $F_1$, it seems that this process must considerably simplify our task. /274

However, this is so artificial that a priori we conceive doubts concerning its usefulness and we wonder if it is not illusory. This is in fact so, because expressions (14), formed by means of $F_1'-F_1$, are either zero or indeterminate.

Let us assume that we develop $F_1'-F_1$ in the following form:

$$F_1' - F_1 = \Sigma B_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')}.$$

The coefficients $B_{m_1 m_2}$ will be functions of $\beta L$, $\beta' L'$ and of other osculatory elements ( $l$ and $l'$ excepted). Let us give L and L' values such that

$$m_1 n + m_2 n' = 0$$

(calling n and n' the mean motions).

I say that for these values of L and L' the coefficient $B_{m_1 m_2}$ will vanish.

To show this, I shall make use of the following lemma.

Let

$$x_1, x_2, \ldots, x_n; \quad y_1, y_2, \ldots, y_n \tag{2}$$

be a system of variables, conjugate in pairs; let

$$x'_1, x'_2, \ldots, x'_n; \quad y'_1, y'_2, \ldots, y'_n \tag{3}$$

be another system of conjugate variables. Let us assume that these two systems are connected by relationships such that we may pass from one to the other without altering the canonical form of the equations. We must then, according to article 5, have

$$\Sigma(dx_i \delta y_i - dy_i \delta x_i) = \Sigma(dx'_i \delta y'_i - dy'_i \delta x'_i). \tag{4}$$

Let us assume that values $x'_i$ and the $y'_i$ depend on a certain parameter $\mu$ and are developable in terms of the powers of $\mu$; also for $\mu=0$, $x'_i$ and $y'_i$ reduce to $x_i$ and $y_i$.

We will then have

/2

$$\left.\begin{array}{l} x'_i = x_i + \mu \xi_i + \ldots, \\ y'_i = y_i + \mu \eta_i + \ldots, \end{array}\right\} \tag{5}$$

$\xi_i$ and $\eta_i$ being functions of $x_i$ and $y_i$.

Then the expression

$$\Sigma(\xi_i dy_i - \eta_i dx_i) = dS$$

will be an exact differential. This is a necessary consequence of identity (4), which obviously implies the following:

$$\Sigma(d\xi_i \delta y_i - dy_i \delta \xi_i + dx_i \delta \eta_i - d\eta_i \delta x_i) = 0.$$

Let us now consider the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

where

$$F = F_0(x_i, y_i) + \mu F_1(x_i, y_i) + \ldots.$$

Let us change variables and take the variables (3) as new variables; it will follow that

$$F = F'_0(x'_i, y'_i) + \mu F'_1(x'_i, y'_i) + \ldots.$$

If we replace $x_i'$ and $y_i'$ by their values (5), it will follow that

$$F_0'(x_i', y_i') = F_0'(x_i, y_i) + \mu \Sigma \left( \frac{dF_0'}{dx_i} \xi_i + \frac{dF_0'}{dy_i} \eta_i \right) + \text{terms divisible by } \mu^2,$$

$$F_1'(x_i', y_i') = F_1'(x_i, y_i) + \text{terms divisible by } \mu,$$

hence, identifying the two developments,

$$F_0(x_i, y_i) = F_0'(x_i, y_i),$$

$$F_1(x_i, y_i) = F_1'(x_i, y_i) + \sum_i \left( \frac{dF_0'}{dx_i} \xi_i + \frac{dF_0'}{dy_i} \eta_i \right).$$

If we observe that $F_0(x_i, y_i) = F_0'(x_i, y_i)$ and that

$$\xi_i = \frac{dS}{dy_i}, \qquad \eta_i = -\frac{dS}{dx_i},$$

we will be able to write

$$F_1 - F_1' = [F_0, S]. \tag{6}$$

Let us assume that $F_0$ depends only on two variables $x_1$ and $x_2$ and that $F_1$, /276
$F_2$ are periodic of period $2\pi$ with respect to $y_1$ and $y_2$. This is what happened in all the problems which we have dealt with up until now.

Let us also assume that S is periodic in $y_1$ and $y_2$ and let

$$S = \Sigma A e^{\sqrt{-1}(m_1 y_1 + m_2 y_2)},$$

A depending on $x_1, x_2, \ldots, x_n; y_3, y_4, \ldots, y_n$.

Let us assume that we wish to develop $F_1'$ and $F_1 - F_1'$ in the same form, and let

$$F_1 - F_1' = \Sigma B e^{\sqrt{-1}(m_1 y_1 + m_2 y_2)}.$$

Equation (6) shows that

$$B = \sqrt{-1} A \left( m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} \right).$$

If therefore we give $x_1$ and $x_2$ values such that

$$m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} = 0,$$

we will similarly have

$$B = 0.$$

Let us apply this result to the case which concerns us.

Let

$$\left.\begin{matrix} \beta L, & \beta G, & \beta\theta, & \beta' L', & \beta' G', & \beta'\theta'; \\ l, & g, & \theta, & l', & g', & \theta' \end{matrix}\right\} \tag{7}$$

be the variables (4) of article 11 relative to the two former osculatory orbits, B with respect to A, C with respect to D.

Let

$$\left.\begin{matrix} \beta_1 L_1, & \beta_1 G_1, & \beta_1\theta_1, & \beta_1' L_1', & \beta_1' G_1', & \beta_1'\theta_1'; \\ l_1, & g_1, & \theta_1, & l_1', & g_1', & \theta_1' \end{matrix}\right\} \tag{8}$$

be the variables (4) of article 11 relative to the two new orbits (B with respect to E, C with respect to A).

These variables (8) may replace the variables (7) without altering the canonical form of the equations; they will depend on the variables (7) and on $\mu$; they will be developable in terms of the powers of $\mu$; they will reduce to the variables (7) for $\mu=0$.

We will therefore find the conditions where the preceding result is applicable and we must conclude that, if we set

$$F_1' - F_1 = \Sigma B_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_1' l')},$$

$B_{m_1 m_2}$ vanishes for

$$m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} = 0.$$

This result can be directly verified without difficulty. Let us in fact refer to the expressions given by Tisserand in his Mécanique céleste (Vol. I, p. 312).

The result which must be verified, translated into the notations of Tisserand, can be stated thus (I recall that Tisserand designates by $\sigma$ the cosine of the angle between the two radii vectors.

If we set

$$\sigma\left(\frac{r}{r'^2} - \frac{r'}{r^2}\right) = \Sigma B_{n,n'} e^{\sqrt{-1}(n\zeta + n'\zeta')},$$

$B_{n,n'}$ vanishes for

$$\frac{n}{a^{\frac{3}{2}}} + \frac{n'}{a'^{\frac{3}{2}}} = 0;$$

and, in fact, in returning to the expressions of the page which I have just cited, we find

$$B_{n,n'} = \left(\frac{n'a}{na'^2} - \frac{na'}{n'a^2}\right)C,$$

C depending only on the eccentricities, inclinations and longitudes of the perihelions and of the nodes; this expression will therefore vanish for

$$\frac{n^2}{a^3} = \frac{n'^2}{a'^3},$$

and consequently for

$$\frac{n}{a^{\frac{3}{2}}} + \frac{n'}{a'^{\frac{3}{2}}} = 0.$$

Q. E. D.

I have nevertheless thought it necessary to connect this theorem with a more general theorem, which perhaps will permit us to discover other analogous positions. /278

## Principles of the Method of Darboux

93. After this digression, I resume my principal subject. It is first proper to recall the results of Darboux, which must serve us as a point of departure.

(1) Let there be a series

$$\varphi(x) = \Sigma a_n x^n,$$

admitting r as radius of convergence.

We will have, when n increases indefinitely,

$$\lim a_n \rho^n = 0 \quad \text{si } \rho < r,$$
$$\lim a_n \rho^n = \infty \quad \text{si } \rho > r.$$

(2) Let us consider now that the function

$$\varphi(x) = \Sigma a_n x^n$$

remains finite on a circumference of radius r, as do its first p derivatives; the product $n^{p+1} a_n r^n$ will not increase beyond all limits when n increases.

(3) If we have

$$\varphi(x) = (1 - 2x)^k = \Sigma a_n x^n,$$

$$a_n = \frac{n^{1-k} 2^n}{\Gamma(-k)} : \qquad (1)$$

[illegible] that the ratio of the two members of the equality (1) will tend [illegible] increases indefinitely.

(4) [illegible] now consider that the function $\varphi(x)$ has two singular points [illegible] circumference of radius r; that in the neighborhood of point [illegible]

$$\varphi(x) = A_1\left(1 - \frac{x}{\alpha}\right)^{\gamma_1} + A_2\left(1 - \frac{x}{\alpha}\right)^{\gamma_2} + \ldots + A_h\left(1 - \frac{x}{\alpha}\right)^{\gamma_h} + \psi(x)$$

and in the neighborhood of point $\beta$ /279

$$\varphi(x) = B_1\left(1 - \frac{x}{\beta}\right)^{\delta_1} + B_2\left(1 - \frac{x}{\beta}\right)^{\delta_2} + \ldots + B_k\left(1 - \frac{x}{\beta}\right)^{\delta_k} + \psi_1(x),$$

$\psi(x)$ and $\psi_1(x)$ as well as their first p derivatives remaining fixed. It will then follow that for $n = \infty$

$$\lim n^{p+1} r^n \left[a_n - \Sigma A_i \frac{n^{1-\gamma_i}}{\alpha^{\gamma_i}} \frac{1}{\Gamma(-\gamma_i)} - \Sigma B_i \frac{n^{1-\delta_i}}{\alpha^{\delta_i}} \frac{1}{\Gamma(-\delta_i)}\right] = 0,$$

from which we deduce the approximate values of $a_n$.

(5) If we have

$$\varphi(x) = \log(1 - x),$$

we will have

$$a_n = -\frac{1}{n};$$

if

$$\varphi(x) = \log(1 - x)(1 - x)^k,$$

we will have approximately

$$a_n = \frac{-n^{1-k} \log n}{\Gamma(-k)}.$$

This last formula is applicable only if k is not a positive integer; in this case, we would have

(6) Let

$$a_n = \frac{(-1)^{k+1} k!}{n(n-1)-(n-k)}.$$

$$\varphi(x) = \Sigma a_n x^n + \Sigma a_{-n} x^{-n};$$

be a series containing positive and negative powers which is convergent provided that

$$|x| < R \quad |x| > r.$$

Let $\alpha$ and $\beta$ be two singular points of the function $\varphi(x)$ situated on the circumference $|x|=R$; let $\gamma$ and $\delta$ be two singular points of $\varphi(x)$ on the circumference $|x|=r$. Let us assume that $\varphi(x)$ has no other singular point on these two circumferences.

Let

$$\psi(x) = \Sigma b_n x^n, \qquad \psi_1(x) = \Sigma c_n x^n$$

be two convergent series for /280

$$|x| < R.$$

Let

$$\psi_2(x) = \Sigma b_{-n} x^{-n}, \qquad \psi_3(x) = \Sigma c_{-n} x^{-n}$$

be two convergent series for

$$|x| > r.$$

If the differences $\varphi-\psi, \varphi-\psi_1, \varphi-\psi_2, \varphi-\psi_3$ as well as their first p derivatives are finite, the first in the neighborhood of point $x=\alpha$, the second in the domain of point $x=\beta$, the third in that of point $x=\gamma$, the fourth when x is near $\delta$, we will have

$$\left.\begin{aligned} \lim n^{p+1} R^n (a_n - b_n - c_n) = 0 \\ \lim n^{p+1} r^{-n} (a_{-n} - b_{-n} - c_{-n}) = 0 \end{aligned}\right\} \text{ for } n=\infty).$$

The approximate values of the coefficients $a_n$ therefore depend solely on the singularities which the function $\varphi(x)$ presents on the circumferences of its circle of convergence.

## Extension to Functions of Several Variables

94. Let us apply these principles to the case which concerns us.

It is a question of developing a certain function $F_1^0$ of the two mean anomalies $l$ and $l'$ in the following form:

$$F_1^0 = \Sigma A_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')};$$

We therefore have

$$4\pi^2 A_{m_1 m_2} = \int_0^{2\pi} \int_0^{2\pi} F_1^0 e^{-\sqrt{-1}(m_1 l + m_2 l')} dl\, dl'.$$

It is a matter of finding an approximate value of the coefficient $A_{m_1 m_2}$ when, the ratio $\frac{m_1}{m_2}$ being given and finite, the two numbers $m_1$ and $m_2$ are very great or more generally when we have

$$m_1 = an + b, \qquad m_2 = cn + d,$$

a, b, c, d being finite integers and n a very large integer; a and c are first among these. /23

If I then say that we have approximately

$$A_{m_1 m_2} = \varphi(n), \qquad (m_1 = an + b,\ m_2 = cn + d),$$

this equality will signify that the ratio

$$\frac{A_{m_1 m_2}}{\varphi(n)}$$

tends toward unity when n increases indefinitely and a, b, c, d remain finite.

The problem to be resolved being thus defined, I will make use of the following notations.

Let us set

$$e^{\sqrt{-1}\, l} = t^c, \qquad e^{\sqrt{-1}\, l'} = t^{-a} z^{\frac{1}{c}};$$

it will follow that

$$F_1^0 = \Sigma A_{m_1 m_2} t^{m_1 c - m_2 a} z^{\frac{m_2}{c}}.$$

If we then set for brevity

$$F(z, t) = F_1^0 t^{ad - bc - 1} z^{-\frac{d}{c}},$$

it will follow that

$$F(z, t) = \Sigma A_{m_1 m_2} t^{\alpha} z^{\frac{m_2 - d}{c}};$$

in setting for brevity

$$\alpha = m_1 c - m_2 a + ad - bc - 1.$$

Now let

$$\Phi(z) = \frac{1}{2i\pi}\int F(z, t)\,dt,$$

the integral being taken with respect to t along the circumference $|t|=1$. We will have

$$\Phi(z) = \sum \frac{A_{m_1 m_2}}{2i\pi} z^{\frac{m_2-d}{c}} \int t^{\alpha}\,dt.$$

All the integrals are zero except those for which $\alpha=-1$ and which are equal to $2i\pi$.

If $\alpha=-1$, we will have /282

$$m_1 = an + b, \qquad m_2 = cn + d, \qquad \frac{m_2 - d}{c} = n.$$

It then follows that

$$\Phi(z) = \Sigma A_{m_1 m_2} z^n.$$

If therefore we develop $\Phi(z)$ in the form

$$\Phi(z) = \Sigma a_n z^n + \Sigma a_{-n} z^{-n},$$

the coefficient $A_{m_1 m_2}$ will be nothing other than $a_n$ if $m_1=an+b$, $m_2=cn+d$.

We are therefore led to seek the approximate expression of $a_n$ for very large n and consequently to study the singularities of the function $\Phi(z)$.

95. The function $\Phi(z)$ is defined as an integral taken with respect to t along the circumference $|t|=1$. We can replace this circumference by an arbitrary contour C, but on one condition.

Let us for a moment regard z as a constant and $F(z,t)$ as a function of t. This function will admit a certain number of singular points.

It is necessary that between the circumference $|t|=1$ and contour C there be none of these singular points.

Let us now vary z in a continuous manner; these singular points will be displaced in a continuous manner. If at the same time we deform contour C in a continuous manner such that it passes through no singular point, function $\Phi(z)$ will remain holomorphic.

The function $\Phi(z)$ can therefore cease to be continuous only if it becomes impossible to deform contour C such that it does not pass through a singular point. This is how this can occur: let us consider that for a certain value of z we have two singular points $\alpha$ and $\beta$, one outside and the other inside the contour C. If, in varying z in a continuous manner, one of them, $\alpha$ for example, comes near contour C, we will be able to deform C by making it recede, so to say, before this singular moving point, in such a manner that this point $\alpha$ can never /28
attain this contour. Thus $\alpha$ will always remain outside C and $\beta$ inside C. However, let us now assume that $\alpha$ and $\beta$ approach each other indefinitely; contour C, taken, so to say, between two fires, will no longer be able to recede before these two moving points and the function $\Phi(z)$ will no longer be holomorphic.

Consequently, in order to obtain all singular points of $\Phi(z)$, it suffices to express that two of these singular points of $F(z,t)$ considered as a function of t will merge into only one.

The series

$$\Phi(z) = \Sigma a_n z^n + \Sigma a_{-n} z^n$$

will be convergent in a region bounded by two circumferences

$$|z| = R, \quad |z| = r;$$

these two circumferences will pass through one or several of the singular points which I have just defined.

However, if we wish to know which of these singular points are on these circumferences and which consequently define the limits of convergence of our series, a deeper discussion is necessary.

All the singular points will in fact not be expedient for the problem. This is true for several reasons.

In the first place, the function $F(z,t)$ is not uniform; if two singular points $\alpha$ and $\beta$ of this function F considered as a function of t coincide for a certain value of z, it is necessary, for this value to be a true singular point of $\Phi(z)$, that $\alpha$ and $\beta$ belong to the same determination of F and in addition that this determination be the same as that which figures in the integral

$$\frac{1}{2i\pi}\int F\,dt,$$

which, taken along C, defines function $\Phi$.

It is also necessary that these two points $\alpha$ and $\beta$ not be on the same side of contour C before merging into one.

Let H be a path traced in the plane of z and going from a point $z_0$ of modulus 1 to singular points $z_1$ defined above. Let us assume that we follow this path /2 from $z_0$ to $z_1$ and study the variations of $\Phi(z)$, taking for an initial value

$$\Phi(z_0) = \Sigma a_n z_0^n + \Sigma a_{-n} z_0^{-n}.$$

Although function $\Phi(z)$ may not be and generally is not uniform, the particular determination of $\Phi(z)$ which we have seen is thus entirely defined, since we are given the initial value and the path traveled.

It is then a matter of knowing if point $z_1$ is truly a singular point for this particular determination of $\Phi(z)$.

The function $F(z,t)$ not being uniform, it is necessary to vary t not on a plane but on a Riemann surface S possessing as many sheets as function F possesses determinations (this number can be infinite).

When z varies along the path H, the singular points will be displaced and the Riemann surface S will be deformed.

It is on this Riemann surface that we must assume that contour C is traced.

For $z=z_0$ this contour will reduce to the circle $|t|=1$ traced on one of the sheets of S; when the surface S is deformed, we must deform contour C as well such that no singular point will ever be found upon it. A special even delicate discussion will then show if, for a value of z very near $z_1$, the two singular points of $F(z,t)$ which merge for $z=z_1$ are separated by contour C, which is the necessary and sufficient condition for point $z=z_1$ to be a singular point for the particular determination of $\Phi(z)$ under consideration.

How can we now recognize if point $z_1$ is found on one of the circumferences

$$|z| = R, \quad |z| = r$$

which bound the domain of convergence of the series

$$\Sigma a_n z^n + \Sigma a_{-n} z^{-n},$$

and if consequently it is one of those on which the approximate value we seek depends?

Let us trace path H going from point $z_0$ of modulus 1 to point $z_1$ such that the modulus of z varies constantly in the same sense. If point $z_1$ belongs to one of our two circumferences, it must be a singular point for the determination of $\Phi(z)$ defined by path H, and which we will recognize by the method just explained. /285

If point $z_1$ satisfies this condition, I will say that this singular point is admissible.

This granted, among all the admissible singular modulus points greater than 1, those which lie upon the circumference $|z|=R$ will be of smallest modulus.

Similarly, among all admissible singular modulus points smaller than 1, those which lie upon the circumference $|z|=r$ will be of greatest modulus.

In conclusion, I will add that function $\Phi(z)$ possesses several determinations which interchange either when two of the determinations of $F(z,t)$ are interchanged or when two of the singular points of $F(z,t)$ rotate about each other.

I shall first seek to determine the singular points of $\Phi(z)$; I will then determine by special discussion those which are apposite to the question.

Investigation of Singular Points

96. Let us restrict ourselves to the case where the motion occurs in a plane.

Let u and u' be the eccentric anomalies, $\sin\varphi$ and $\sin\varphi'$ the eccentricities, $L^2$ and $L'^2$ the major axes, $\varpi$ and $\varpi'$ the longitudes of the perihelions.

We will have

$$l = u - \sin\varphi \sin u, \qquad l' = u' - \sin\varphi' \sin u'.$$

The coordinates of the first planet with respect to the major axis of its ellipse and to a perpendicular passing thru the focus will be

$$L^2(\cos u - \sin\varphi) \quad \text{and} \quad L^2 \cos\varphi \sin u:$$

these will therefore be the real and imaginary parts of $\xi L^2$. If we set

$$\xi = \cos u - \sin\varphi + \sqrt{-1}\cos\varphi \sin u.$$

If we set as well /280

$$\eta = \cos u' - \sin\varphi' + \sqrt{-1}\cos\varphi' \sin u',$$

the coordinates of the second planet, related to the same axes as the first, will be the real and imaginary parts of

$$\eta L'^2 e^{\sqrt{-1}(\varpi' - \varpi)}.$$

Let

$$\beta = L'^2 L^{-2} e^{\sqrt{-1}(\varpi' - \varpi)},$$

let

$$\xi_0 = \cos u - \sin\varphi - \sqrt{-1}\cos\varphi \sin u,$$

$$\eta_0 = \cos u' - \sin\varphi' - \sqrt{-1}\cos\varphi'\sin u',$$
$$\beta_0 = L'^2 L^{-2} e^{-\sqrt{-1}(\varpi'-\varpi)};$$

it will follow that

$$L^2 F_1^0 = \frac{1}{\sqrt{(\xi - \beta\eta)(\xi_0 - \beta_0\eta_0)}}.$$

The singular points of F(z,t) are the same as those of $F_1^0$, because F(z,t) differs from $F_1^0$ only by a power of t and the point t=0, which in addition will not enter the discussion, is already a singular point of $F_1^0$.

The singular points of $F_1^0$ will be those for which u and u' and consequently $\xi, \eta, \xi_0, \eta_0$, cease being uniform functions of i and i' and consequently of z and of t, and in addition those for which

$$\xi = \beta\eta \quad \text{or} \quad \xi_0 = \beta_0\eta_0.$$

I am going to set

$$x = e^{iu}, \quad y = e^{iu'},$$

whence

$$\cos u = \frac{1}{2}\left(x + \frac{1}{x}\right), \quad \cos u' = \frac{1}{2}\left(y + \frac{1}{y}\right),$$
$$i\sin u = \frac{1}{2}\left(x - \frac{1}{x}\right), \quad i\sin u' = \frac{1}{2}\left(y - \frac{1}{y}\right).$$

From this we will deduce

$$l = u - \frac{\sin\varphi}{2i}\left(x - \frac{1}{x}\right), \quad l' = u' - \frac{\sin\varphi'}{2i}\left(y - \frac{1}{y}\right)$$

and /287

$$e^{il} = x e^{\frac{\sin\varphi}{2}\left(\frac{1}{x} - x\right)}, \quad e^{il'} = y e^{\frac{\sin\varphi'}{2}\left(\frac{1}{y} - y\right)}.$$

We will then have

$$t = e^{\frac{il}{c}} = x^{\frac{1}{c}} e^{\frac{\sin\varphi}{2c}\left(\frac{1}{x} - x\right)}, \quad z = e^{icl'} t^{ca} = y^c x^a e^{\omega},$$

in setting, for brevity,

$$\omega = \frac{a\sin\varphi}{2}\left(\frac{1}{x} - x\right) + \frac{c\sin\varphi'}{2}\left(\frac{1}{y} - y\right).$$

We will have in addition

$$\xi = \frac{1}{2}\left(x + \frac{1}{x}\right) - \sin\varphi + \frac{\cos\varphi}{2}\left(x - \frac{1}{x}\right).$$
$$\eta = \frac{1}{2}\left(y + \frac{1}{y}\right) - \sin\varphi' + \frac{\cos\varphi'}{2}\left(y - \frac{1}{y}\right).$$

The singular points of F(z,t) are given us by

$$\frac{dl}{du} = 1 - \sin\varphi \cos u = 0,$$
$$\frac{dl'}{du'} = 1 - \sin\varphi' \cos u' = 0;$$
$$H = \xi - \beta \eta = 0,$$
$$H_0 = \xi_0 - \beta_0 \eta_0 = 0.$$

We can transcribe these equations by making use of the variables x and y; they then become algebraic; the first two are in fact written

$$2x - \sin\varphi (x^2 + 1) = 0; \quad (1)$$

$$2y - \sin\varphi'(y^2 + 1) = 0, \quad (2)$$

and the last two, after clearing of the denominators,

$$\left.\begin{aligned} &y[(x^2+1) - 2x\sin\varphi - \cos\varphi(x^2-1)] \\ &\quad = \beta\, x[(y^2+1) - 2y\sin\varphi' + \cos\varphi'(y^2-1)], \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} &y[(x^2+1) - 2x\sin\varphi + \cos\varphi(x^2-1)] \\ &\quad = \beta_0 x[(y^2+1) - 2y\sin\varphi' - \cos\varphi'(y^2-1)] \end{aligned}\right\} \quad (4)$$

In order to find the singular points of $\Phi(z)$, it is sufficient to express the condition that two of the singular points of F(z,t) coalesce. However, this can occur in two ways:

Either a singular point defined by one of the four equations $\frac{dl}{du}=0$, $\frac{dl'}{du'}=0$, $H=0$, $H_0=0$ will coalesce with a singular point defined by another of these four equations: we will thus obtain the singular points of $\Phi(z)$ of the first type; on the other hand two of the singular points defined by one of these four equations will coalesce into one: we will thus obtain the singular points of $\Phi(z)$ of the second type.

In order to have the points of the first type, it is sufficient to combine in pairs the four equations (1), (2), (3), (4). We see that these points in no way depend on the integers a and c.

In order to have the second-type points, this is how it must be done:

Let f(z,t)=0 be one of the four equations (1), (2), (3), (4); in order to express that two of the singular points defined by this equation coalesce it suffices for me to write

$$f = 0, \quad \frac{df}{dt} = 0$$

If we change variables which express z and t and consequently f as functions of $l$ and of $l'$, it follows that

$$\frac{df}{dt} = \frac{-i}{t}\left(c\frac{df}{dl} - a\frac{df}{dl'}\right),$$

so that the equation $\frac{df}{dt}$=0 can be replaced by

$$c\frac{df}{dl} - a\frac{df}{dl'} = 0$$

or just as well

$$\frac{c}{1-\sin\varphi\cos u}\frac{df}{du} - \frac{a}{1-\sin\varphi'\cos u'}\frac{df}{du'} = 0.$$

The first members of equations (1) and (2) depend only on u or u': we can set them aside.

However, we have singular points which will be given us by two equations /289

$$H = 0, \quad \frac{dH}{dt} = 0$$

or again by the two equations

$$H_0 = 0, \quad \frac{dH_0}{dt} = 0.$$

We have

$$H = \cos u - \sin\varphi + i\cos\varphi\sin u - \beta(\cos u' - \sin\varphi' + i\cos\varphi'\sin u').$$

The equation $\frac{dH}{dt}$=0 can therefore be replaced by the following:

$$\frac{c(-\sin u + i\cos\varphi\cos u)}{1-\sin\varphi\cos u} + \frac{a\beta(-\sin u' + i\cos\varphi'\cos u')}{1-\sin\varphi'\cos u'} = 0$$

or

$$\frac{c[\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2+1)}+\frac{a\beta[\cos\varphi'(y^2+1)+(y^2-1)]}{2y-\sin\varphi'(y^2+1)}=0. \tag{5}$$

Equally, the equation $\frac{dH}{dt}=0$ can be replaced by the following:

$$\frac{c[-\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2-1)}+\frac{a\beta_0[-\cos\varphi'(y^2+1)+(y^2-1)]}{2y-\sin\varphi'(y^2+1)}=0. \tag{6}$$

The singular points of the second type are therefore given by equations (3) and (5) or by equations (4) and (6); contrary to those of the first type, they therefore depend on the ratio of the integers a and c.

All singular points of $\Phi(z)$ are therefore given by equations.

These algebraic equations are simplified when we assume that $\varphi'=0$. It is then permissible to assume that $\varpi'=\varpi$ and consequently $\beta_0=\beta$.

Equation (1) does not change, equation (2) reduces to y=0 and there is no need to consider it; equations (3) and (4) become

$$(x^2+1)-2x\sin\varphi+\cos\varphi(x^2-1)=2\beta xy, \tag{3}$$

$$y[(x^2+1)-2x\sin\varphi-\cos\varphi(x^2-1)]=2\beta x. \tag{4}$$

Equations (5) and (6) become /23

$$\frac{c[\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2+1)}+a\beta y=0, \tag{5}$$

$$\frac{c[-\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2+1)}-\frac{a\beta}{y}=0. \tag{6}$$

The combination of equations (3) and (5) gives

$$\left.\begin{aligned}&\frac{2c[\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2+1)}\\&+\frac{a[(x^2+1)-2x\sin\varphi+\cos\varphi(x^2-1)]}{x}=0,\end{aligned}\right\} \tag{7}$$

and that of equations (4) and (6) gives

$$\left.\begin{aligned}&\frac{2c[-\cos\varphi(x^2+1)+(x^2-1)]}{2x-\sin\varphi(x^2+1)}\\&-\frac{a[(x^2+1)-2x\sin\varphi-\cos\varphi(x^2-1)]}{x}=0.\end{aligned}\right\} \tag{8}$$

Equations (7) and (8) give us the values of x corresponding to the points of the second type; equation (1) gives us the values of x corresponding to certain points of the first type. It remains for us to speak of the points of the first type defined by equations (3) and (4), for which equation (2) becomes illusory.

Equations (3) and (4) are written

$$\xi - \beta\eta = \xi_0 - \beta_0\eta_0 = 0.$$

If they are satisfied at the same time, we will have

$$\xi\xi_0 = \beta\beta_0\eta\eta_0 = \beta^2.$$

But

$$\xi\xi_0 = (1 - \sin\varphi\cos u)^2.$$

It therefore remains

$$1 - \sin\varphi\cos u = \pm\beta,$$

so that the values of x corresponding to this type of singular points will be given by the two equations

$$2x - \sin\varphi(x^2 + 1) = 2\beta x, \qquad (9)$$

$$2x - \sin\varphi(x^2 - 1) = -2\beta x. \qquad (10)$$

The values of x which correspond to the singular points will be given us by five equations (1), (7), (8), (9), (10). Let us observe that equations (1), (9) and (10) are reciprocal and that equations (7) and (8) change one into the other when we change x into $\frac{1}{x}$. If x is a singular point, it will therefore be the same for $\frac{1}{x}$. This is something which was easy to predict. /291

If we set $\varphi=0$, our equations reduce to x=0; therefore, when $\varphi$ tends toward 0, the roots of equations (1), (7) and (8) tend toward 0 or infinity.

If we set

$$\text{tang}\frac{\varphi}{2} = \tau,$$

equations (3), (4), (5), (6), (7) and (8) become

$$y = \frac{(x-\tau)^2}{\beta(1+\tau^2)x}, \qquad (3)$$

$$y = \frac{\beta(1+\tau^2)x}{(1-x\tau)^2}, \qquad (4)$$

$$\frac{c(x+\tau)}{1-\tau x} + a\xi y = 0, \tag{5}$$

$$\frac{c(1-x\tau)}{x-\tau} - \frac{c\xi}{y} = 0, \tag{6}$$

$$\frac{c(x+\tau)}{1-\tau x} + \frac{a(x-\tau)^2}{(1+\tau^2)x} = 0, \tag{7}$$

$$\frac{c(1+\tau x)}{x-\tau} + \frac{a(1-\tau x)^2}{(1+\tau^2)x} = 0. \tag{8}$$

on the other hand, equation (1) gives us a solution

$$x = \tau, \quad x = \frac{1}{\tau}.$$

We have seen that when $\varphi$ and $\tau$ are very small, the values of x are very small, or very large and as the equations do not change when we change x into $\frac{1}{x}$, we must conclude that there are precisely as many very small values as very large ones.

Our equations and the corresponding values of x are somewhat simplified when, /29 assuming $\varphi$ very small, we neglect the square of this quantity.

Equations (1), (9) and (10) then give us respectively for x three very small values which are approximately

$$x = \frac{\varphi}{2}, \quad x = \frac{\varphi}{2}\frac{1}{1-\beta}, \quad x = \frac{\varphi}{2}\frac{1}{1+\beta}, \tag{11}$$

and three very large values, which are approximately

$$x = \frac{2}{\varphi}, \quad x = \frac{2(1-\beta)}{\varphi}, \quad x = \frac{2(1+\beta)}{\varphi}. \tag{11a}$$

Equation (7) gives us two very small values, approximately defined by the equation

$$4x^2(a+c) + 2x\varphi(c-2a) + a\varphi^2 = 0, \tag{12}$$

and a very large value, which is approximately

$$x = \frac{2}{\varphi}\frac{a+c}{a}. \tag{13a}$$

Equation (8) gives us two very large values, defined by

$$4(a+c)+2x\varphi(c-2a)+ax^2\varphi^2=0, \qquad (12a)$$

and one very small, which is written

$$x=\frac{\varphi}{2}\,\frac{a}{a+c}. \qquad (13)$$

It is easy to verify that equations (12) and (12a) have real roots when $\frac{c}{a}<0$. If therefore c and a are of opposite sign and $\varphi$ is sufficiently small, equations (7) and (8) will have real roots.

The values of x corresponding to the various singular points being thus defined, it remains to determine the values of y and z.

I first observe that, if we have a singular point corresponding to certain values of x, y and z, the inverse values $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ will correspond to another singular point, which I will call the reciprocal of the first. We in fact state /293
that our system of equations does not change when we change x, y, z into $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$, which was also easy to predict.

The values of x and y will be defined by the following pairs of equations:

(1),(3); (1),(4); (7),(3); (8),(4); (9),(3) or (4); (10),(3) or (4).

These equations show us that if $\varphi$ is very small and can be regarded as an infinitesimal of the first order, y is very small if x is very small, and very large if x is very large.

We have on the other hand

$$z=y^c x^a e^{\frac{\alpha \sin\varphi}{2}\left(\frac{1}{x}-x\right)}.$$

If $\varphi$ is an infinitesimal of the first order, x is an infinitesimal (or infinitely large) of the same order; the same is true for y; the exponent $\frac{\alpha \sin\varphi}{2}\left(\frac{1}{x}-x\right)$ is then finite; consequently z is an infinitely small (or infinitely large) quantity of the order a+c. I will distinguish among the singular points that which is defined by $x=\tau$ [solution of equation (1)] and by equation (3).

For this point, in fact, y and z are zero.

Similarly, for the point defined by $x=\frac{1}{\tau}$ [another solution of (1)] and by equation (4), and which is the reciprocal of the first, the values of y and z are infinite.

We therefore need not concern ourselves with these two singular points in the discussion to follow.

Discussion

97. Here is the question which it remains for me to resolve.

In all I have 14 singular points, seven of which correspond to very small values of x and y, and 7 of which correspond to very large values of x and y.

From another point of view, 7 of these points correspond to very small values of z, and 7 to very large values of z. It is a question of knowing which among the first 7 is that for which the modulus of z is greatest. (This will at the same time teach us, since the values of z are reciprocal together as are those of x and y, which among the last 7 is that for which the modulus of z is smallest.) /29

If the corresponding singular points are admissible, it will be these which will define the circumferences

$$|z| = R, \quad |z| = r \quad \text{(here we have } R=\frac{1}{r}\text{)}.$$

In order not to prolong the discussion by examination of too great a number of different cases, I am going to make some particular hypotheses. I will assume

$$\beta > 1.$$

I will likewise assume that the ratio $\frac{c}{a}$ is nearly equal to the ratio of the mean motions with changed signs, that is, we have nearly (designating these mean motions by n and n')

$$an + cn' = 0.$$

The most interesting terms are in fact those which correspond to small divisors.

We then have nearly

$$\frac{c}{a} = -(\beta)^{\frac{3}{2}},$$

which shows that c and a are of opposite sign; for example I will assume c positive and a negative; as $\beta$ is greater than 1, c+a will be positive.

Thanks to these hypotheses all values of x are real. This makes possible a simple geometric representation which will permit the discussion to proceed more easily.

In the following figure we represent each singular point by a point of the plane whose rectangular coordinates are x and y.

I have given two figures (fig. 1 and fig. 2), the first representing the quadrant of the plane between the axis of the positive x and that of the positive y; and the second representing the quadrant between the axis of the negative x and the axis of the negative y. /2

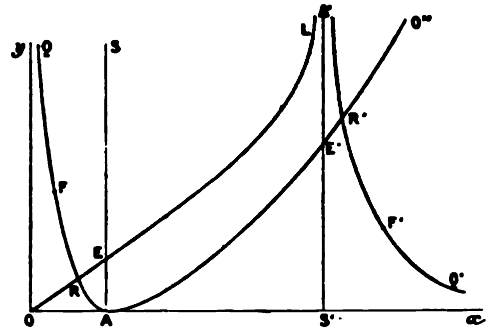


Figure 1.

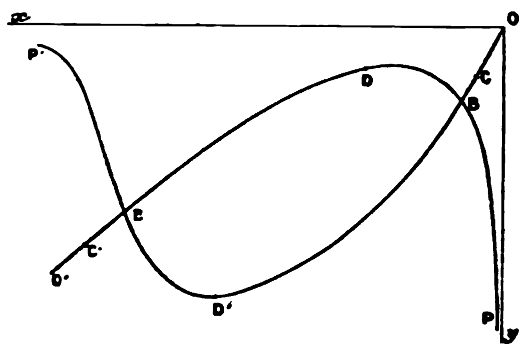


Figure 2.

The straight lines AS and A'S' have respectively for equations

$$x = \tau, \qquad x = \frac{1}{\tau}.$$

The two branches of the curve C'B'DBP and QFAE'R' have for equation

$$y = \frac{(x-\tau)^2}{\beta(1+\tau^2)x},$$

that is, equation (3); the two branches of the curve /296

**B'D'BCOREL** and **R'F'Q'**

have for this equation

$$y = \frac{\beta(1+\tau^2)x}{(1-\tau x)^2}. \qquad (4)$$

The various singular points are represented in the figure by the following points:

A . . . . . . . . . . . . . . . . equations (1) and (3) ($x=\tau$), B . . . . . . . . . . . . . . . . . . (9), (3) and (4) [second equation (11)]

C . . . . . . . . . . . . . . . . . . (8) and (4) [(13)],

D . . . . . . . . . . . . . . . . . . (7) and (3) [(12) negative root],

E . . . . . . . . . . . . . . . . . . (1) and (4) ($x=\tau$),

F . . . . . . . . . . . . . . . . . . (7) and (3) [(12) positive root],

R . . . . . . . . . . . . . . . . . . (10),(3) and (4) [third equation (11)];

and by the points A', B', C', D', E', F' and R', respectively, reciprocals of the first.

It is easy to verify that if $\varphi$ is small enough these points are well set up in the order of the figure, i.e., that the abscissae of the points C'B'D'DBCFREE'R'F' are increased.

Let us compare the values of z corresponding to these various points. We first see that for the points of figure 1 (where $x>0$, $y>0$) z is real and positive and that for the points of figure 2 (where $x<0$, $y<0$) the argument of z is equal to $(c+a)\pi$, and that of $z^{\frac{1}{c}}$ equal to $(1+\frac{a}{c})\pi$. It remains to be seen how the modulus of z varies. If we follow one of curves (3) or (4), the maxima and minima of $|z|$ correspond to the contact points of curves (3) and (4) with the curves

$$z = y^c x^a e^{\frac{a \sin\varphi}{2}\left(\frac{1}{x}-x\right)} = \text{const.},$$

that is, to the points C', D, F, A for curve (3), and to the points D', C, F' for curve (4).

Here is how $|z|$ varies: /297

1) When we follow curve (3)

| | | | |
|---|---|---|---|
| At O' . . . . . . . . . | $\|z\|=0$ | At Q . . . . . . . . . . . | $\|z\|=0$ |
| From O' to C' . . . . . | increases | From Q to F . . . . . . | increases |
| At C' . . . . . . . . . | max. | At F . . . . . . . . . . . | max. |
| From C' to D . . . . . | decreases | From F to A . . . . . . | decreases |
| At D . . . . . . . . . | min. | At A . . . . . . . . . . . | $\|z\|=0$ |
| From D to P . . . . . . | increases | From A to O" . . . . . . | increases |
| At P . . . . . . . . . | $\|z\|=\infty$ | At O" . . . . . . . . . . | $\|z\|=\infty$ |

2) When we follow curve (4)

| | | | |
|---|---|---|---|
| At P' . . . . . . . . . . | $\|z\|=0$ | At O . . . . . . . . . . . | $\|z\|=0$ |
| From P' to D' . . . . . | increases | From O to L or to A' . . | increases |

| | | | |
|---|---|---|---|
| At D' . . . . . . . . . | max. | At A' . . . . . . . . . . | $\|z\|=\infty$ |
| From D' to C . . . . | decreases | From A' to F' . . . . . | decreases |
| At C . . . . . . . . . | min. | At F' . . . . . . . . . . | min. |
| From C to O' . . . . | increases | From F' to Q' . . . . . | increases |
| At O . . . . . . . . . | $\|z\|=\infty$ | At Q' . . . . . . . . . . | $\|z\|=\infty$ |

From this we conclude that the z of point B is greater than that of point C, and that of point E is greater than that of point R.

Similarly, the z of point D is smaller than that of point B, and the z of R is smaller than that of F.

We have seen that, the function $F(z,t)$ not being uniform, it was necessary to trace the integration contours on the corresponding Riemann surface whose number of sheets is infinite. In order to avoid consideration of this Riemann surface, one can change variables. Let us in fact observe that the square $F_1^0$ is a uniform function of x and y and consequently the square of $F(z,t)$ is a uniform function of $x^{\frac{1}{c}}$ and $z^{\frac{1}{c}}$.

If we therefore agree to give $z^{\frac{1}{c}}$ a determined value which we momentarily considered constant, to each point of the plane of the $x^{\frac{1}{c}}$ there will correspond only two values of $F(z,t)$ equal and of opposite sign. It will then be advantageous to trade our contours of integration on the plane of $x^{\frac{1}{c}}$.

Let us first give $z^{\frac{1}{c}}$ an initial value $\zeta_0$ whose modules is equal to 1. We /298 are agreed, in defining $\Phi(z)$, that the contour of integration along which we must take the integral

$$\Phi(z)=\int F(z,t)\,dt$$

must reduce to the circle $|t|=1$ for values of z of modulus 1.

For $z^{\frac{1}{c}}=\zeta_0$ we must therefore take for contour in the plane of the t the circle $|t|=1$ and in the plane of the $x^{\frac{1}{c}}$ the circle $|x^{\frac{1}{c}}|=1$.

Here, therefore, is the rule to recognize if a singular point of $\Phi(z)$ is admissible. Let $\alpha_i$ be the value of $x^{\frac{1}{c}}$ and $\zeta_i$ the value of $z^{\frac{1}{c}}$ which corresponds to this singular point. We will assume, for example, that the modulus of $\zeta_i$ is smaller than 1; we know equally well that among the singular points of $\Phi(z)$ half have their modulus smaller than 1. We are going to vary $z^{\frac{1}{c}}$ in the following manner: its argument must remain constant and constantly equal to that of $\zeta_i$ and its modulus will increase from $|\zeta_i|$ to 1. In other words, points $z^{\frac{1}{c}}$ will describe a segment of the straight line $\Delta$ limited to the points $\zeta_i$ and $\frac{\zeta_i}{|\zeta_i|}$.

For each of the values of $z^{\frac{1}{c}}$, $F(z,t)$, considered as a function of $x^{\frac{1}{c}}$, presents a certain number of singular points; for $z^{\frac{1}{c}}=\zeta_i$, two of these singular points merge into one with $\alpha_i$. When $z^{\frac{1}{c}}$ describes the straight line $\Delta$, these two singular points vary in a continuous and perfectly defined manner. When $z^{\frac{1}{c}}$ attains the final value $\frac{\zeta_i}{|\zeta_i|}$, it may occur that the final positions of these two singular points are either both inside or both outside the circle $|x^{\frac{1}{c}}|=1$, and the considered point is then inadmissible, or that these final positions are one outside and the other inside this circle, and the point considered is then admissible.

The function $F(z,t)$ is multiplied by a c-th root of unity when $x^{\frac{1}{c}}$ is multiplied by a c-th root of unity. Let us therefore assume that for a known value of /29
$z^{\frac{1}{c}}$ the point

$$x^{\frac{1}{c}} = \gamma$$

is a singular point of $F(z,t)$ considered a function of $x^{\frac{1}{c}}$. This will be true also for points

$$x^{\frac{1}{c}} = \gamma e^{\frac{2i\pi}{c}}, \quad x^{\frac{1}{c}} = \gamma e^{\frac{4i\pi}{c}}, \quad \ldots, \quad x^{\frac{1}{c}} = \gamma e^{\frac{2(c-1)i\pi}{c}}.$$

We have seen that the values of x which correspond to the singular points of $\Phi(z)$ are all real, and consequently have 0 or $\pi$ for argument. The corresponding values of $x^{\frac{1}{c}}$ will therefore have $\frac{k\pi}{c}$ for argument, k being an integer. Therefore let $\alpha_i$ be one of these values. I will be able to write

$$\alpha_i = \alpha'_i e^{\frac{2ki\pi}{c}},$$

$\alpha_i$ having 0 or $\frac{\pi}{c}$ for argument and k being an integer.

If $\alpha_i$ corresponds to a singular point of $\Phi(z)$ [that is, to two coalesced singular points of $F(z,t)$], the same will be true for $\alpha'_i$.

I say that the necessary and sufficient condition for point $\alpha_i$ to be admissible is that the point $\alpha'_i$ be so.

In fact, let us apply the rule: when the point $z^{\frac{1}{c}}$ describes the straight line $\Delta$, the two singular points first coalesced into $\alpha_i$ will have $\gamma$ and $\gamma'$ for final positions; likewise, the two singular points first coalesced into $\alpha'_i$ will have for final positions

$$\gamma e^{-\frac{2ki\pi}{c}} \quad \text{and} \quad \gamma' e^{-\frac{2ki\pi}{c}}$$

It is obviously sufficient, in order to demonstrate the stated theorem, to observe that

$$|\gamma| = \left|\gamma e^{-\frac{2ki\pi}{c}}\right|, \quad |\gamma'| = \left|\gamma' e^{-\frac{2ki\pi}{c}}\right|.$$

It will be sufficient to examine the singular points which correspond to real and /300 positive values of $x^{\frac{1}{c}}$, that is, to points F, E, R and A of the figure, and the singular points which correspond to the value $\frac{\pi}{c}$ of the argument of $x^{\frac{1}{c}}$, that is to say, to the points D, B, and C of the figure.

Point E is inadmissible; in fact, the corresponding value of $\alpha_i$ is

$$\alpha_i = \tau^{\frac{1}{c}},$$

when the point $z^{\frac{1}{c}}$ describes the straight line $\Delta$, the two singular points originally coalesced into $\alpha_i$ will remain real. To each of them will correspond a value of x and of y, and consequently a representative point on our figure.

One of these representative points will then describe the straight line ES and the other the curve EL.

One of the singular points will therefore remain fixed and equal to $\tau^{\frac{1}{c}}$ and will consequently have its modulus always smaller than 1.

The initial value of $\zeta_i$ of $z^{\frac{1}{c}}$ is real and positive; the straight line $\Delta$ will therefore be a portion of the axis of the real quantities and the final value $\frac{\zeta_i}{|\zeta_i|}$ will be equal to 1.

The second singular point (which corresponds to the representative point which followed the curve EL) has a real and positive value which I call $\gamma$; we must know if $\gamma$ is smaller or larger than 1.

When this representative point describes the curve EL from E to L, the modulus of z will continue increasing from a certain very small value to infinity;

it will therefore pass once and only once through value 1. It is a question of showing that the corresponding value $\gamma^2$ of x is smaller than 1. To do so, it is sufficient to show that when the abscissa x of this representative point reaches the value 1, $|z|$ is greater than 1.

Now we find that for x=1

$$z = y^c x^a e^{\frac{a \sin \varphi}{2}\left(\frac{1}{x} - x\right)} = y^c.$$

It therefore remains to demonstrate that y>1.

Now it is clear that /30

$$y = \frac{\beta(1+\tau^2)}{(1-\tau^2)} > 1.$$

Therefore $\gamma < 1$.

Therefore point E is inadmissible.

Q. E. D.

Point F is inadmissible; here even the straight line Δ will be a portion of the axis of the real quantities since $\zeta_i$ will be real. The singular points originally coalesced into $\alpha_i$ will not remain real, but they will remain imaginary conjugates; they therefore have the same modulus; it is therefore impossible that when $z^{\frac{1}{c}}$ attains its final value $\frac{\zeta_i}{|\zeta_i|}=1$ one of these points is larger than 1 and the other smaller than 1 in absolute value.

Q. E. D.

However, it will be useful for us to know if, when $z^{\frac{1}{c}}$ attains its final value 1, the common modulus of these two singular points is larger or smaller than 1. If it is originally smaller than 1, it will not cease being so except when passing through the value 1. It would therefore be necessary that, for an imaginary value of x having modulus 1, $z^{\frac{1}{c}}$ have a real and positive value.

In the plane of the x let us therefore construct the lines of equal argument of the function

$$z = \frac{(x-\tau)^{2c}}{[\beta(1+\tau^2)x]^c} x^a e^{\frac{a \sin \varphi}{2}\left(\frac{1}{x} - x\right)}.$$

These lines are represented in figure 3 at least in the only part of the plane which interests us and which is a neighborhood of point O.

The significant points are the point $x=0$, corresponding to point O of figure 1, point $x=\tau$, corresponding to point A, and two points which correspond to the points D and F. These points are in addition designated in figure 3 by the same letters.

Among the lines of equal argument, the ones regarded as significant are represented by solid lines. These are the axis of the real quantities on one hand, and on the other hand, lines going from point O to point F and from point A to point D.

The other lines of equal argument ending in either point A or point O or both, are represented by dotted lines. /302

When the point $z^{\frac{1}{c}}$ describes the straight line $\Delta$, point x describes the curve in solid lines FO in our figure 3.

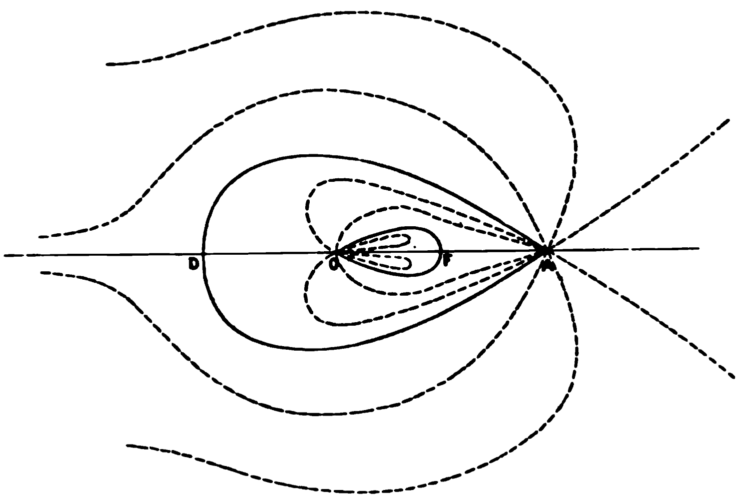


Figure 3.

We therefore see that the modulus of x will always remain very small and that we will have

$$|\tau| < 1.$$

Point R is inadmissible; in fact, when the point $z^{\frac{1}{c}}$ describes the straight line $\Delta$, the two singular points originally coalesced will first remain real; the two representative points will describe the two branches of curve RE and RF; when the first of these points attains point E, the corresponding singular point will coalesce with another; the two points thus coalesced will then separate and the corresponding representative points will describe the curve EL and ES; in speaking of point E we have seen that the final values of $x^{\frac{1}{c}}$ are real and smaller than 1.

Likewise, when the second representative point attains F, the singular /303 corresponding point will coalesce with another, then separate; the final values, as we have seen in speaking of point F, are imaginary conjugates and smaller in modulus than 1.

We therefore have here no longer 2, but 4 final values; all four are smaller than 1 in absolute value.

Q. E. D.

Point B is inadmissible. The two singular points originally coalesced separate, but the corresponding values of x remain real. The two representative points describe the branches of curve BP and BD'. For the first, which describes BP, the absolute value of x continues to diminish; it therefore remains smaller than 1; let us consider the second, which describes BD'; it remains for me to demonstrate that although the absolute value of x increases it remains smaller than 1, while the modulus of z is itself less than 1.

To do so, it is necessary to show that for x=-1, $|z|>1$; now, for x=-1,

$$|z| = |y|^c.$$

Now

$$|y| = \frac{\beta(1+\tau^2)}{(1+\tau)^2} > 1 \quad \text{(if } \tau \text{ is sufficiently small).}$$

Point C is inadmissible. The two singular points originally coalesced separate, x remaining real; the first representative point describes CO, the second CB. For the first, $|x|$ constantly diminishes: its final value is therefore smaller than 1. Let us examine the second singular point, which describes CB. When it arrives at B, it merges with another singular point and again separates; the two representative points will describe the two curves BP and BD'; according to what we have just seen, the final values of $|x|$ are smaller than 1. Thus we have not two but three final values, all smaller than 1.

Q. E. D.

Point D is admissible. The two values of x remain real and the first re- /304
presentative point describes DB; having reached B, the representative curve forks into BP and BD', and the final values of x are smaller than 1, as we have just seen.

The second representative point describes DB'; I say that the final value of $|x|$ is greater than 1. For this, it is necessary to see that for x=<1 we have $|z|<1$; now, for x=-1,

$$|z| = |y|^c; \quad |y| = \frac{(1+\tau)^2}{\beta(1+\tau^2)} < 1 \text{ (if } \tau \text{ is sufficiently small).}$$

Of our three final absolute values, two are smaller and one larger than 1. Therefore the point is admissible.

Q. E. D.

In summary, of the six points BCDEFR, point D alone is admissible.

The same is true for the six reciprocal points B'C'D'E'F'R': point D' alone is admissible.

If therefore one of the eccentricities is sufficiently small, the other zero, the orbit inclination zero, the major axis of the circular orbit larger than that of the elliptic orbit; if the ratio $\frac{c}{a}$ differs little from that of the mean motions, it is the points D and D' which determine the radii of convergence r and $R=\frac{1}{r}$.

In order to facilitate understanding of this discussion, I have constructed a fourth figure in which I have represented the variation of the singular points by taking x for abscissa, if x is real, and $|x|$ if x is imaginary, and for ordinate $|z|$. I have nevertheless represented only those singular points which play a role in the discussion. The straight lines shown by dots and dashes are the two axes of coordinates x=0 and $|z|=0$ and the straight lines x=+1, $|z|=1$. The curves in a solid line represent the variation of the real singular points, and the curves in dots are that of the singular imaginary points. According to the conventions made above, each of the points of these dotted curves represents two conjugate imaginary singular points.

The various significant points are designated by the same letters as the corresponding points of the other figures. In order to find the various final values obtained upon leaving a given singular point, it is necessary to follow the solid or dotted curves, always descending (since in the figure the axis of the positive $|z|$ is directed toward the bottom) to the straight line $|z|=1$. /305

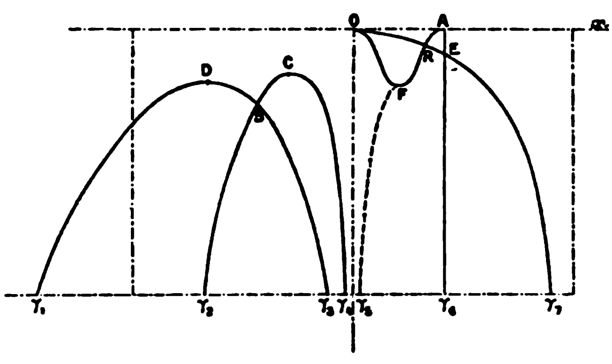


Figure 4.

We thus find that

For the point D the final values are $\gamma_1$, $\gamma_2$ and $\gamma_3$,

" B " $\gamma_2$ and $\gamma_3$,

" C " $\gamma_2$, $\gamma_3$ and $\gamma_4$,

" F " $\gamma_5$,

" R " $\gamma_5$, $\gamma_6$ and $\gamma_7$,

" E " $\gamma_6$ and $\gamma_7$.

I recall that $\gamma_5$ represents two final conjugate imaginary values. We see that of these final values, all except $\gamma_1$ are smaller than 1 in absolute value.

## Discussion of the General Case

98. The limits which are imposed on me here do not permit me to repeat this discussion in the more general case; however, I may indicate in a few words the manner in which it may be conducted.

When one varies the elements of the orbits in a continuous manner, the singular points of $\Phi(z)$ will also vary in a continuous manner. Let us assume that one also varies the elements such that the orbits remain real and intersect themselves at a real point, such that also at no time do two singular points of $\Phi(z)$ coalesce. Let us consider a singular point of $\Phi(z)$; it varies in a continuous manner and, as we assume that it never coalesces with any other, we will be able to follow it in its variations without fear of any ambiguity. /30

This granted, I say that if this point is admissible at a certain moment, it will always remain admissible and vice versa, except in a case to which we will return.

In fact, to say that the singular point is admissible is to say that among the final values of x corresponding to this point, there are those whose modulus is greater than 1 and others whose modulus is smaller than 1. But it is important to be more precise. In fact, in the particular case dealt with in the preceding article, $F(z,t)$ was a uniform function of $z^{\frac{1}{c}}$ and $x^{\frac{1}{c}}$, which permitted us to represent the singular points of $F(z,t)$ on the plane of the $x^{\frac{1}{c}}$.

In the general case it is not the same and such a sample representation is no longer possible. It is necessary to represent the singular points of $F(z,t)$ (considered a function of t) on a particular Riemann surface which I earlier called S; this surface can be defined as follows: we have

$$z = x^a e^{\frac{a \sin \varphi}{2}\left(\frac{1}{x}-x\right)} y^c e^{\frac{c \sin \varphi'}{2}\left(\frac{1}{y}-y\right)}. \tag{1}$$

If we regard z as given, this equation defines a relationship between x and y which an infinity of systems of values of x and y satisfies, or of $x^{\frac{1}{c}}$ and $y^{\frac{1}{a}}$; each of these systems of values represents what we may call an analytical point. One and only one of these analytical points will correspond to each of the points of the Riemann surface, and vice versa.

When we vary z, this Riemann surface S also varies, since then the singular points of $F(z,t)$ are displaced. Let $S_0$ be what S becomes when z reaches a modulus value 1. On the surface $S_0$ we will be able to trace a circle which I will call $C_0$ and whose equation will be /30

$$|x| = |y| = 1.$$

(In fact, if we give x an arbitrary modulus value 1, we can always choose a value of y likwise of modulus 1, such that z has whatever value having modulus that one wishes.)

This circle $C_0$ splits the Riemann surface $S_0$ into two parts.

I will call $R_0$ that of the two regions which contains the neighboring points of $C_0$ and for which $|x|<1$, and I will call $R_0'$ the other region.

Let us therefore assume that we have the straight line Δ of the preceding article follow to point $z^{\frac{1}{c}}$ and that we study the variations of the singular points of F(z,t); when z varies, these points are displaced on the surface S at the same time as this surface S itself varies. Two of these points first coalesced into one (which is a singular point of $\Phi(z)$) separate; when the modulus of z reaches the value 1 and S is reduced to $S_0$ they reach two final positions on this surface. (The discussion in the preceding article has shown us cases where one of these singular points splits into two others; then there are more than two final positions, but what I am going to say remains applicable.) If all these final positions belong to the same of the two regions determined on the surface $S_0$ by the circle $C_0$, the corresponding singular point of $\Phi(z)$ is inadmissible; in the opposite case, it is admissible.

One sees the nuance which distinguishes this statement from that which I had first given and which was proper in the particular case of the preceding article. The bearers of two points can become the one greater and the other smaller than 1 in absolute value, and these two points nevertheless can belong to the same one of the two regions defined above if they are not part of the same sheet of the Riemann surface.

This granted, I say that when we vary the elements of the two orbits, a singular point at first admissible cannot in general become inadmissible or vice versa. In fact, let us consider the variations of the surface $S_0$ and of what we have called the final values. In order for a singular point to in fact cease being admissible or to become so, it would be necessary for the final corresponding value to escape the circle $C_0$ in order to pass from one of the two regions to the other. Now what is the significance of the equations of this circle $C_0$ /308

$$|x| = |y| = 1 \ ?$$

They signify that the two eccentric anomalies are real. To each point M on the Riemann surface S, and in particular on the surface $S_0$, there corresponds upon the two orbits a pair of points P and P' defined by the values of the eccentric anomalies or what goes back to the same thing by x and y.

If point M is on the circle $C_0$, the points P and P' are real. The point M can be singular only if the distance PP' is zero, or if one of the points P and P' is at a zero distance from the Sun. This second circumstance cannot occur if the points P and P' are real; nor the first if, as we have assumed, the two orbits do not intersect themselves at a real point.

It is therefore impossible that a point of the circle $C_0$ be singular: that is to say, one of the final values passes over this circle: that is to say also that finally a singular point of $\Phi(z)$ loses or gains the character of admissibility.

However, there is a case which I must yet speak of and where this reasoning would be found at fault. I assume that we make the point $z^{\frac{1}{c}}$ follow the straight line $\Delta$ and that we study the corresponding variations of the singular points of $F(z,t)$. In the beginning, two of these points are coalesced together and consequently coalesce to a singular point A of $\Phi(z)$; they can separate: let $\alpha$ be one of them; it may occur (and we have seen examples of this in the preceding article) that, for a certain value of z, the point $\alpha$ coalesces with another singular point of $F(z,t)$ (generally different from that with which it first coalesced) and consequently with a singular point B of $\Phi(z)$. It then separates from it such that the singular point A admits not two but three final values.

I will say in this case, for brevity in language, that point B is subordinate to point A; in order for this to be so, it is necessary for the z of point B to have the same argument and modulus more closely approaching one than $|z|$ of point A.

Then let A and B be two singular points of $\Phi(z)$ and let us assume that their z first have different arguments. Let us vary in a continuous manner the elements of the two orbits and consequently points A and B; if at a certain moment point B becomes subordinate to point A, it may occur that at this moment, in exception to the general rule formulated above, point A becomes admissible or ceases to do so.

Let us see how this circumstance can occur. Let us first observe that the values of x which correspond to the singular points of $\Phi(z)$ are furnished us by a certain number of algebraic equations. If the two points A and B are thus defined by one and the same irreducible equation, I will say that they are of the same nature and, in the contrary case, that they are of a different nature. One can see without difficulty that if points A and B are of a different nature, point B can become subordinate to A without point A being able to lose or gain the character of admissibility.

I now assume that points A and B are of the same nature. If point B is inadmissible, it can still become subordinate to A without this last point either becoming admissible or ceasing to do so. If, on the contrary, point B is admissible, it will occur in general, at the moment when B becomes subordinate to A, that A will cease being admissible if it was so, and will become so if it was not. In addition, point B still retains its character of admissibility or inadmissibility.

The preceding considerations therefore furnish us the means to recognize which are admissible by varying the elements of the orbits in a continuous manner, and by following the variations of the singular points. This is so provided we restrict ourselves to varying the elements in such a manner that two singular points at no moment have a z of the same argument, so as to avoid the necessary discussion of knowing if they are really subordinate one to another, or provided that we do not restrict ourselves to this condition in resigning ourselves to making this discussion.

We can vary not only the elements of the orbits, but the ratio $\frac{c}{a}$, forgetting for an instant that it must be commensurable, which we have assumed only for a particular end which is in no way related to the discussion of the admissibility of singular points. This ratio $\frac{c}{a}$ must nevertheless remain real and pass through zero nor infinity in order for what we have just said to remain applicable.

It therefore suffices to know which are the admissible points for certain values of the elements so as to be able to apply the preceding considerations. What I have said in the preceding article for a particular case therefore seems sufficient for us; in this particular case, however, certain singular points reduce to zero or to infinity and I have left this to one side in the discussion.

It is for this reason that I have still some points to add. Let us first assume that the inclination remains zero, the two eccentricities being finite. Let

$$\text{tang}\frac{\varphi}{2} = \tau, \qquad \text{tang}\frac{\varphi'}{2} = \tau'.$$

The singular points of F(z,t) will then be defined by the following equations:

$$x = \tau, \qquad x = \frac{1}{\tau}, \qquad y = \tau', \qquad y = \frac{1}{\tau'}.$$

$$\frac{y(x-\tau)^2}{1+\tau^2} = \beta x \frac{(y-\tau')^2}{1+\tau^2}, \tag{3}$$

$$\gamma \frac{(1-\tau x)^2}{1+\tau^2} = \beta_0 x \frac{(1-\tau' y)^2}{1+\tau'^2}. \tag{4}$$

Curves (3) and (4) are of the third order; in order for them to be real, it is necessary and sufficient that the major axes of the two orbits coincide, i.e., that the difference $\varpi-\varpi'$ be equal to 0 or $\pi$.

Let us assume $\varpi-\varpi'$; curve (3) will present a double point

$$x = \tau, \qquad y = \tau'.$$

If $\tau'$ is very small, the curve will present three branches: the first, which I will call $\gamma_1$, will differ little from the branch B'DBP of figure 1; the second which I will call $\gamma_2$, will pass through the origin and through the double point. It will first be asymptotic to the axis of the negative x and will deviate very little from this axis; after having passed through the double point, it will differ little from the branch AO' of figure 1; the third, which I will call $\gamma_3$, /311 is asymptotic to the axis of the y and at first differs very little from the branch CRA of figure 1; it then passes through the double point and deviates very little from the axis of the x to which it is asymptotic. I will henceforth say that two points are reciprocal when we pass from one to the other changing x to $\frac{1}{x}$, y to $\frac{1}{y}$, z to $\frac{1}{z}$ and $\sqrt{-1}$ to $-\sqrt{-1}$. The two curves (3) and (4) are then reciprocal to each other. If $\varpi=\varpi'$ and consequently our curves are real, this definition does not differ from that of the preceding article.

We have as singular points:

(1) The intersections of curves (3) and (4) differing very little from points B, B', R, R' of figure 1 and which I may always designate by the same letters. We have seen that they are inadmissible.

(2) The intersections of $x=\tau$ and curve (4), of $x=\frac{1}{\tau}$ and curve (3), differing very little from points E and E' of figure 1; they are thus inadmissible.

(3) Three points situated on curve (3) and differing very little from points D, F and C' of figure 1; only the first is admissible.

(4) Three reciprocal points of the first situated on curve (4); what differs little from D' is alone admissible.

(5) A point defined by equations (3) and (5) situated on the branch $\gamma_2$ and reducing to y=0 and $x=-\tau$ for $\tau'=0$. This point, of which there was no question in the preceding article, demands a special discussion. This discussion would prove that this point, which I will call T, is admissible; the two singular points of F(z,t), at first coalesced with it, separate when $z^{\frac{1}{c}}$ describes the straight line $\Delta$ and are first imaginary conjugates ; then they coalesce again at only one point, which corresponds to point D, and separate only to again become real. We see that the final values of T are the same as those of D; therefore T is admissible, as is D.

(6) A point T', reciprocal to T and consequently admissible as it is.

(7) The double point $x=\tau$ and $y=\tau'$, which I will call U; through this point /3
pass two of the branches of curve (3) and the two straight lines $x=\tau$ and $y=\tau'$. Four final values correspond to this point, for when $z^{\frac{1}{c}}$ describes the straight line $\Delta$, four singular points of F(z,t), at first coalesced into one, separate in such a way that the four representative points describe respectively the two branches of (3) and the two straight lines $x=\tau$, $y=\tau'$; among these final values three are smaller than 1 in absolute value or, more precisely, belong to the region $R_0$ of the Riemann surface $S_0$. The fourth final value, that which corresponds to the branch of curve $\gamma_2$, belongs to the other region. The point is therefore admissible.

(8) The point U', reciprocal to U, that is to say, the double point of (4), will be admissible for the same reason.

(9) There still remain the points of intersection of the straight line $y=\tau'$ with curve (4) which I call V and W' and those of the straight line $y=\frac{1}{\tau}$ with curve (3) which I call V' and W, and to which I will join the two reciprocal points one from the other

$$\left(x=\tau,\quad y=\frac{1}{\tau'}\right) \quad \text{and} \quad \left(x=\frac{1}{\tau},\quad y=\tau'\right),$$

which I will call X and X'. Point X is inadmissible and the two final values corresponding respectively to the two straight lines $x=\tau$ and $y=\frac{1}{\tau}$ belong to the region $R_0$.

Let us pass to point V (that of intersections of $y=\tau'$ with (4), which is very near the origin); when the point $z^{\frac{1}{c}}$ describes $\Delta$, the two representative points corresponding to the two singular points which separate follows: the first from curve (4) to point R and the second from the straight line $y=\tau'$ to U. The points R and U are therefore subordinate to V, and V admits as final values the total of the final values of R and U. All those of R belong to $R_0$; those of U which are admissible belong to the two regions. Therefore the point V is admissible but it ceases to be so when the difference $\varpi-\varpi'$ becomes very small instead of being zero. In this case, in fact, R and U cease being subordinate to V, and the only final values which V retains are, on one hand, a final value little different /313 from one of those of R, and another little different from one of those of U (which correspond to $y=\tau'$) and which both belong to $R_0$.

Finally, W is inadmissible (it is that of the intersections of (3) with $y=\frac{1}{\tau}$ which is near the axis of the y). In fact, F and X, whose final values belong to $R_0$, are subordinate to this point.

In summary, if the inclination is zero, the difference $\varpi-\varpi'$ very small, the eccentricity $\varphi$ small, and the eccentricity $\varphi'$ very small with respect to $\varphi$, the only admissible points will be D, T, U and their reciprocals.

Let us now assume that the inclination is not zero, but very small.

If we write that the distance between the two planets is zero, we will no longer obtain, as in the preceding case, two distinct equations (3) and (4), but one unique equation

$$\Theta(x, y)=0$$

which will represent a curve of the sixth order if we consider (as in figure 1) x and y as the coordinates of a point in a plane.

This curve decomposes into two third-order curves (3) and (4) when the inclination is zero; in order for them to be real, it is necessary and sufficient that the major axes of the orbits be perpendicular to the line of the nodes.

If the inclination is very small, the singular points will be:

(1) Points very little different from E, D, F, C, T, V, W, X and their reciprocals; I will designate them by the same letters ; it is clear that D and T are alone admissible with their reciprocals.

(2) Two points $B_1$ and $B_2$, very little different from B; two points $R_1$ and $R_2$, very little different from R and their reciprocals. All inadmissible.

(3) Nine points little different from U, namely $x=\tau$, $y=\tau'$; two intersections of $x=\tau$ with $\Theta=0$, two of $y=\tau'$ with $\Theta=0$, four points of $\Theta=0$. A special discussion would be necessary.

Having thus recognized which are the admissible points, it remains to see /31 which it is proper to retain, that is, to see which corresponds to the values of $|z|$ closest to 1.

If the eccentricity which corresponds to the larger of the two major axes and the inclination are small with respect to the other eccentricity and if the difference $\varpi-\varpi'$ is small, the point which is suitable for us is point D.

Forced to restrict myself, I stop this discussion, which I have only succeeded in outlining. However, it seems to me that the importance of the subject can tempt more than one investigator; beyond this discussion, he would have to give a practical and rapid method of solving the algebraic equations to which we are led in considering the smallness of certain quantities, and we can content ourselves most often with a mediocre approximation. His task would also be greatly facilitated by a complete analytical study of the function $\Phi(z)$ and its different determinations.

## Application of the Method of Darboux

99. Let us now assume that through the preceding discussion we have determined the singular point of $\Phi(z)$ which is suitable to the problem, and that consequently we know what are the two circumferences

$$|z| = r, \qquad |z| = R = \frac{1}{r}$$

which limit the domain where $\Phi(z)$ is developable in a Laurent series and are the singular points situated on these circumferences. In general, there will be only one on each of them.

Therefore let $z_0$ be the singular point found on the circumference $|z|=r$.

Let $x_0$, $y_0$ and $t_0$ be the corresponding values of x, y and t. We easily see that $x_0$ and $y_0$ are perfectly determined by the algebraic equations which we have discussed above; on the other hand

$$t_0 = x_0^{\frac{1}{c}} e^{\frac{\sin\varphi}{2c}\left(\frac{1}{x_0} - x_0\right)}$$

is not entirely determined, but is susceptible of c values which I will call /315

$$t_0, \; jt_0, \; j^2 t_0, \; \dots \; j^{c-1} t_0,$$

j being a c-th primitive root of unity.

Let us apply the method of Darboux to the development of $\Phi(z)$. To do so, it is necessary for us to know how this function behaves in the neighborhood of the singular points $z=z_0$.

When z is very near $z_0$, the function $F(z,t)$ admits two singular points $t_1$ and $t_2$ very near $t_0$; likewise it will admit c-1 other pairs of singular points

$$jt_1 \text{ and } jt_2, \; j^2t_1 \text{ and } j^2t_2, \; \dots, \; j^{c-1}t_1 \text{ and } j^{c-1}t_2$$

respectively very near

$$jt_0, \; j^2t_0, \; \dots, \; j^{c-1}t_0.$$

The integration contour C along which we must calculate the integrals

$$\Phi(z) = \frac{1}{2i\pi}\int F(z, t)\,dt$$

must pass between the points $t_1$ and $t_2$ and similarly between the points $jt_1$ and $jt_2$, .... In addition, we will be able to assume that this contour presents the following symmetry: it will be formed from c arcs $C_0$, $C_1$, ..., $C_{c-1}$, and we will pass from the arc $C_0$ to arc $C_k$, changing t to $tj^k$, as

$$F(z, tj^k) = j^{-k}F(z, t);$$

the integral taken along the c arcs $C_0$, $C_1$, ..., $C_{c-1}$ will be the same, and we will have

$$\Phi(z) = \frac{c}{2i\pi}\int_{C_0} F(z, t)\,dt.$$

The arc $C_0$, which is our new path of integration, will then pass only between the singular points $t_0$ and $t_1$; in addition, let us decompose the arc $C_0$ into three partial arcs $C'_0$, $C''_0$ and $C'''_0$; I will call $\alpha$ and $\beta$ the extremities of the arc $C'_0$, $\beta$ and $\gamma$ those of $C''_0$, and $\gamma$ and S those of $C'''_0$. I will assume that it is $C''_0$ which passes between $t_1$ and $t_2$ and that, when z tends toward $z_0$, none of the four points $\alpha, \beta, \gamma, \delta$ tends toward $t_0$, so that these four points are at a finite distance from $t_1$ and $t_2$.

Our integral taken along $C_0$ is the sum of three others taken respectively /318 along $C'_0$, $C''_0$ and $C'''_0$. The first and third remain holomorphic functions of z in the neighborhood of the point $z=z_0$, since the points $t_1$ and $t_2$ are at a finite distance from the arcs $C'_0$ and $C'''_0$. It is therefore only the second integral, taken along $C''_0$ which admits $z_0$ as a singular point; it is therefore the study of this second integral which will let us know the behavior of the function $\Phi(z)$ in the neighborhood of $z=z_0$.

Let us therefore see how the function F(z,t) will behave in the neighborhood of $z=z_0$, $t=t_0$. This depends, as is well understood, on the nature of the singular point considered. I will first examine the hypothesis that this point is one of those which we have designated by D, F, T, C and by the same letters primed, or in the case where the inclination is not zero, even one of those which we have designated by $B_1$, $B_2$, $R_1$, $R_2$ or their reciprocals. This is the most important hypothesis, for we have seen that, if the inclination and one of the eccentricities are very small, it is the point D which is applicable for us.

In this hypothesis $[F(z)]^{-2}$ can be developed in increasing powers of $z-z_0$ and $t-t_0$.

I therefore have

$$F(z, t) = \frac{1}{\sqrt{\psi(z, t)}},$$

designating by $\psi(z,t)$ a series developed in increasing powers of z and t.

I will assume that z is sufficiently near $z_0$, and that the points which I have just called $\beta$ and $\gamma$ (extremities of $C''_0$) are near enough to $t_0$ (although their distance at this point $t_0$ has been assumed finite) for the series $\psi$ to converge for $t=\beta$ and for $t=\gamma$.

What will now be the form of this series $\psi$? In the first place, for

$$t = t_0, \quad z = z_0,$$

we must have

$$\psi = 0, \quad \frac{d\psi}{dt} = 0.$$

Therefore, if in $\psi$ we have $z=z_0$, the first term of the development of $\psi$ will be a term in $(t-t_0)^2$. It follows from this and from a theorem of Weierstrass that we have identically /317

$$\psi = [(t-h)^2 + k]\psi_1,$$

where $\psi_1$ is a series developed in powers of $z-z_0$ and $t-t_0$ and not vanishing for $z=z_0$, $t=t_0$; where h and k are two ordered series in powers of $z-z_0$ reducing respectively to $t_0$ and to 0 for $z=z_0$ (Weierstrass, Abhandlungen aus der Functionenlehre, Berlin, Springer, 1886, p. 107 and following; also see Poincaré, Thèse inaugurale, Paris, Gauthier-Villars, 1879).

We can then set

$$\frac{1}{\sqrt{\psi_1}} = \theta, \quad \text{whence} \quad F(z,t) = \frac{\theta}{\sqrt{(t-h)^2+k}},$$

$\theta$ being developable in increasing powers of $z-z_0$ and $t-t_0$.

Let us proceed to a second hypothesis which will be the one where the singular point $z_0$ will be 1 of the points B, R, B' or R' for zero inclination. We would then see that $F(z,t)$ is still of the same form; however, there is one difference. Under the first hypothesis, k is divisible by $z-z_0$ but not by $(z-z_0)^2$; under the second, k is divisible by $(z-z_0)$.

The last hypotheses which remain for us to examine are those where we have either $x_0=\tau$ or $\frac{1}{\tau}$, or $y_0=\frac{1}{\tau'}$. In this case, it may be useful to make a change of variables.

Let us first assume

$$x_0 = \tau \text{ or } \frac{1}{\tau}, \quad y_0 \gtrless \tau', \quad y_0 \gtrless \frac{1}{\tau'}.$$

For new variables we will then take no longer t and z, but x and z; in the neighborhood of the singular point considered, y can be developed in increasing powers of $t-t_0$ and $z-z_0$ and consequently in those of $x-x_0$ and $z-z_0$. The quantity $[F_1^0]^{-2}$ is likewise developable in powers of $z-z_0$ and $x-x_0$.

If therefore we set

$$\psi = [tF(z, t)]^{-2}, \tag{1}$$

$\psi$ will be developable in powers of $x-x_0$ and $z-z_0$ and we will have /318

$$2i\pi\Phi(z) = \int \frac{dx}{cx^2\sqrt{\psi}} \frac{(x-\tau)(1-\tau x)}{1+\tau^2} = \int H(z, x)\,dx.$$

The function $H(z,x)$ under the sign $\int$ presents a singular point only if $\psi=0$.

In order for $\Phi(z)$ to present a singular point, it is necessary that two of the singular points of $H(z,x)$ coalesce. Now this will occur only if we have at the same time

$$\psi = \frac{d\psi}{dx} = 0.$$

The equation $\psi=0$ corresponds to the curves (3) and (4) of the preceding article (or to the curve of the sixth degree which replaces them when inclination is not zero). The equations $\psi=\frac{d\psi}{dx}=0$ correspond to the singular points studied in the first two hypotheses.

From which this consequence: point E and its reciprocal are for the function $\Phi(z)$ only apparent singular points, and we need never concern ourselves with them.

Let us assume

$$x_0 \gtrless \tau, \quad x_0 \gtrless \frac{1}{\tau}, \quad y_0 = \frac{1}{\tau'} \quad \text{or} \quad \tau'.$$

We then will take y and z as new variables; we will find, reserving to $\psi$ the significance which was given it in equation (1),

$$2i\pi\Phi(z) = \int \frac{-dy}{ay^2\sqrt{\psi}} \frac{(y-\tau')(1-\tau' y)}{1+\tau'^2}.$$

From this we would conclude that the points defined by the equations

$$y_0 = \frac{1}{\tau'} \quad \text{or} \quad \tau', \quad \psi = 0$$

(and for which we do not have at the same time $\frac{d\psi}{dy}=0$), i.e., the points V, W and their reciprocals, are for the function $\Phi(z)$ only apparent singular points. /319

In the case where we have at the same time

$$x_0 = \tau \quad \text{or} \quad \frac{1}{\tau}, \quad y_0 = \tau' \quad \text{or} \quad \frac{1}{\tau'},$$

the choice of change of **variables**, which can in addition be made in an infinity of ways, is more delicate. Here is how we can make this choice.

We have

$$z = x^a e^{\frac{a\sin\varphi}{2}\left(\frac{1}{x}-x\right)} y^c e^{\frac{c\sin\varphi'}{2}\left(\frac{1}{y}-y\right)},$$
$$z_0 = x_0^a e^{\frac{a\sin\varphi}{2}\left(\frac{1}{x_0}-x_0\right)} y_0^c e^{\frac{c\sin\varphi'}{2}\left(\frac{1}{y_0}-y_0\right)}.$$

Let us set

$$x^a e^{\frac{a\sin\varphi}{2}\left(\frac{1}{x}-x\right)} = x_0^a e^{\frac{a\sin\varphi}{2}\left(\frac{1}{x_0}-x_0\right)}(1+\xi^2),$$
$$y^c e^{\frac{c\sin\varphi'}{2}\left(\frac{1}{y}-y\right)} = y_0^c e^{\frac{c\sin\varphi'}{2}\left(\frac{1}{y_0}-y_0\right)}(1+\eta^2).$$

Then x will be developable in powers of $\xi$, and y in those of $\eta$; we will have $x=x_0$ for $\xi=0$ and $y=y_0$ for $\eta=0$. On the other hand it will follow that

$$\frac{z}{z_0} - 1 = \xi^2 + \eta^2 + \xi^2\eta^2,$$

whence

$$\eta = \sqrt{\frac{z - z_0 - \xi^2}{z_0(1+\xi^2)}}.$$

In general, $F(z,t)$ and t will be functions developable in powers of $\xi$ and $\eta$ [there would nevertheless be an exception in the case where the inclination is zero and where we would have

$$x_0 = \tau, \qquad y_0 = \tau'$$

or as well

$$x_0 = \frac{1}{\tau}, \qquad y_0 = \frac{1}{\tau'};$$

this point $x=\tau$, $y=\tau'$, which we have called U, as a double point in fact belongs to curve (3); this case would merit special discussion]. /320

Taking z and $\xi$ for independent variables, we therefore have $\Phi(z)=\int \varphi(z,\xi)d\xi$, $\varphi(z,\xi)$ being developable in powers of $\xi$, $z-z_0$ and $\sqrt{z-z_0-\xi^2}$, which permits us to write

$$\Phi(z) = \int \varphi_1\, d\xi + \int \frac{\varphi_2\, d\xi}{\sqrt{z - z_0 - \xi^2}},$$

$\varphi_1$ and $\varphi_2$ being developable in powers of $\xi$ and $z-z_0$.

The first integral is a holomorphic function of z in the neighborhood of the point $z=z_0$; as for the second, it is completely of the same form as the integral

$$\int \frac{\theta\, dt}{\sqrt{(t-h)^2+k}},$$

which we have been led to consider under the first two hypotheses. We must therefore conclude that the points

$$x_0 = \tau \text{ or } \frac{1}{\tau}, \quad y_0 = \tau' \text{ or } \frac{1}{\tau'}$$

are for the function $\Phi(z)$ true singular points and not only apparent ones.

One might be astonished at first glance at the difference between singular points such as E, V, W, etc., which are only apparent, and points such as $x=\tau$, $y=\tau'$, or such as D, etc., which are true singular points.

Their origin appears completely the same; however we obtain these points by writing that two of the singular points $t_1$ and $t_2$ of the function $F(z,t)$ coalesce. But let us examine the matter somewhat more closely. Let us give z a value very near $z_0$, such that the two points $t_1$ and $t_2$ are very little different from each other, and let us study the behavior of the function $F(z,t)$ in the neighborhood of these two points. The difference between these two cases is then very great.

First Case. The point $z_0$ is a point such as D or $x=\tau$, $y=\tau'$; i.e., a true singular point of $F(z)$. /32

Then two values of $F(z,t)$ are exchanged when we turn about point $t_1$, and these two same values are again exchanged when we turn about point $t_2$. If we construct a curve taking t for abscissa and $F(z,t)$ for ordinate, this curve will vary naturally when we vary z, and for $z=z_0$ it will have a double point.

Second Case. The point $z_0$ is a point such as E, i.e., an apparent singular point of $\Phi(z)$.

Then four values of $F(z,t)$ are exchanged when we turn about $t_1$ and $t_2$, namely the first with the second, the third with the fourth when we turn about $t_1$, and the second with the third when we turn about $t_2$.

Let us therefore construct the Riemann surface relative to the function $F(z,t)$, that is to say a Riemann surface having as many sheets as this function $F(z,t)$ has determinations. In the first case, the order of connectivity of this surface will be lowered by two units when z becomes equal to $z_0$; in the second case it will remain the same. This is the true reason for the difference between the two cases.

This circumstance that certain singular points are only apparent may considerably simplify the discussion of the two preceding articles if applied carefully.

100. Nothing is now easier than to determine the behavior of the function $\Phi(z)$ in the neighborhood of the point $z=z_0$.

We have in fact

$$\Phi(z) = \Phi_1(z) + \frac{1}{2i\pi}\int \frac{\theta\, dt}{\sqrt{(t-h)^2+k}},$$

$\Phi_1(z)$ remaining holomorphic for $z=z_0$ and the integral being taken along $C''_0$.

As $\theta$ can be developed in powers of $t-t_0$ and $z-z_0$, and h in those of $z-z_0$, we can write

$$\theta = \theta_0 + \theta_1(t-h) + \theta_2(t-h)^2 + \ldots + \theta_n(t-h)^n + \ldots,$$

such that setting /322

$$2i\pi J_n = \int \frac{(t-h)^n\, dt}{\sqrt{(t-h)^2+k}},$$

it follows that

$$\Phi(z) = \Phi_1(z) + \Sigma\theta_n J_n.$$

On the other hand,

$$2i\pi J_0 = \int_\beta^\gamma \frac{dt}{\sqrt{(t-h)^2+k}} = \log\frac{\gamma - h + \sqrt{(\gamma-h)^2+k}}{\beta - h + \sqrt{(\beta-h)^2+k}}.$$

From this we will conclude (observing that the path of integration passes between $t_1=h+\sqrt{k}$ and $t_2=h-\sqrt{k}$) that

$$2i\pi J_0 = \lambda_0(z) + \log(z-z_0),$$

$\lambda_0(z)$ being holomorphic for $z=z_0$.

In the case where k would be divisible by $(z-z_0)^2$, it would be necessary to say 2 log $(z-z_0)$ (second hypothesis of the preceding article) and not log $(z-z_0)$.

It then follows that

$$2i\pi J_1 = \int_\beta^\gamma \frac{(t-h)dt}{\sqrt{(t-h)^2+k}} = \text{holomorphic function of } z,$$

$$nJ_n + k(n-1)J_{n-2} = \text{holomorphic function of } z.$$

Therefore $J_n$ remains holomorphic in z if n is odd. Now if n is even and if we were to set

$$\alpha_n = \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2},$$

we would have

$$2i\pi J_n = \lambda_n(z) + (-k)^{\frac{n}{2}} \alpha_n \log(z - z_0),$$

$\lambda_n(z)$ being holomorphic in z.

Therefore it finally follows that

$$\Phi(z) = \sum_{n=0}^{n=\infty} \frac{\theta_{2n}(-k)^n \alpha_n}{2i\pi} \log(z - z_0) + \Phi_2(z),$$

$\Phi_2$ remaining holomorphic in z for $z=z_0$.

I can then still write /323

$$\Phi(z) = \Phi_2(z) + \Phi_3(z)\log(z - z_0),$$

$\Phi_2$ and $\Phi_3$ remaining holomorphic for $z=z_0$.

We have

$$\Phi(z) = \Sigma A_{an+b,cn+d} z^n.$$

If therefore

$$\Phi_3(z) = \delta_0 + \delta_1(z - z_0) + \delta_2(z - z_0)^2 + \ldots$$

and if

$$(z - z_0)^h \log(z - z_0) = \Sigma \gamma_{n,h} z^n,$$

for very large n we will have approximately

$$A_{an+b,cn+d} = \delta_0\gamma_{n,0} + \delta_1\gamma_{n,1} + \ldots + \delta_p\gamma_{n,p}.$$

In general, we will be able to content ourselves with taking the first term

$$\delta_0\gamma_{n,0} = \frac{\theta_{0,0}}{2i\pi}\frac{-1}{n z_0^n},$$

$\theta_{0,0}$ being the value of $\theta_0$ for $z=z_0$, or even that of $\theta$ for $z=z_0$, $t=t_0$.

Now if I call $\Delta$ the square of the distance between the two planets, we have

$$F(z, t) = \frac{\theta}{\sqrt{(t-h)^2 + k}} = \frac{t^{ad-bc-1} z^{-\frac{d}{c}}}{\sqrt{\Delta}}.$$

Therefore

$$\theta_{0,0} = \frac{1}{2} t_0^{ad-bc-1} z_0^{-\frac{d}{c}} \frac{d^2\Delta}{dt^2},$$

on the condition, of course, that we make $t=t_0$, $z=z_0$ in $\frac{d^2\Delta}{dt^2}$.

What I have just said is applicable to the first and second hypotheses of the preceding article. If we assumed

$$x_0 = \tau \quad \text{or} \quad \frac{1}{\tau}, \quad y_0 = \tau' \quad \text{or} \quad \frac{1}{\tau'},$$

an analogous method would be applicable since in this case we have reduced $\Phi(z)$ to an integral

$$\int \frac{\varphi_2 \, d\xi}{\sqrt{z - z_0 - \xi^2}},$$

which is of the same form as /324

$$\int \frac{\theta \, dt}{\sqrt{(t-h)^2 + k}}.$$

The coefficient $A_{m_1 m_2}$ which we have just calculated is that which enters the development of the principal part of $F_1^0$ of the perturbative function. We have, in fact, set

$$F_1^0 = \Sigma A_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')}.$$

It would now be possible to take account of the complementary part $F_1 - F_1^0$ of the perturbative function. Let us therefore set

$$F_1 = \Sigma B_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')},$$

then

$$F'(z, t) = F_1 t^{ad-bc-1} z^{-\frac{d}{c}},$$

$$2 i \pi \Phi'(z) = \int F'(z, t) dt.$$

If we assume $m_1 = an+b$ and $m_2 = cn+d$, then $B_{m_1 m_2}$ will be the coefficient of $z^n$ in $\Phi'(z)$ just as $A_{m_1 m_2}$ was the coefficient of $z^n$ in $\Phi(z)$.

The function $F'(z,t) - F(z,t)$ has no other singular points than those of the straight lines

$$x = \tau, \quad x = \frac{1}{\tau}, \quad y = \tau', \quad y = \frac{1}{\tau'}.$$

The function $\Phi'(z)-\Phi(z)$ will therefore have only four singular points, namely

$$x=\tau \text{ or } \frac{1}{\tau}, \quad y=\tau' \text{ or } \frac{1}{\tau'}.$$

From this it results that, if the singular point which is proper for the problem is not one of these four points, that is to say under the first two hypotheses of article 99 (which is the most ordinary case), the difference $B_{m_1m_2}-A_{m_1m_2}$ will be negligible compared to $A_{m_1m_2}$ and the approximate value of $B_{m_1m_2}$ will be the same as that of $A_{m_1m_2}$.

If, on the contrary, the singular point $z_0$, which is proper for the problem, is one of these four points, it will be necessary to consider the difference $B_{m_1m_2}$ /32 $-A_{m_1m_2}$, which however presents no difficulty.

## Application to Astronomy

101. Most often we will be able to content ourselves with a rather rough approximation; what is in fact proposed is to recognize if certain terms, whose order is very high but which in consequence of the close commensurability of the mean motions are affected by very small divisors, if these terms, I say, are or are not negligible. More often they will be, and it will suffice to have an idea of their order of size.

As an example I will take the celebrated inequality of Pallas. In order to study it we must calculate by taking

$$a=2, \quad b=1, \quad c=-1, \quad d=0, \quad n=8,$$

from which

$$m_1=17, \quad m_2=-8.$$

It appears that we might attempt to again find in this way the result of Le Verrier.

## Application to Demonstration of the Non-Existence of Uniform Integrals

102. But this is not the principal aim which I have proposed in undertaking the work. We recall that it is to fill the gap which I indicated at the end of the preceding chapter in the demonstration of the non-existence of uniform integrals.

In article 85, I established in effect that which follows. Let

$$F_1=\Sigma B_{m_1m_2}e^{\sqrt{-1}(m_1l+m_2l')},$$

$B_{m_1m_2}$ depend at the same time on the two major axes, of the two eccentricities the orbits' inclination, longitudes of the two perihelions (measured from the node), that is to say seven variables.

Let

$$m_1 = an, \qquad m_2 = cn,$$

a, c and n being integers, a and c first among them and of opposite sign. Let us give the two major axes determined values chosen such that the ratio of the mean motions is equal to $-\frac{c}{a}$. The coefficients $B_{m_1 m_2}$ will depend on only five variables. As in the preceding chapter, let us set /326

$$D_n = B_{an,cn}\zeta^n;$$

$D_n$ will depend on six variables which are the two eccentricities the longitudes of the perihelions, the inclination and $\zeta$.

Thus, if there existed a uniform integral there would be a relationship among six arbitraries of the quantities $D_n$ and the various quantities

$$D_{-n}, \ \ldots, \ D_{-1}, \ D_0, \ D_1, \ D_2, \ \ldots, \ D_n, \ \ldots$$

could be expressed as functions of only five and not six variables.

Now we have

$$\Phi'(z) = \Sigma B_{an,cn} z^n$$

and consequently

$$\Phi'(z\zeta) = \Sigma D_n z^n.$$

If there were therefore a uniform integral, the coefficients of the development of $\Phi'(z\,\zeta)$ would depend on only five parameters.

In applying the rules of the preceding articles, we would find that for very large n, we have approximately

$$D_n = \left(\frac{\zeta}{z_0}\right)^n \left(\frac{E_1}{n} + \frac{E_2}{n^2} + \frac{E_3}{n^3} + \ldots\right).$$

One would then see without difficulty that if the $D_n$ are expressed by means of only five variables, the same must hold for

$$\frac{\zeta}{z_0}, \ E_1, \ E_2, \ \ldots,$$

and consequently, the $E_i$ depend on only four variables. One would recognize afterward that this is not so.

This was my first plan but it is simpler to operate otherwise. /327

The singular points of $\Phi'(z\,\zeta)$ obviously depend only on the coefficients $D_n$: they should therefore depend on only five variables.

Let

$$z_1, \quad z_2, \quad \ldots, \quad z_6$$

be six singular points of $\Phi'(z)$; the corresponding singular points of $\Phi'(z\,\zeta)$ will be

$$\frac{z_1}{\zeta}, \quad \frac{z_2}{\zeta}, \quad \ldots, \quad \frac{z_6}{\zeta},$$

and they will depend on $\zeta$ and on our five other variables, eccentricities, inclination, longitude of the perihelions, which for the moment I will call

$$\alpha_1, \quad \alpha_2, \quad \ldots, \quad \alpha_5.$$

If there were a uniform integral, they would need only depend on five variables and the functional determinant

$$\frac{\partial\left(\frac{z_1}{\zeta}, \frac{z_2}{\zeta}, \ldots, \frac{z_6}{\zeta}\right)}{\partial(\zeta, \alpha_1, \alpha_2, \ldots, \alpha_5)}$$

would have to be zero.

But this determinant is equal to

$$-\frac{z_1^6}{\zeta^7}\,\frac{\partial\left(\frac{z_2}{z_1}, \frac{z_3}{z_1}, \ldots, \frac{z_6}{z_1}\right)}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_5)}.$$

Now z is not zero nor is $\zeta$ infinite; one would therefore have to have

$$\frac{\partial\left(\frac{z_2}{z_1}, \frac{z_3}{z_1}, \ldots, \frac{z_6}{z_1}\right)}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_5)} = 0.$$

In other words, pairwise the ratios of the singular points of $\Phi'(z)$ would have to depend on only four variables, which I will call $\beta_1, \beta_2, \beta_3, \beta_4$. Now these singular points are of two types.

We have first those which are given us by the equations 13

$$x = \tau \text{ or } \frac{1}{\tau}, \qquad y = \tau' \text{ or } \frac{1}{\tau'},$$

$$z = x^a e^{\frac{e\sin\varphi}{2}\left(\frac{1}{x} - x\right)} y^c e^{\frac{e'\sin\varphi'}{2}\left(\frac{1}{y} - y\right)};$$

I call them $z_1$, $z_2$, $z_3$ and $z_4$.

We see immediately that $z_1$, $z_2$, $z_3$ and $z_4$ depend on only two eccentricities that is to say on $\tau$ and $\tau'$;

$$z_1 z_3 = z_2 z_4 = 1.$$

The ratio $\frac{z_1}{z_3}$ would only depend on our four variables $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$; now this ratio is equal to $z_1^2$. Therefore $z_1$ and similarly $z_2$, $z_3$, $z_4$ would depend only on the four variables $\beta_i$.

Therefore it would be so of $\tau$ and $\tau'$, which are manifestly functions of $z_1$ and $z_2$.

Let us proceed to the singular points of the second kind, which are furnished us by the equations

$$\Delta = 0, \quad \frac{d\Delta}{dt} = 0.$$

When we take x and y as variables in these equations, they become algebraic. The equation $\Delta=0$ then defines, as we have seen, a curve of the sixth degree which, for zero inclination, divides into two third-degree curves (3) and (4); if the inclination is zero, we can deduce from the equation $\frac{d\Delta}{dt}=0$ combined with $\Delta=0$, two others, which are equations (5) and (6) of article 96.

Let $z_0$ be one of the roots of equations

$$\Delta = \frac{d\Delta}{dt} = 0, \tag{1}$$

the ratios $\frac{z_0}{z_1}, \frac{z_0}{z_3}$ and consequently $z_0$ would depend only on the four variables $\beta_i$.

If therefore $z_0$, $z_0'$, $z_0''$ are three roots of equations (1), $z_0$, $z_0'$, $z_0''$, $\tau$ and $\tau'$ would depend only on these four variables, such that the functional determinant /329

$$\frac{\partial(\tau, \tau', z_0, z_0', z_0'')}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)}$$

is zero. Let us assume for example that $\alpha_1$ and $\alpha_2$ are the two eccentricities; $\tau$ will depend only on $\alpha_1$ and $\tau'$ on $\alpha_2$, such that this functional determinant is equal to

$$\frac{d\tau}{d\alpha_1}\frac{d\tau'}{d\alpha_2}\frac{\partial(z_0, z_0', z_0'')}{\partial(i, \varpi, \varpi')},$$

since the last three variables are the inclination i and the longitudes of the perihelions $\varpi$ and $\varpi'$.

We would therefore have

$$\frac{\partial(z_0, z'_0, z''_0)}{\partial(i, \varpi, \varpi')} = 0,$$

which would mean that the roots of equation (1)(when we regard the two eccentricities and consequently $\tau$ and $\tau'$ as constant) and depend only on two variables

It remains for me to demonstrate that this is not so.

103. Let us begin with the case where inclination is zero. In this case, the roots of equations (1) depend only on the major axes, the eccentricities and the difference $\varpi-\varpi'$. If we regard the major axes and the eccentricities as constants as we have just done, these roots will depend only on the difference $\varpi-\varpi'$.

In recalling what we said in article 85 and in reasoning as we have just done in the preceding article, we would see that in order for the Problem of Three Bodies in the plane to admit a uniform integral (other than those of vis viva and area), it would be necessary that these roots not depend on $\varpi-\varpi'$ and that they remain constant when the major axes and the eccentricities themselves remain constant and the inclination zero.

Now it is clear that this is not so, for $z_0$ is real when $\varpi-\varpi'$ is zero and imaginary, in general, in the opposite case.

Let us now return to the case where the inclination is not zero. /33

Let us enumerate the singular points given by equations

$$\Delta = 0, \quad \frac{d\Delta}{dt} = 0. \tag{1}$$

To do so, let us assume the inclination to be very small; we will see, in referring to what has been said in article 98, that there exist:

(1) Eight singular points little different from D, C, F, T and their reciprocals;

(2) Eight singular points of which two differ very little from B, two others very little from R and two others very little from each of their reciprocals;

(3) Four points very little different from U ($x=\tau$, $y=\tau'$) and in fact, when the inclination is zero, the two curves $\Delta=0$, $\frac{d\Delta}{dt}=0$ have a double point at U;

(4) Four points very little different from U' ($x=\frac{1}{\tau}$ $y=\frac{1}{\tau'}$).

In all, 24 singular points.

We may arrive at the same result in a different manner.

We see that

$$x^2y^2\Delta = P$$

is an integral polynomial of the sixth order in x and y, such that the equation

$$P=0$$

is that of a curve of the sixth order which divides into two others, (3) and (4), when the inclination is zero.

On the other hand, the equation $\frac{d\Delta}{dt}=0$ can be replaced by the following:

$$Q = cx^2(1+\tau^2)(y-\tau')(1-\tau'y)\frac{dP}{dx} - ay^2(1+\tau'^2)(x-\tau)(1-\tau x)\frac{dP}{dy} = 0.$$

This equation Q=0 is that of a curve of the ninth order, and the singular points will be the intersections of these two curves, less those which are rejected at the origin or at infinity.

The curve P=0 admits the origin as a double point and the axes as double asymptotes; the curve Q=0 admits the origin as a triple point and the two axes as triple asymptotes. /331

But there is more. We may note that P is the sum of three squares, such that I may write

$$P = U_1^2 + U_2^2 + U_3^2 = \Sigma U^2,$$

with

$$U = Ax^2y + Bxy^2 + Cxy + Dx + Ey.$$

On the other hand, we may set

$$V = x\frac{dU}{dx} - U = Ax^2y - Ey,$$

whence

$$x\frac{dP}{dx} = 2\Sigma VU + P.$$

In considering P=0, we then arrive at

$$\begin{aligned} Q = {} & 2cxy(1+\tau^2)(y-\tau')(1-\tau'y)\Sigma(Ax^2-E)U \\ & - 2axy(1+\tau'^2)(x-\tau)(1-\tau x)\Sigma(By^2-D)U, \end{aligned}$$

so that in suppressing the factor 2xy the system

$$P = Q = 0$$

can be replaced by the following:

$$P = 0,$$
$$R = c(1+\tau'^2)(y-\tau')(1-\tau' y)\Sigma(Ax^2-E)U - a(1+\tau^2)(x-\tau)(1-\tau x)\Sigma(By^2-D)U = 0.$$

The curve R=0 is only of the seventh order; it no longer has only a simple point as origin; it admits as asymptotes the two axes, two straight lines other than the axis of the x and parallel to this axis, two straight lines other than the axis of the y and parallel to this axis, and one straight line not parallel to the axes.

The two curves R=0 and P=0 have 42 intersections in all. Among these intersections there are two at the origin. Let us see how many there are at infinity in the direction of the axis of the x.

The curve R has three asymptotes parallel to the axis of x, among which there is this axis itself; curve P admits this axis as double asymptote; in general, this would make seven points of intersection. In general, in fact, if there is a double asymptote, there is a "point of retrogression at infinity". This is not so for the curve P, but it presents two distinct branches of curves touching at infinity, which gives not seven, but eight points of intersection.

We therefore have at infinity eight points in the direction of the axis of the x, and eight in that of the axis of the y.

There remain then

$42 - 2 - 8 - 8 = 24$ singular points.

This granted, is it possible that the z of these 24 singular points depend on only two variables? Let us call $\gamma_1$ and $\gamma_2$ these two variables. We can choose a third $\gamma_3$ such that i, $\varpi$ $\varpi'$ are functions of $\gamma_1, \gamma_2, \gamma_3$. Then, when we vary $\gamma_3$, the two other variables $\gamma_1$ and $\gamma_2$ remaining constant, the z must not vary.

One has by hypothesis

$$\Delta = 0, \quad \frac{d\Delta}{dt} = 0.$$

Differentiating the first of these two equations, we find

$$\frac{d\Delta}{dt}dt + \frac{d\Delta}{dz}dz + \frac{d\Delta}{d\gamma_3}d\gamma_3 = 0.$$

Now $\frac{d\Delta}{dt}=0$ and on the other hand dz would be zero since z was not varied. It would therefore result that

$$\frac{d\Delta}{d\gamma_3} = 0. \tag{2}$$

Let us see what this equation indicates. If we vary $\gamma_3$, the curve $\Delta=0$ (or what goes back to the same thing, the curve P=0) varies; let us consider the curves)

$$\Delta + \frac{d\Delta}{d\gamma_3} d\gamma_3 = 0,$$

infinitely little different from P=0; I will call it curve P'. Equation (2) would signify that this curve P' would have to pass through the 24 singular points.

Now these two curves P and P' are of the sixth order; they therefore cannot admit more than 36 points of intersection without being identical. /333

They have four at the origin where they both have a double point.

They admit the axis of the x as double asymptote, which makes (bearing in mind the remark made above on the subject of the nature of this double asymptote) eight intersections at infinity in the direction of the axis of x. There are likewise eight in the direction of the axis of y.

This would make in all

$$24 + 4 + 8 + 8 = 44 \text{ intersections.}$$

The two curves would therefore have to coalesce.

Thus, if we were to vary $\gamma_3$, the curve P=0 need not vary.

Let us interpret this result.

Let us consider the two ellipses described by the two planets. These two ellipses will be invariable in size and form since we have agreed to regard the major axes and the eccentricities as constants, but when we vary i, $\varpi$ and $\varpi'$ these two ellipses are displaced with respect to each other. I may assume that one of the ellipses, E, is fixed and the other, E', is moving.

To say that the curve C=0 does not change when $\gamma_1$ and $\gamma_2$ remain constant is to say that we may find a law of the motion of E' such that if at any moment a point M' of E' is at a zero distance from a point M of E (needless to recall that these two points, being imaginary, can be at zero distance without coinciding), the distance of these two points will remain constantly zero.

Let $M'_0$ be the position of the point M' at any instant. On E there are four points $M_1$, $M_2$, $M_3$, $M_4$ which are at a zero distance from $M'_0$; these four points cannot be a straight line. The point M' would therefore have to remain on four spheres of the radius zero having their centers at $M_1$, $M_2$, $M_3$, $M_4$; but, as these centers are not in a straight line, these four spheres can have only two common points at a finite distance. It is therefore impossible that the point M' move while remaining on these four spheres.

The non-existence of the uniform integrals is thus rigorously demonstrated.

104. Let /335

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n) \tag{1}$$

be n simultaneous differential equations. The X are functions of x and t.

With respect to x, they may be developed as a power series.

With respect to t, they are periodic and of period 2π.

Let

$$x_1 = x_1^0, \quad x_2 = x_2^0, \quad \ldots, \quad x_n = x_n^0$$

be a particular periodic solution of these equations. The $x_i^0$ will be periodic functions of t of period 2π. Let us set

$$x_i = x_i^0 + \xi_i.$$

It will follow that

$$\frac{d\xi_i}{dt} = \Xi_i. \tag{2}$$

The Ξ are functions of ξ and t, periodic with respect to t and developed in powers of ξ; however, there will be no more terms independent of ξ.

If ξ are very small and we neglect their squares, the equations reduce to

$$\frac{d\xi_i}{dt} = \frac{dX_i}{dx_1^0}\xi_1 + \frac{dX_i}{dx_2^0}\xi_2 + \cdots + \frac{dX_i}{dx_n^0}\xi_n, \tag{3}$$

which are the equations of variation of equations (1).

They are linear and have periodic coefficients. We know the form of their general solution; we find /336

$$\begin{aligned}
\xi_1 &= A_1 e^{\alpha_1 t}\varphi_{11} + A_2 e^{\alpha_2 t}\varphi_{21} + \ldots + A_n e^{\alpha_n t}\varphi_{n1},\\
\xi_2 &= A_1 e^{\alpha_1 t}\varphi_{12} + A_2 e^{\alpha_2 t}\varphi_{22} + \ldots + A_n e^{\alpha_n t}\varphi_{n2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\xi_n &= A_1 e^{\alpha_1 t}\varphi_{1n} + A_2 e^{\alpha_2 t}\varphi_{2n} + \ldots + A_n e^{\alpha_n t}\varphi_{nn},
\end{aligned}$$

A being constants of integration, α fixed constants which are called characteristic exponents, and φ being periodic functions of t.

If we then set

$$\begin{aligned} \xi_1 &= \eta_1\varphi_{11} + \eta_2\varphi_{21} + \ldots + \eta_n\varphi_{1n}, \\ \xi_2 &= \eta_1\varphi_{12} + \eta_2\varphi_{22} + \ldots + \eta_n\varphi_{2n}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \xi_n &= \eta_1\varphi_{1n} + \eta_2\varphi_{2n} + \ldots + \eta_n\varphi_{nn}, \end{aligned}$$

equations (2) will become

$$\frac{d\eta_i}{dt} = H_i, \tag{2'}$$

where $H_i$ are functions of t and of $\eta$ of the same form as $\Xi$.

In addition we may write

$$\frac{d\eta_i}{dt} = H_i^1 + H_i^2 + \ldots + H_i^p + \ldots; \tag{2''}$$

$H_i^p$ represents the total of the terms of $H_i$ which are of degree p with respect to $\eta$.

As for equations (3), they become

$$\frac{d\eta_i}{dt} = H_i^1 = \alpha_i\eta_i. \tag{3'}$$

Let us now seek the form of the general solutions of equations (2) and (2').

I say that we must find:

$\eta_i$ = function developed in powers of $A_1e^{\alpha_1 t}$, $A_2e^{\alpha_2 t}$, ..., $A_ne^{\alpha_n t}$ whose coefficients are periodic functions of t.

We may then write /337

$$\eta_i = \eta_i + \eta_i^2 + \ldots + \eta_i^p + \ldots, \tag{4'}$$

$\eta_i^p$ representing the total of the terms of $\eta_i$ which are of degree p with respect to A.

We will replace $\eta_i$ by their values in $H_i^p$ and we will find

$$H_i^p = H_i^{p,p} + H_i^{p,p+1} + \ldots + H_i^{p,q} + \ldots,$$

$H_i^{p,q}$ designating the total of the terms which are of degree q with respect to A.

We will then find

$$\frac{d\eta_i^1}{dt} = \alpha_i \eta_i^1, \qquad \eta_i^1 = A_i e^{\alpha_i t},$$

$$\frac{d\eta_i^2}{dt} - \alpha_i \eta_i^2 = H_i^{2,2}, \qquad \frac{d\eta_i^3}{dt} - \alpha_i \eta_i^3 = H_i^{2,3} + H_i^{3,3},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$\frac{d\eta_i^q}{dt} - \alpha_i \eta_i^q = H_i^{2,q} + H_i^{3,q} + \dots + H_i^{q,q} = K_q.$$

These equations will permit successively calculating by recurrence

$$\eta_i^2, \quad \eta_i^3, \quad \dots, \quad \eta_i^q, \quad \dots.$$

In fact, $K_q$ depends only on $\eta^1$, $\eta^2$, ..., $\eta^{q-1}$. If we assume that these quantities have been previously calculated, we may write $K_q$ in the following form:

$$K_q = \Sigma A_1^{\beta_1} A_2^{\beta_2} \dots A_n^{\beta_n} e^{t(\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n)} \psi,$$

$\beta$ being positive integers whose sum is q and $\psi$ a periodic function

We may again write

$$\psi = \Sigma C e^{\gamma t\sqrt{-1}},$$

C being a generally imaginary coefficient and $\gamma$ a positive or negative integer. For brevity, we will write

$$A_1^{\beta_1} A_2^{\beta_2} \dots A_n^{\beta_n} = A^q, \qquad \alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n = \Sigma\alpha\beta,$$

and it will follow that /338

$$\frac{d\eta_i^q}{dt} - \alpha_i \eta_i^q = \Sigma C A^q e^{t(\gamma\sqrt{-1} + \Sigma\alpha\beta)}.$$

Now we can satisfy this equation by setting

$$\eta_i^q = \sum \frac{C A^q e^{t(\gamma\sqrt{-1} + \Sigma\alpha\beta)}}{\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i}.$$

An exception would be the case where we would have

$$\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i = 0,$$

in which case it would introduce itself into the formulas for the terms in t. We will reserve this case, which does not in general present itself.

105. We must now treat the question of the convergence of these series. In addition, the sole difficulty results, as we will see, from the divisors

$$\gamma\sqrt{-1}+\Sigma\alpha\beta-\alpha_i. \tag{5}$$

Let us replace equations (2') by the following:

$$\epsilon\eta_i = \epsilon A_i e^{\alpha_i t} + \overline{H}_i^2 + \overline{H}_i^3 + \dots + \overline{H}_i^p + \dots \tag{2''}$$

Let us define $\overline{H}_i^p$. We see without difficulty that $H_i^p$ has the following form:

$$H_i^p = \Sigma C \eta_1^{\beta_1}\eta_2^{\beta_2}\dots\eta_n^{\beta_n} e^{\gamma t\sqrt{-1}}.$$

C is an arbitrary constant, β are positive integers whose sum is p, and γ is a positive or negative integer. We will then take

$$\overline{H}_i^p = \Sigma |C| \eta_1^{\beta_1}\eta_2^{\beta_2}\dots\eta_n^{\beta_n}.$$

The series thus obtained will be convergent provided the trigonometric series which define the periodic functions which the H depend on are absolutely and uniformly convergent; now this will always occur because these periodic functions /33 are analytic. As for ϵ, it is a positive constant.

We can extract η from equations (2") in the following form:

$$\eta_i = \Sigma M \epsilon^{-\delta} A_1^{\beta_1}A_2^{\beta_2}\dots A_n^{\beta_n} e^{(\Sigma\alpha\beta)t}. \tag{4''}$$

In addition several terms may correspond to the same exponents β, and δ is a positive integer. If we compare this with the series taken from (2'), which are written

$$\eta_i = \Sigma N \frac{A_1^{\beta_1}A_2^{\beta_2}\dots A_n^{\beta_n}}{\Pi} e^{t(\Sigma\alpha\beta+\gamma\sqrt{-1})},$$

here is what is observed: 1) M is real, positive and greater than |N|. 2) It designates the product of the divisors (5) whose number is at most equal to δ.

If series (4") therefore converges and if none of the divisors (5) is smaller than ϵ, series (4') likewise converges. This, therefore, is how we can express the condition of convergence.

The series converges if the expression

$$\gamma\sqrt{-1}+\Sigma\alpha\beta-\alpha_i$$

cannot become smaller than any given quantity ϵ for integral and positive values of β and integral (positive or negative) values of γ; that is to say, if neither of the two convex polygons which contain, first the α and $+\sqrt{-1}$ and the second the α and $-\sqrt{-1}$, at the same time contains the origin; or if all quantities α have their real parts of the same sign and if none has its real part zero.

What will we then do if this is not so?

Let us assume, for example, that k of quantities $\alpha$ have their real parts positive and that n-k have real parts which are negative or zero. It will then occur that series (4") will remain convergent if we annul the constants A which correspond to an $\alpha$ whose real part is negative or zero, so that these series will no longer give us the general solution of the proposed equations, but a solution containing only k arbitrary constants. This solution is represented by a series (4') developed in powers of /340

$$A_1 e^{\alpha_1 t}, \ A_2 e^{\alpha_2 t}, \ \ldots, \ A_k e^{\alpha_k t},$$

because, by hypothesis, the real parts of

$$\alpha_1, \ \alpha_2, \ \ldots, \ \alpha_k$$

are positive and the exponentials

$$e^{\alpha_1 t}, \ e^{\alpha_2 t}, \ \ldots, \ e^{\alpha_k t}$$

tend toward 0 when t tends toward $-\infty$. Therefore the same is true for the quantities $\eta_i$, which means that when t tends toward $-\infty$, the solution represented by series (4') asymptotically approaches the periodic solution considered. We will for this reason call it an asymptotic solution.

We will obtain a second system of asymptotic solutions by annulling in series (4') all coefficients A which correspond to exponents $\alpha$ whose real part is positive or zero. This series is then developed in powers of

$$A'_1 e^{\alpha'_1 t}, \ A'_2 e^{\alpha'_2 t}, \ \ldots, \ A'_k e^{\alpha'_k t},$$

the exponents $\alpha'_1$, $\alpha'_2$, ..., $\alpha'_k$ having their real part negative. If we then make t tend toward $+\infty$, the corresponding solution will asymptotically approach the considered periodic solution.

If we assume that the given equations go back to the equations of Dynamics, we have seen that n is even and that the $\alpha$ are equal to each other and of opposite sign.

Then, if k among them have their real part positive, k will have their real part negative and n-2k will have their real part zero. In first taking the $\alpha$, which have their real part positive, we will obtain a particular solution containing k arbitrary constants; from this we will obtain a second, taking the $\alpha$ which have their real part negative.

In the case where none of the $\alpha$ has its real part zero and in particular if all $\alpha$ are real, we have in addition

$$k = \frac{n}{2}.$$

106. Let us assume that in equations (1) the X depend on a parameter $\mu$ and the functions X are developable in powers of this parameter. /341

Let us imagine that, for $\mu=0$, the characteristic exponents $\alpha$ are all distinct so that these exponents, being defined by an equation $G(\alpha, \mu)=0$ (analogous to that of article 74, but such that the equation $G(\alpha, 0)=0$ has all its roots distinct) are themselves developable in powers of $\mu$ by virtue of articles 30 and 31.

Let us assume finally that one has, as we have just said, annulled all constants A which correspond to an $\alpha$ whose real part is negative or zero.

Series (4') which define the quantities $\eta_i$ then depend on $\mu$. I propose to establish that these series can be developed not only in powers of the $A_i e^{\alpha_i t}$ but also in powers of $\mu$.

Let us consider the inverse of one of the divisors (5)

$$(\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i)^{-1}.$$

I say that this expression can be developed in powers of $\mu$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the k characteristic exponents whose real part is positive for $\mu=0$ and for small values of $\mu$ and which we have agreed to retain. Each is developable in powers of $\mu$. Let $\alpha_i^0$ be the value of $\alpha_i$ for $\mu=0$; we may take $\mu_0$ sufficiently small so that $\alpha_i$ differs as little as we wish from $\alpha_i^0$ when $|\mu|<\mu_0$. Then let h be a positive quantity smaller than the smallest of the real parts of the k quantities $\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0$; we can take $\mu_0$ sufficiently small so that when $|\mu|<\mu_0$, the k exponents $\alpha_1, \alpha_2, \ldots, \alpha_k$ have their real part greater than h.

The real part of $\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i$ will then be greater than h (if $\beta_i>0$), so that we will have

$$|\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i| > h. \qquad (6)$$

Thus, if $|\mu|<\mu_0$, the function

$$(\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i)^{-1}$$

remains uniform, finite, continuous and smaller in absolute value than $\frac{1}{h}$. /342

From this we will conclude, according to a well-known theorem, that this function is developable in powers of $\mu$ and that the coefficients of the development are smaller in absolute value than those of the development of

$$\frac{1}{h\left(1-\frac{\mu}{\mu_0}\right)}.$$

It is to be remarked that the numbers h and $\mu_0$ are independent of the integers $\beta$ and $\gamma$.

There would be an exception in the case where $\beta_i$ is zero. The real part of the divisor (5) could then be smaller than h and could even be negative. It is in fact equal to the real part of $\Sigma\,\alpha\,\beta$, which is positive, less the real part of $\alpha_i$ which is likewise positive and which can be greater than that of $\Sigma\,\alpha\,\beta$ if $\beta_i$ is zero.

Let us assume that the real part of $\alpha_i$ remains smaller than a certain number $h_2$ so long as $|\mu|<\mu_0$. Then, if

$$\Sigma\beta > \frac{h_1}{h} + 1 \tag{7}$$

the real part of (5) is certainly greater than h; therefore there can only be difficulty for those of divisors (5) for which inequality (7) does not occur.

Let us now assume that the imaginary part of quantities $\alpha_1, \alpha_2, \ldots, \alpha_k$ remains constantly smaller in absolute value than a certain positive number $h_2$; if we then have

$$|\gamma| > h_2\Sigma\beta + h, \tag{8}$$

the imaginary part of (5) and consequently its modulus will still be greater than h; so that there can be difficulty only for those of the divisors (5) for which none of the inequalities (7) and (8) takes place. But these divisors, which satisfy none of these inequalities, are finite in number.

According to a hypothesis which we made above, none of them vanishes for the values of $\mu$ which we consider; we can therefore take h and $\mu_0$ sufficiently small /343 for the absolute value of any one of them to remain greater than h when $|\mu|$ remains smaller than $\mu_0$.

Then the inverse of an arbitrary divisor (5) is developable in powers of $\mu$, and the coefficients of development are smaller in absolute value than those of

$$\frac{1}{h\left(1-\frac{\mu}{\mu_0}\right)}.$$

We have written further above

$$H_i^p = \Sigma C \eta_1^{\beta_1} \eta_2^{\beta_2} \cdots \eta_n^{\beta_n} e^{\gamma t \sqrt{-1}}.$$

According to our hypotheses, C can be developed in powers of $\mu$ so that I may set

$$C = \Sigma E \mu^{\prime}, \qquad H_i^p = \Sigma E \mu^{\prime} \eta_1^{\beta_1} \eta_2^{\beta_2} \cdots \eta_n^{\beta_n} e^{\gamma t \sqrt{-1}}.$$

Let us now return to equations (2"), here getting

$$\varepsilon = h\left(1 - \frac{\mu}{\mu_0}\right),$$

$$\overline{H}_i^p = \Sigma |E| \mu^{\prime} \eta_1^{\beta_1} \eta_2^{\beta_2} \cdots \eta_n^{\beta_n}.$$

The second members of equations (2") will then be convergent series ordered in powers of $\mu$, $\eta_1$, $\eta_2$, ..., $\eta_n$.

From this we will obtain the $\eta_i$ in the form of series (4"), convergent and ordered in powers of $\mu$, $A_1 e^{\alpha_1 t}$, $A_2 e^{\alpha_2 t}$, ..., $A_k e^{\alpha_k t}$.

From equations (2') we would on the other hand obtain the $\eta_i$ in the form of the series (4'), ordered in powers of $\mu$, $A_1 e^{\alpha_1 t}$, $A_2 e^{\alpha_2 t}$, ..., $A_k e^{\alpha_k t}$, $e^{t\sqrt{-1}}$, $e^{-t\sqrt{-1}}$. Each of the terms of (4') is smaller in absolute value than the corresponding term of (4"), and as series (4") converge, the same will be true for series (4').

## Asymptotic Solutions of the Equations of Dynamics

107. Let us return to equations (1) of article 13 /344

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots, n), \tag{1}$$

and the hypotheses made concerning them in this article.

We have seen in article 42 that these equations admit periodic solutions and we can conclude from this that, provided one of the corresponding characteristic exponents $\alpha$ is real, these equations will also admit asymptotic solutions.

At the end of the preceding article we considered the case where in equations (1) of article 104 the second members $X_i$ are developable in powers of $\mu$, but where the characteristic exponents remain distinct from one another for $\mu=0$.

In the case of the equations which are now going to concern us, that is to say, equations (1) of article 13, the second members are still developable in terms of the powers of $\mu$; however, all the characteristic exponents are zero for $\mu=0$.

From this there results a large number of important differences.

In the first place, the characteristic exponents $\alpha$ are not developable in powers of $\mu$, but in those of $\sqrt{\mu}$ (cf. article 74). Similarly, the functions which I have called $\varphi_{i,k}$ at the beginning of article 104 (and which in the particular case of the equations of Dynamics which here concern us are nothing other than the functions $S_i$ and $T_i$ of article 79) are developable not in powers of $\mu$, but in powers of $\sqrt{\mu}$.

Then in equations (2') of article 104

$$\frac{d\eta_i}{dt} = H_i;$$

the second member $H_i$ is developed in powers of $\eta$, $e^{t\sqrt{-1}}$, $e^{-t\sqrt{-1}}$ and $\sqrt{\mu}$ (and not of $\mu$).

From this we will obtain the $\eta_i$ in the form of the series obtained in article 104

$$\eta_i = \sum N A_1^{\beta_1} A_2^{\beta_2} \dots A_n^{\beta_n} e^{t(\sum \alpha\beta + \gamma\sqrt{-1})}$$

and N and $\Pi$ will be developed in powers of $\sqrt{\mu}$.

A certain number of questions then present themselves naturally: /345

1) We know that N and $\Pi$ are developable in powers of $\sqrt{\mu}$; is the same true for quotient $\frac{N}{\Pi}$?

2) If this is so, there exist series ordered in powers of $\sqrt{\mu}$, $A_i e^{\alpha_i t}$, $e^{t\sqrt{-1}}$ and $e^{-t\sqrt{-1}}$ which formally satisfy the proposed equations; are these series convergent?

3) If they are not convergent, what part can we extract from them for calculation of the asymptotic solutions?

## Development of These Solutions in Powers of $\sqrt{\mu}$

108. I propose to demonstrate that we can develop $\frac{N}{\Pi}$ in powers of $\sqrt{\mu}$ and that consequently there exist series ordered in powers of $A_i e^{\alpha_i t}$ and which formally satisfy equations (1). One might doubt this to be so; in fact, $\Pi$ is the product of divisors (5) of article 104. All these divisors are developable in powers of $\sqrt{\mu}$. However, some of them, those for which $\gamma$ is zero, vanish with $\sqrt{\mu}$. It may occur that $\Pi$ vanish with $\mu$ and contain as a factor a certain power of $\sqrt{\mu}$. If N did not then contain this same power as a factor, the $\frac{N}{\Pi}$ would still develop in terms of the increasing powers of $\sqrt{\mu}$, but the development would begin with negative powers.

I say that it is not so and that the development of $\frac{N}{\Pi}$ contains only positive powers of $\sqrt{\mu}$.

Let us see by what mechanism these negative powers of $\sqrt{\mu}$ disappear. Let us set $A_i e^{\alpha_i t} = w_i$ and let us consider x and y as functions of the variables t and **w**.

Before proceeding, it is important to make the following remark: among the 2n characteristic exponents $\alpha$, two are zero and the others are in pairs equal and of opposite sign. At most, we will retain only n-1 of these exponents, agreeing to regard as zero the coefficients $A_i$ and the variables $w_i$ which correspond to the rejected n+1 exponents. We will retain only those exponents whose real part is positive.

This granted, equations (1) become

$$\frac{dx_i}{dt} + \Sigma_k \alpha_k w_k \frac{dx_i}{dw_k} = \frac{dF}{dy_i}, \tag{2}$$

$$\frac{dy_i}{dt} + \Sigma_k \alpha_k w_k \frac{dy_i}{dw_k} = -\frac{dF}{dx_i}. \tag{3}$$

In leaving these equations, let us attempt to develop $x_i$ and $y_i - n_i t$ in increasing powers of $\sqrt{\mu}$ and of w so that the coefficients are periodic functions of t.

We may write

$$\alpha_k = \alpha_k^1 \sqrt{\mu} + \alpha_k^2 \mu + \ldots = \Sigma \alpha_k^p \mu^{\frac{p}{2}},$$

for we have in article 74 how we may develop the characteristic exponents in powers of $\sqrt{\mu}$.

On the other hand, let us write

$$x_i = x_i^0 + x_i^1 \sqrt{\mu} + \ldots = \Sigma x_i^p \mu^{\frac{p}{2}},$$

$$y_i - n_i t = y_i^0 + y_i^1 \sqrt{\mu} + \ldots = \Sigma y_i^p \mu^{\frac{p}{2}},$$

the $x_i^p$ and the $y_i^p$ being functions of t and of w, periodic with respect to t and developable in powers of w.

If in equations (2) and (3) we substitute these values in place of $\alpha_k$, $x_i$ and $y_i$, the two members of these equations will be developed in powers of $\sqrt{\mu}$.

In the two members of equations (2) let us equate the coefficients of $\mu^{\frac{p+1}{2}}$, and in the two members of equations (3) let us equate the coefficients of $\mu^{\frac{p}{2}}$; we will obtain the following equations:

$$\left.\begin{aligned} \frac{dx_i^{p+1}}{dt} + \Sigma_k \alpha_k^1 w_k \frac{dx_i^p}{dw_k} &= Z_i^p + \sum_k \frac{d^2 F_1}{dy_i^0\, dy_k^0} y_k^{p-1}, \\ \frac{dy_i^p}{dt} + \Sigma_k \alpha_k^1 w_k \frac{dy_i^{p-1}}{dw_k} &= T_i^p - \sum_k \frac{d^2 F_0}{dx_i^0\, dx_k^0} x_k^p, \end{aligned}\right\} \qquad (4)$$

where $Z_i^p$ and $T_i^p$ depend only on /347

$$x_i^0,\ x_i^1,\ \ldots,\ x_i^{p-1},$$
$$y_i^0,\ y_i^1,\ \ldots,\ y_i^{p-2},$$

Let us agree, as we did above, to represent the mean value of U by [U] if U is a periodic function of t.

From equations (4) we will then be able to deduce the following:

$$\left.\begin{aligned} \Sigma_k \alpha_k^1 w_k \frac{d[x_i^p]}{dw_k} &= [Z_i^p] + \sum_k \left[\frac{d^2 F_1}{dy_i^0\, dy_k^0} x_k^{p-1}\right], \\ \Sigma_k \alpha_k^1 w_k \frac{d[y_i^{p-1}]}{dw_k} &= [T_i^p] - \sum_k \frac{d^2 F_0}{dx_i^0\, dx_k^0} [y_k^p]. \end{aligned}\right\} \qquad (5)$$

Let us now assume what a previous calculation has given us:

$$x_i^0,\ x_i^1,\ \ldots,\ x_i^{p-1},\ x_i^p - [x_i^p],$$
$$y_i^0,\ y_i^1,\ \ldots,\ y_i^{p-2},\ y_i^{p-1} - [y_i^{p-1}].$$

Equations (5) will permit us to calculate $[x_i^p]$ and $[y_i^{p-1}]$ and consequently $x_i^p$ and $y_i^{p-1}$. Equations (4) will then permit us to determine

$$x_i^{p+1} - [x_i^{p+1}] \text{ and } y_i^p - [y_i^p],$$

such that this process will furnish us by recurrence all coefficients of the developments of $x_i$ and $y_i$.

The sole difficulty is the determination of $[x_i^p]$ and $[y_i^{p-1}]$ by equations (5).

The functions $[x_i^p]$ and $[y_i^{p-1}]$ are developed in increasing powers of w, and we are going to calculate the various terms of these developments beginning with the terms of the lowest degree.

To do so we are going to return to the notations of article 79, that is to say, we are going to set

$$-\frac{d^2 F_0}{dx_i^0\, dx_k^0} = C_{ik}^0 \text{ and } \left[\frac{d^2 F_1}{dy_i^0\, dx_k^0}\right] = b_{ik}$$

(for zero values of w).

If now we call $\xi_i$ and $\eta_i$ the coefficients of /348

$$w_1^{m_1} w_2^{m_2} \ldots w_{n-1}^{m_{n-1}}$$

in $[x_i^p]$ and $[y_i^{p-1}]$ in order to determine these coefficients, we will have the following equations:

$$\begin{aligned} \Sigma_k h_{ik} \eta_k - S\xi_i &= \lambda_i, \\ \Sigma_k C_{ik} \xi_k - S\eta_i &= \mu_i. \end{aligned} \qquad (6)$$

In these equations (6), $\lambda_i$ and $\mu_i$ are given quantities because they depend only on

$$x_i^0, \; x_i^1, \; \dots, \; x_i^{p-1}, \; x_i^p - [x_i^p],$$
$$y_i^0, \; y_i^1, \; \dots, \; y_i^{p-2}, \; y_i^{p-1} - [y_i^{p-1}]$$

or on terms of $[x_i^p]$ and $[y_i^{p-1}]$ whose degree with respect to w is smaller than

$$m_1 + m_2 + \dots + m_{n-1}.$$

In addition, for brevity we have set

$$S = m_1 \alpha_1^1 + m_2 \alpha_2^1 + \dots + m_{n-1} \alpha_{n-1}^1.$$

For calculation of the coefficients $\xi_i$ and $\eta_i$ we therefore have a system of linear equations; there could be difficulty here only if the determinant of these equations was zero but this determinant is equal to

$$S^2[S^2 - (\alpha_1^1)^2][S^2 - (\alpha_2^1)^2] \dots [S^2 - (\alpha_{n-1}^1)^2].$$

It could vanish only for

$$S = 0, \qquad S = \pm \alpha_i^1,$$

that is to say, for

$$m_1 + m_2 + \dots + m_{n-1} = 0 \text{ or } 1.$$

One could therefore encounter difficulty only in the calculation of the terms of degree 0 or 1 with respect to w.

However, we need not go back to the calculation of these terms; in fact, we have learned to calculate the terms independent of w in article 44 and the coefficients of /323

$$w_1, \; w_2, \; \dots, \; w_{n-1}$$

in article 79.

The terms independent of w are in fact none other than series (2) of article 44, and the coefficients of

$$w_1, \; w_2, \; \dots, \; w_{n-1}$$

are none other than the series $S_i$ and $T_i$ of article 79.

It remains for me to say a word on first approximations.

We will give the $x_i^0$ constant values which are none other than those which we have designated thus in article 44.

We will then have the following equations:

$$\left.\begin{aligned} \frac{dy_i^0}{dt} = 0, \quad \frac{dx_i^1}{dt} = 0, \quad \frac{dy_i^1}{dt} + \Sigma_k \alpha_k^1 w_k \frac{dy_i^0}{dw_k} = -\sum_k \frac{d^2 F_0}{dx_i^0\, dx_k^0} x_k^1, \\ \frac{dx_i^2}{dt} + \Sigma_k \alpha_k^1 w_k \frac{dx_i^1}{dw_k} = \frac{dF_1}{dy_i^0}. \end{aligned}\right\} \tag{7}$$

In $F_0$, which depends only on the $x_i$, these quantities must be replaced by $x_i^0$. In $F_1$ the $x_i$ are replaced by $x_i^0$ and the $y_i$ by $n_i{}^t$. $F_1$ then becomes **a periodic** function of t whose period is T. We will designate $\psi$ the mean value of this periodic function $F_1$; $\psi$ is then a periodic function of period $2\pi$ with respect to $y_i^0$.

The first two equations (7) show that $y_i^0$ and $x_i^1$ depend only on w. In equating in the last two equations (7) the mean values of the two members, it follows that

$$\left.\begin{aligned} \Sigma \alpha_k^1 w_k \frac{dy_i^0}{dw_k} = \Sigma C_{ik}^0 x_k^1, \\ \Sigma \alpha_k^1 w_k \frac{dx_i^1}{dw_k} = \frac{d\psi}{dy_i^0}. \end{aligned}\right\} \tag{8}$$

These equations (8) should serve to determine the $y_i^0$ and the $x_i^1$ as functions of w.

Can these equations be satisfied by substitution of series developed in powers of w in place of the $y_i^0$ and the $x_i^1$?

In order to make this clear, let us consider the following differential equations:

$$\left.\begin{aligned} \frac{dy_i^0}{dt} = \Sigma C_{ik}^0 x_k^1, \\ \frac{dx_i^1}{dt} = \frac{d\psi}{dy_i^0}. \end{aligned}\right\} \tag{9}$$

These differential equations where the unknown functions are the $y_i^0$ and the $x_i^1$ will admit a periodic solution /350

$$x_i^1 = 0, \quad y_i^0 = \varpi_i,$$

$\varpi_i$ being the quantity so designated in article 44.

The characteristic exponents relative to this periodic solution are precisely the quantities $\alpha_k^1$. Among these quantities we have agreed to retain only those whose real part is positive. Equations (9) admit a system of asymptotic solutions and it is easy to see that these solutions are presented in the form of series developed in powers of w. These series will then satisfy equations (8). These equations can therefore be solved.

The $x_i^1$ and the $y_i^0$ being thus determined, the remainder of the calculation no longer presents any difficulty as we have seen. There therefore exist series ordered in powers of $\sqrt{\mu}$, w and $e^{+t\sqrt{-1}}$ which formally satisfy equations (1).

This proves that the development of $\frac{N}{\Pi}$ never begins with a negative power of $\sqrt{\mu}$. The analysis in articles 110 and 111 will furnish us a new demonstration of this.

### Divergence of the Series of Article 108

109. Unfortunately, the series thus obtained are not convergent.

Let, in fact,

$$\frac{1}{\sqrt{-1}\,\gamma + \Sigma\alpha\beta - a_i}.$$

If $\gamma$ is not zero this expression is developable in powers of $\sqrt{\mu}$, but the radius of convergence of the series thus obtained tends toward 0 when $\frac{\gamma}{\Sigma\beta}$ tends toward 0.

If we therefore develop the various quantities $\frac{1}{\Pi}$ in powers of $\sqrt{\mu}$, we will 13

always be able to find an infinity of these quantities, for which the radius of convergence of the development is as small as we wish.

We might still hope, improbable as this may seem, that the same is not true for the developments of the various quantities $\frac{N}{\Pi}$; however, the demonstration which I have given in volume XIII of Acta mathematica (p. 222) and to which I will later return shows that this is not so in general; it is therefore necessary to give up this faint hope and conclude that the series which we have just formed are divergent.

But although they are divergent, can part of them not be extracted?

Let us first consider the following series, which is simpler than those which we have seen:

$$F(w, \mu) = \sum_n \frac{w^n}{1 + n\mu}.$$

This series converges uniformly when $\mu$ remains positive and w remains smaller in absolute value than a positive number $w_0$ smaller than 1 but otherwise arbitrary.

Likewise the series

$$\frac{1}{\underline{|p}} \frac{d^p F(w, \mu)}{d\mu^p} = \pm \sum \frac{n^{p-1} w^n}{(1 + n\mu)^p}$$

converges uniformly.

If now we attempt to develop $F(w, \mu)$ in powers of $\mu$, the series to which we are led

$$\Sigma w^n (-n)^p \mu^p \tag{10}$$

does not converge. If in this series we neglect all terms where the exponent of $\mu$ exceeds p, we obtain a certain function

$$\Phi_p(w, \mu).$$

It is easy to see that the expression

$$\frac{F(w, \mu) - \Phi_p(w, \mu)}{\mu^p}$$

tends toward 0 when $\mu$ tends toward 0 through positive values, so that the series (10) asymptotically represents the function $F(w, \mu)$ for small values of $\mu$ in the /352
same way that the Stirling series asymptotically represents the Eulerian function for large values of x.

In the following articles I propose to establish that the divergent series which we learned to form in article 108 are completely analogous to series (10).

Let us in fact consider one of the series

$$\sum \frac{N}{\Pi} w_1^{\beta_1} w_2^{\beta_2} \dots w_k^{\beta_k} e^{\gamma t \sqrt{-1}} = F(\sqrt{\mu}, w_1, w_2, \dots, w_k, t); \tag{10'}$$

the reasoning of article 105 has shown that these series are uniformly convergent provided w remains below certain limits in absolute value and $\sqrt{\mu}$ remain real.

If we develop $\frac{N}{\Pi}$ in powers of $\sqrt{\mu}$, the series (10') are divergent, as we have said. Let us assume that in the development we neglect the terms where the exponent of $\sqrt{\mu}$ exceeds p. We will obtain a certain function

$$\Phi_p(\sqrt{\mu}, w_1, w_2, \dots, w_k, t)$$

which will be developable in powers of w and $e^{\pm t\sqrt{-1}}$ and which will be a polynomial of degree p in $\sqrt{\mu}$.

We will see later that the expression

$$\frac{F - \Phi_p}{\sqrt{\mu^p}}$$

tends toward 0 when μ tends toward 0 through positive values, and this is true however large p may be.

In fact, if we designate by $\Pi_p$ the sum of the terms in the development of Π where the exponent of $\sqrt{\mu}$ is at most equal to p, we have

$$\frac{F - \Phi_p}{\sqrt{\mu^p}} = \sum \frac{1}{\sqrt{\mu^p}} \left( \Pi - \Pi_p \right) w_1^{\beta_1} w_2^{\beta_2} \dots w_k^{\beta_k} e^{\gamma t \sqrt{-1}},$$

and I will show that the series of the second member is uniformly convergent and that all terms tend toward 0 when μ tends toward 0.

We can therefore say that the series which we obtained in article 108 represent the asymptotic solutions for small values of μ in the same way that the Stirling series represents the Eulerian functions. /355

New Demonstration of the Proposition of Article 108

110. In order to demonstrate this fact, I am going to submit the equations to a transformation which will at the same time furnish me a new demonstration of the theorem which was the subject of article 108. To fix the ideas, let us assume only two degrees of freedom; then we will retain only one of the quantities w and we may write our equations in the following form:

$$\frac{dx_i}{dt} + \alpha w \frac{dx_i}{dw} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} + \alpha w \frac{dy_i}{dw} = -\frac{dF}{dx_i} \qquad (i = 1, 2)$$

suppressing the indexes of α and w which have become unnecessary.

We know that α is developable in odd powers of $\sqrt{\mu}$ and consequently $\alpha^2$ in powers of μ; inversely, μ is developable in powers of $\alpha^2$; we can replace μ by this development so that F will be developed in powers of $\alpha^2$. For α=0, F reduces to $F_0$ which depends only on $x_1$ and $x_2$.

Let

$$x_i = \varphi_i(t), \qquad y_i = \psi_i(t)$$

be the periodic solution which serves us as a point of departure. As in article 79, let us set

$$x_i = \varphi_i(t) + \xi_i, \qquad y_i = \psi_i(t) + \eta_i;$$

our equations will become

$$\frac{d\xi_i}{dt} + \alpha w \frac{d\xi_i}{dw} = \Xi_i, \qquad \frac{d\eta_i}{dt} + \alpha w \frac{d\eta_i}{dw} = H_i. \tag{11}$$

The values $\Xi_i$ and $H_i$ are developed in powers of $\xi_i$, $\eta_i$ and $\alpha^2$, and the coefficients are periodic functions of t.

For $\alpha$=0, $\frac{dF}{dy_i}$ and consequently $\Xi_i$ vanish; therefore $\Xi_i$ is divisible by $\alpha^2$ /354
and I may set

$$\Xi_i = \alpha^2 X_i + \alpha^2 X'_i,$$

$\alpha^2 X_i$ representing the sum of terms of the first degree with respect to the $\xi$ and $\eta$, and $\alpha^2 X'_i$ representing the sum of the terms of higher degree.

Likewise, when $\alpha$ is zero, $\frac{dF}{dx_i}$ and consequently $H_i$ depend only on $\xi_i$ and not on $\eta_i$.

I may then set

$$H_i = Y_i + Y'_i + \alpha^2 Q_i + \alpha^2 Q'_i,$$

$Y_i + \alpha^2 Q_i$ representing the sum of the terms of the first degree with respect to $\xi$ and $\eta$, while $Y'_i + \alpha^2 Q'_i$ represent the sum of the terms of a degree higher than the first. I also assume that $Y_i$ and $Y'_i$ depend only on $\xi_1$ and $\xi_2$.

Let us set

$$\xi_1 = \alpha\zeta_1, \qquad \xi_2 = \alpha\zeta_2,$$

$Y_i$ will become divisible by $\alpha$ and $Y'_i$ by $\alpha^2$, such that I will be able to set

$$Y_i + \alpha^2 Q_i = \alpha Z_i, \qquad Y'_i + \alpha^2 Q'_i = \alpha^2 Z'_i$$

and our equations will become

$$\left.\begin{aligned} \frac{d\zeta_i}{dt} + \alpha w \frac{d\zeta_i}{dw} &= \alpha X_i + \alpha X'_i, \\ \frac{d\eta_i}{dt} + \alpha w \frac{d\eta_i}{dw} &= \alpha Z_i + \alpha^2 Z'_i. \end{aligned}\right\} \tag{12}$$

Let us consider the equations

$$\left.\begin{aligned} \frac{d\zeta_i}{dt} &= \alpha X_i, \\ \frac{d\eta_i}{dt} &= \alpha Z_i. \end{aligned}\right\} \qquad (13)$$

These equations are linear with respect to the unknowns $\zeta_i$ and $\eta_i$. They do not differ from equations (2) of article 79 except because $\xi_1$ and $\xi_2$ are replaced by $\alpha\zeta_1$ and $\alpha\zeta_2$. According to what we have seen in articles 69 and 74, the equation which defines the characteristic exponents admits four roots, one equal to $+\alpha$, the other to $-\alpha$ and the other two to 0. /355

To the first root, namely the root $+\alpha$, there corresponds a solution of equations (2) of article 79, which we learned to form in this article and which we have written thus:

$$\xi_i = e^{\alpha t} S_i, \qquad \eta_i = e^{\alpha t} T_i.$$

I recall that $S_i^0$ is zero, and consequently, $S_i$ is divisible by $\alpha$.

To the second root, $-\alpha$, there corresponds in the same way another solution of equation (2) and we write

$$\xi_i = e^{-\alpha t} S'_i, \qquad \eta_i = e^{-\alpha t} T'_i.$$

Finally, to the two zero roots, there correspond two solutions of equations (2), which we will write (cf. article 80)

$$\begin{aligned} \xi_i &= S''_i, & \eta_i &= T''_i. \\ \xi_i &= S'''_i + \alpha t S''_i, & \eta_i &= T'''_i + \alpha t T''_i; \end{aligned}$$

$T'_i$, $T''_i$, $T'''_i$, $S'_i$, $S''_i$, $S'''_i$ are periodic functions of t, as are $S_i$ and $T_i$.

According to what we have seen in articles 79 and 80, $S'_i$, $S''_i$ and $S'''_i = \alpha S_i^*$ will like $S_i$ be divisible by $\alpha$.

Let us then set

$$\left.\begin{aligned} \alpha\zeta_1 &= S_1\theta_1 + S'_1\theta_2 + S''_1\theta_3 + S'''_1\theta_4, \\ \alpha\zeta_2 &= S_2\theta_1 + S'_2\theta_2 + S''_2\theta_3 + S'''_2\theta_4. \\ \eta_1 &= T_1\theta_1 + T'_1\theta_2 + T''_1\theta_3 + T'''_1\theta_4. \\ \eta_2 &= T_2\theta_1 + T'_2\theta_2 + T''_2\theta_3 + T'''_2\theta_4. \end{aligned}\right\} \qquad (13a)$$

The functions $\theta_i$ thus defined will play a role analogous to that of function $\eta_i$ of article 105. Equations (12) then become

$$\left.\begin{aligned}&\frac{d\theta_1}{dt}+\alpha w\frac{d\theta_1}{dw}-\alpha\theta_1=\alpha\Theta_1, \quad \frac{d\theta_2}{dt}+\alpha w\frac{d\theta_2}{dw}+\alpha\theta_2=\alpha\Theta_2,\\ &\frac{d\theta_3}{dt}+\alpha w\frac{d\theta_3}{dw}=\alpha\theta_4+\alpha\Theta_3, \quad \frac{d\theta_4}{dt}+\alpha w\frac{d\theta_4}{dt}=\alpha\Theta_4.\end{aligned}\right\} \tag{14}$$

$\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$ are functions developed in powers of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ and $\alpha$, all terms of which are of at least the second degree with respect to $\theta$, and whose coefficients are periodic functions of t. In addition, $\theta$ must be periodic functions of t and the terms of the first degree in w, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ must reduce to w, 0, 0 and 0. /356

These equations (14) are analogous to equations (2') of article 105.

We in fact find

$$\alpha X_i' = \theta_1 S_i + \theta_2 S_i' + \theta_3 S_i'' + \theta_4 S_i''',$$
$$\alpha Z_i' = \theta_1 T_i + \theta_2 T_i' + \theta_3 T_i'' + \theta_4 T_i''',$$

which gives us four equations from which we can obtain the four functions $\Theta$, since the S, T, X' and Z' are known functions. I say that we will find

$$\Theta_i = U_{i,1}X_1' + U_{i,2}X_2' + U_{i,3}Z_1' + U_{i,4}Z_2',$$

the U being periodic functions of t developable in increasing and positive powers of $\alpha$. In fact, for this it suffices that the

$$\Delta = \begin{vmatrix} \frac{1}{\alpha}S_1 & \frac{1}{\alpha}S_1' & \frac{1}{\alpha}S_1'' & \frac{1}{\alpha}S_1''' \\ \frac{1}{\alpha}S_2 & \frac{1}{\alpha}S_2' & \frac{1}{\alpha}S_2'' & \frac{1}{\alpha}S_2''' \\ T_1 & T_1' & T_1'' & T_1''' \\ T_2 & T_2' & T_2'' & T_2''' \end{vmatrix}$$

not be divisible by $\alpha$, that is to say, not vanish for $\alpha$=0.

For $\alpha$=0, $\frac{S_i}{\alpha}$ reduces to the quantity which we have called $S_i^1$ in article 79 and $T_i$ to $T_i^0$, and these quantities satisfy equations (9) and (10) of this article 79.

Here we develop not in powers of $\sqrt{\mu}$, but in those of $\alpha$, so that the quantity which we called $\alpha_1$ in article 79 is equal to 1. Equations (9) of article 79 are therefore to be written

$$T_i = \frac{C_{i1}^0 S_1}{\alpha} + \frac{C_{i2}^0 S_2}{\alpha},$$
$$\frac{S_i}{\alpha} = b_{i1}T_1 + b_{i2}T_2,$$

and these should be satisfied for $\alpha=0$.

Concerning the second solution, the exponent is equal to $-\alpha$ and consequently $\alpha_1$ is equal to -1, so that these equations become /35

$$-T'_i = \frac{C^0_{i1} S'_1}{\alpha} + \frac{C^0_{i2} S'_2}{\alpha},$$

$$-\frac{S'_i}{\alpha} = b_{i1} T'_1 + b_{i2} T'_2,$$

which permits assuming

$$T'_i = T_i, \qquad S'_i = -S_i.$$

$S''_i$ being divisible by $\alpha^2$, $\frac{S''_i}{\alpha}$ vanishes for $\alpha=0$. At the same time, for $\alpha=0$ we have

$$T''_i = n_i = -\frac{dF_0}{dx_i}.$$

For $\alpha=0$, $T'''_i = \alpha T^*_i$ vanishes and we have

$$\frac{S'''_i}{\alpha} = S^*_i \gtrless 0;$$

we find

$$n_i = C^0_{i1} S^*_1 + C^0_{i2} S^*_2.$$

From this we can conclude that for $\alpha=0$ the determinant $\Delta$ reduces to

$$\Delta = 2 \begin{vmatrix} \frac{S_1}{\alpha} & \frac{S'''_1}{\alpha} \\ \frac{S_2}{\alpha} & \frac{S'''_2}{\alpha} \end{vmatrix} \begin{vmatrix} T_1 & n_1 \\ T_2 & n_2 \end{vmatrix}.$$

We find in addition

$$\begin{vmatrix} T_1 & n_1 \\ T_2 & n_2 \end{vmatrix} = \begin{vmatrix} \frac{S_1}{\alpha} & \frac{S'''_1}{\alpha} \\ \frac{S_2}{\alpha} & \frac{S'''_2}{\alpha} \end{vmatrix} \begin{vmatrix} C^0_{11} & C^0_{12} \\ C^0_{21} & C^0_{22} \end{vmatrix}.$$

The determinant of these $C^0_{ik}$, which is nothing other than the Hessian of $F_0$, does not vanish in general, so that $\Delta$ can vanish only if we have

$$\frac{T_1}{n_1} = \frac{T_2}{n_2};$$

but if we observe that /35

$$n_1 b_{i1} + n_2 b_{i2} = 0,$$

we would deduce from this that

$$\frac{S_1}{\alpha} = \frac{S_2}{\alpha} = 0,$$

which cannot hold true.

The determinant $\Delta$ is therefore not zero. We can also establish it in the following manner. Let us consider the following equations:

$$\frac{d\xi_i}{dt} = b_{i1}\eta_1 + b_{i2}\eta_2,$$
$$\frac{d\eta_i}{dt} = C^0_{i1}\xi_1 + C^0_{i2}\xi_2.$$

These are linear equations with constant coefficients. They admit four linearly independent solutions, namely

$$\xi_i = e^t \frac{S_i}{\alpha}, \qquad \eta_i = e^t T_i;$$
$$\xi_i = e^{-t} \frac{S'_i}{\alpha}, \qquad \eta_i = e^{-t} T'_i;$$
$$\xi_i = \frac{S''_i}{\alpha}, \qquad \eta_i = T''_i;$$
$$\xi_i = \frac{S'''_i}{\alpha} + t\frac{S''_i}{\alpha}, \qquad \eta_i = T'''_i + tT''_i.$$

It goes without saying that in $T_i$ and $\frac{S_i}{\alpha}$ it is necessary to make $\alpha=0$, so that these quantities reduce to constants.

These four solutions being linearly independent, their determinant for t=0 must not vanish; now this determinant is precisely $\Delta$. Therefore $\Delta$ is not zero.

Q. E. D.

We thus see that the functions $\theta_i$ enjoy many of the required properties.

111. The preceding analysis immediately extends to the case where there are more than 2 degrees of freedom.

If we set

$$\xi_i = \sqrt{\mu}\zeta_i,$$

the equations may then be written, as in the preceding article, /359

$$\left.\begin{aligned} \frac{d\zeta_i}{dt} + \Sigma a_k w_k \frac{d\zeta_i}{dw_k} &= \sqrt{\mu}X_i + \sqrt{\mu}X'_i \\ \frac{d\eta_i}{dt} + \Sigma a_k w_k \frac{d\eta_i}{dw_k} &= \sqrt{\mu}Z_i + \mu Z'_i \end{aligned}\right\} (i=1, 2, \ldots, n).$$

The functions $X_i$, $X'_i$, $Z_i$ and $Z'_i$ enjoy the same properties as in the preceding article, i.e., they are developable in powers of the $\eta_i$, $\zeta_i$ and $\sqrt{\mu}$, and are periodic with respect to t. In addition, $X_i$ and $Z_i$ are linear with respect to $\eta_i$ and $\zeta_i$ and $X'_i$ and $Z'_i$ contain only terms of at least the second degree with respect to these variables.

Let us consider the equations

$$\frac{d\zeta_i}{dt} = \sqrt{\mu} X_i, \qquad \frac{d\eta_i}{dt} = \sqrt{\mu} Z_i;$$

they will admit 2n-2 linearly independent solutions corresponding to the 2n-2 characteristic exponents which are not zero; these solutions may be written

$$\sqrt{\mu}\zeta_i = \xi_i = e^{\alpha_k t} S_{ik}, \qquad \eta_i = e^{\alpha_k t} S_{ik} \qquad (k = 1, 2, \ldots, 2n-2);$$

they will admit also two degenerate solutions defined in article 80 and which I will write

$$\sqrt{\mu}\zeta_i = S_{i,2n-1}, \qquad \eta_i = T_{i,2n-1}$$

and

$$\sqrt{\mu}\zeta_i = S_{i,2n} + \sqrt{\mu}\, t S_{i,2n-1}, \qquad \eta_i = T_{i,2n} + \sqrt{\mu}\, t T_{i,2n-1}.$$

The functions $S_{i,k}$ and $T_{i,k}$ (k=1, 2, ..., 2n) are periodic in t. In addition, $S_{ik}$ is divisible by $\sqrt{\mu}$.

We can then set

$$\sqrt{\mu}\zeta_i = \sum_{k=1}^{k=2n} S_{ik}\theta_k, \qquad \eta_i = \sum_{k=1}^{k=2n} T_{ik}\theta_k,$$

and then we will obtain the equations /360

$$\begin{aligned}
&\frac{d\theta_i}{dt} + \Sigma \alpha_k \varpi_k \frac{d\theta_i}{d\varpi_k} - \alpha_i\theta_i = \sqrt{\mu}\,\Theta_i \qquad (i = 1, 2, \ldots, 2n-2). \\
&\frac{d\theta_{2n-1}}{dt} + \Sigma \alpha_k \varpi_k \frac{d\theta_{2n-1}}{d\varpi_k} - \sqrt{\mu}\,\theta_{2n} = \sqrt{\mu}\,\Theta_{2n-1}, \\
&\frac{d\theta_{2n}}{dt} + \Sigma \alpha_k \varpi_k \frac{d\theta_{2n}}{d\varpi_k} = \sqrt{\mu}\,\Theta_{2n}.
\end{aligned} \tag{14a}$$

The functions $\Theta_k$ are defined by the 2n equations of the first degree

$$X'_i = \sum_{k=1}^{k=2n} \frac{S_{ik}}{\sqrt{\mu}} \Theta_k,$$

$$\sqrt{\mu} Z'_i = \Sigma T_{ik} \Theta_k.$$

The determinant of these 2n equations, that is to say the determinant $\Delta$ formed with $S_{ik}/\sqrt{\mu}$ and $T_{ik}$, does not vanish for $\mu=0$. We could demonstrate this as in the preceding article; the second demonstration in particular can be applied, without change, in the case which concerns us.

From this we will conclude that the functions $\Theta_k$ are periodic with respect to $\tau$ and developable in increasing and positive powers of $\Theta_i$ and $\sqrt{\mu}$.

This granted, it is easy to demonstrate the proposition of article 108.

Let us assume, in effect, that p of the characteristic exponents $\alpha_1$, $\alpha_2$, ..., $\alpha_p$ have their real part positive, and let us attempt to satisfy equations (14a) by replacing the $\theta_i$ by series developed in powers of $w_1$, $w_2$, ..., $w_p$. Therefore let

$$\theta_i = \Sigma[i, \beta_1, \beta_2, \ldots, \beta_p, \gamma] e^{\gamma\sqrt{-1}\,\omega} w_1^{\beta_1} w_2^{\beta_2} \ldots w_p^{\beta_p} \quad ;$$

$\beta_1$, $\beta_2$, ..., $\beta_p$ are positive integers, $\gamma$ a positive or negative integer, and the coefficients $[i, \beta_1, \beta_2, \ldots, \beta_p, \gamma]$, which for brevity I will also write $[i, \beta_k, \gamma]$, are constants which must be determined.

If we substitute these values of $\theta_i$ into $\Theta_i$, it will follow that

$$\Theta_i = \Sigma(i, \beta_1, \beta_2, \ldots, \beta_p, \gamma) e^{\gamma\sqrt{-1}\,\omega} w_1^{\beta_1} w_2^{\beta_2} \ldots w_p^{\beta_p};$$

the coefficients $(i, \beta_1, \beta_2, \ldots, \beta_p, \gamma)$ or $(i, \beta_k, \gamma)$ will be constants which /361
will depend, following a certain law, on the indeterminate coefficients $[i, \beta_k, \gamma]$. I say that the $[i, \beta_k, \gamma]$ and consequently the $(i, \beta_k, \gamma)$ are developable in increasing powers of $\sqrt{\mu}$ and the development contains no negative power.

In effect, equations (14a) give us

$$[i, \beta_k, \gamma] = (i, \beta_k, \gamma) \frac{\sqrt{\mu}}{\gamma\sqrt{-1} + \Sigma\alpha_k\beta_k - \alpha_i}$$

for i=1, 2, ..., 2n-2 and

$$[2n, \beta_k, \gamma] = (2n, \beta_k, \gamma)\frac{\sqrt{\mu}}{\gamma\sqrt{-1} + \Sigma\alpha_k\beta_k}$$

$$[2n-1, \beta_k, \gamma] = \{[2n, \beta_k, \gamma] + (2n-1, \beta_k, \gamma)\}\frac{\sqrt{\mu}}{\gamma\sqrt{-1} + \Sigma\alpha_k\beta_k}.$$

These formulas permit calculating coefficients $[i, \beta_k, \gamma]$ by recurrence. If we in effect agree to say that the coefficient $[i, \beta_k, \gamma]$, just as $(i, \beta_k, \gamma)$, is of the degree

$$\beta_1 + \beta_2 + \ldots + \beta_p,$$

it is easy to see that the quantity $(i, \beta_k, \gamma)$ depends only on the coefficients $[i, \beta_k, \gamma]$ of lesser degree, which can be assumed known by a previous calculation.

Likewise, we can by recurrence demonstrate the stated proposition. In fact, I say that it is true of $[i, \beta_k, \gamma]$ if it is true of the coefficients of lesser degree, for if this is so, it will be true of $(i, \beta_k, \gamma)$, which depends only on these coefficients of lesser degree. It therefore remains to demonstrate that the fraction

$$\frac{\sqrt{\mu}}{\gamma\sqrt{-1} + \Sigma\alpha_k\beta_k - \alpha_i}$$

is developable in positive powers of $\sqrt{\mu}$. Now this is obvious, for if $\gamma$ is not zero, the denominator is not divisible by $\sqrt{\mu}$. If $\gamma$ is zero, the denominator is divisible by $\sqrt{\mu}$, but not by $\mu$; but the same is true for the numerator.

The proposition of article 108 is therefore thus demonstrated anew.

## Transformation of Equations

112. Let us return to the case where there are only 2 degrees of freedom /3
and let us return to equations (14) of article 110.

Let $\Phi$ be a function which, just as $\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$, is developed in powers of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\alpha$, $e^{t\sqrt{-1}}$ and $e^{-t\sqrt{-1}}$, and is such that each of its coefficients is real, positive and greater in absolute value than the coefficient of the corresponding term in $\Theta_1$, $\Theta_2$, $\Theta_3$ and $\Theta_4$; all terms of $\Phi$ will be, in addition, just as those of the $\Theta_i$, of at least the second degree with respect to $\theta$.

Let us observe that the number

$$\frac{n\sqrt{-1}}{\alpha} + p$$

(where n is a positive, negative or zero integer, and where p is a positive integer at least equal to 1) is always greater in absolute value than 1, whatever in addition is assumed of n, p, and $\alpha$. Now, the numbers which will play the role of divisors (5) of article 105 divided by $\alpha$ are precisely of this form.

Let us then form the equations

$$\theta_1 = w + \Phi, \quad \theta_2 = \Phi, \quad \theta_3 = \theta_4 + \Phi, \quad \theta_4 = \Phi, \tag{15}$$

which are similar to equations (2") of article 105.

From equations (14) we can obtain the θ in the form of series ordered in powers of w and $e^{\pm t\sqrt{-1}}$ and which are analogous to series (4') from article 104. From equations (15) we can obtain θ in the form of a series ordered in powers of the same variables and analogous to series (4") of article 105. Each of the terms of these last series is positive and greater in absolute value than the corresponding term of the first series *; if therefore they converge, the same is true for the series obtained from equations (14).

Now it is easy to see that we can find a number $w_0$ independent of $\alpha$, such that if $|w|<w_0$, the series obtained from (15) converge. /363

From this it results that the series ordered in powers of w obtained from (14) converge uniformly however small μ may be, as I stated earlier. This reasoning is in every point similar to that of article 105; the function $\alpha\Phi$ plays the role of $\bar{H}^2_i + \bar{H}^3_i + \ldots$ and $\alpha$ that of $\epsilon$, because all divisors (5) are of the form $n\sqrt{-1}+\alpha p$, and consequently greater in absolute value than $\alpha$.

We now possess the θ in the form of series ordered in powers of w and $e^{\pm t\sqrt{-1}}$; the coefficients are known functions of $\alpha$. If we develop each of these coefficients in powers of $\alpha$, we will obtain the θ in powers of $\alpha$. The series thus obtained are divergent, as we have seen earlier; nevertheless let

$$\theta_i = \theta_i^0 + \alpha\theta_i^1 + \alpha^2\theta_i^2 + \ldots + \alpha^p\theta_i^p + \ldots \tag{16}$$

be these series.

Let us set

$$H_1 = \Theta_1 + \theta_1, \quad H_2 = \Theta_2 - \theta_2, \quad H_3 = \Theta_3 + \theta_4, \quad H_4 = \Theta_4.$$

Let us set

$$\theta_i = \theta_i^0 + \alpha\theta_i^1 + \alpha^2\theta_i^2 + \ldots + \alpha^p\theta_i^p + \alpha^p u_i \tag{17}$$

equating $\theta_i$ to the first p+1 terms of series (16) plus a complementary term $\alpha^p u_i$.

*Later see the demonstration given in detail in a similar case alluding to equations (21) and (21a).

If in $H_i$ we replace the $\theta_i$ by their developments (17), $H_i$ can be developed in powers of $\alpha$ and we can write

$$H_i = \theta_i^0 + \alpha\theta_i^1 + \alpha^2\theta_i^2 + \ldots + \alpha^{p-1}\theta_i^{p-1} + \alpha^p U_i,$$

the $\theta_i^k$ being independent of $\alpha$ while $U_i$ is developable in powers of $\alpha$.

We will then have the equations

$$\left.\begin{array}{l} \frac{d\theta_i^0}{dt} = 0, \quad \frac{d\theta_i^1}{dt} + w\frac{d\theta_i^0}{dw} = \theta_i^0, \\ \frac{d\theta_i^2}{dt} + w\frac{d\theta_i^1}{dw} = \theta_i^1, \quad \ldots, \quad \frac{d\theta_i^p}{dt} + w\frac{d\theta_i^{p-1}}{dt} = \theta_i^{p-1} \end{array}\right\} \tag{18}$$

and then

$$\frac{du_i}{dt} + \alpha w\frac{du_i}{dw} + \alpha w\frac{d\theta_i^p}{dw} = \alpha U_i. \tag{19}$$

Here is the form of the function $U_i$; the quantities $\theta_i^k$ may be regarded as known functions of t and w, defined by equations (18) and by equation (20) which I will write later, while the $u_i$ remain the unknown functions. $U_i$ is then a function developed in powers of w, $e^{\pm t\sqrt{-1}}$, $\alpha$ and $u_i$. In addition, any term of the q-th degree with respect to the $u_i$ is at least of degree p(q-1) with respect to $\alpha$. In fact, $H_i$ and consequently $\alpha^p U_i$ are developable in powers of the $\theta_i$ and consequently of $\alpha^k\theta_i^k$ and $\alpha^p u_i$. Any term of the q-th degree with respect to the $u_i$ will therefore be divisible by $\alpha^{pq}$ in $\alpha^p U_i$ and by $\alpha^{p(q-1)}$ in $U_i$.

Let $U_i^0$ be what $U_i$ becomes when $\alpha$ and the $u_i$ vanish; we will have

$$w\frac{d[\theta_i^p]}{dw} = [U_i^0]. \tag{20}$$

Then, setting

$$U_i' = U_i - w\frac{d\theta_i^p}{dw}$$

then

$$V_1 = U_1' - u_1, \quad V_2 = U_2' + u_2, \quad V_3 = U_3' - u_4, \quad V_4 = U_4',$$

I may set equations (19) in the form

$$\frac{du_1}{dt} + \alpha w \frac{du_1}{dw} - \alpha u_1 = \alpha V_1, \quad \frac{du_2}{dt} + \alpha w \frac{du_2}{dw} + \alpha u_2 = \alpha V_2,$$
$$\frac{du_3}{dt} + \alpha w \frac{du_3}{dw} - \alpha u_4 = \alpha V_3, \quad \frac{du_4}{dt} + \alpha w \frac{du_4}{dw} = \alpha V_4. \qquad (21)$$

We then see that $V_i$ contain only terms of at least the second degree with respect to w and $u_i$. /365

In fact, $\theta_i$ are divisible by w and reduce to w or to 0 when we suppress in them the terms of degree greater than the first in w. From this it results first that $\theta_i^p$ is divisible by $w^2$. On the other hand, the second member of equation (17) will contain only terms which are at least of the first degree with respect to w and $u_i$. Therefore $\Theta_i$ contains only terms of at least the second degree with respect to w and $u_i$. From this it results that the only terms of first degree which can subsist in $U_1$, $U_2$, $U_3$ and $U_4$ reduce respectively to $u_1$, $-u_2$, $u_5$ and 0.

In addition, $w\frac{d\theta_i^p}{dw}$ is divisible by $w^2$; therefore $V_i$ contain only terms of at least the second degree.

Q. E. D.

From equations (21) we can take $u_i$ in the form of a series developed in powers of w and $e^{\pm t\sqrt{-1}}$. In applying to these equations the same reasoning as to equations (14), I will demonstrate that these series converge when $|w|<w_0$ and that the convergence remains uniform however small $\alpha$ may be.

The case will be the same for the series which represent $\frac{du_i}{dw}$, $\frac{d^2u_i}{dw^2}$, ....

From this it will result that we can assign an upper limit independent of $\alpha$ to $u_i$, $\frac{du_i}{dw}$, $\frac{d^2 u_i}{dw^2}$, ..., provided $|w|<w_0$.

I will show later, in articles 116 and 117, that this still holds true for all positive values of w.

In fact, let $\Phi$ be a function developed in powers of $\alpha$, $u_i$, w and $e^{\pm\sqrt{-1}}$ such that we have (for i=1, 2, 3, 4)

$$V_i < \Phi(\text{arg. } u_1, u_2, u_3, u_4, \alpha, w, e^{\pm t\sqrt{-1}}).$$

Let $\Phi'$ be what $\Phi$ becomes when $u_i$, $u_2$, $u_3$, $u_4$, are replaced by $u_1'$, $u_2'$, $u_3'$, $u_4'$.

Let us envision the following equations:

$$u_1' = w + \Phi', \quad u_2' = \Phi', \quad u_3' = u_4' + \Phi', \quad u_4' = \Phi', \qquad (21a)$$

analogous to equations (15). It is clear that these equations will admit a solution such that $u_1'$, $u_2'$, $u_3'$, $u_4'$ be developable in powers of w, $\alpha$ and $e^{\pm t\sqrt{-1}}$ and vanish with w.

These series $u_1'$, $u_2'$, $u_3'$, $u_4'$ will be convergent provided $|w|$ does not exceed /366 a certain limit which I call $w_0$. Let us now compare equations (21), and the functions $u_1$, $u_2$, $u_3$, $u_4$ which satisfy them with equations (21a) and the functions $u_1'$, $u_2'$, $u_3'$, $u_4'$ which satisfy them.

I propose to establish that

$$u_i < u_i'(\text{arg. } w, e^{\pm t\sqrt{-1}}).$$

(I point out that $\alpha$ does not figure among the arguments with respect to which this inequality holds.)

In fact, let $u_i^n$ and $u'^n_i$ be the sum of the terms of $u_i$ and of $u_i'$ which are of degree n, at most, in w; let us assume it established that

$$u_i^n < u_i'^n.$$

I will show that

$$u_i^{n+1} < u_i'^{n+1}.$$

I will then have established by recurrence the inequality to be demonstrated.

If in place of the $u_i$ and $u_i'$ we substitute in $V_i$ and in $\Phi'$ the developments of these quantities in powers of w and $e^{\pm t\sqrt{-1}}$, these functions $V_i$ and $\Phi'$ will themselves become developable in powers of w and $e^{\pm t\sqrt{-1}}$.

Let us also designate by $V_i^n$ and $\Phi'^n$ the sum of the terms of at most degree n in w.

If then $u_i^n < u'^n_i$, we will also have

$$V_i^{n+1} < \Phi'^{n+1}.$$

Then let

$$A w^{n+1} e^{pt\sqrt{-1}}$$

be a term of $\Phi'^{n+1}$ and

$$A_i w^{n+1} e^{pt\sqrt{-1}}$$

the corresponding term of $V^{n+1}_i$; we will have

$$|A_i| < A.$$

Then let

$$B_i w^{n+1} e^{pt\sqrt{-1}} \text{ and } B'_i w^{n+1} e^{pt\sqrt{-1}}$$

be the corresponding terms of $u_i$ and $u'_i$.

Equations (21) and (21a) then give us /367

$$B_1 = \frac{A_1}{\frac{p\sqrt{-1}}{\alpha} + n}, \quad B_2 = \frac{A_2}{\frac{p\sqrt{-1}}{\alpha} + n + 2}, \quad B_4 = \frac{A_4}{\frac{p\sqrt{-1}}{\alpha} + n + 1}$$

$$B_3 = \frac{B_4 + A_3}{\frac{p\sqrt{-1}}{\alpha} + n + 1},$$

$$B'_1 = B'_2 = B'_4 = A, \quad B'_3 = B'_4 + A.$$

As

$$\left|\frac{p\sqrt{-1}}{\alpha} + n\right| > 1,$$

we have

$$|B_i| < B'_i,$$

whence

$$u_i^{n+1} < u_i'^{n+1},$$

and by recurrence

$$u_i < u'_i.$$

Q. E. D.

As this inequality is taken with respect to the arguments w and $e^{t\sqrt{-1}}$, it can be differentiated as much with respect to w as with respect to t, so that we have

$$\frac{du_i}{dt} < \frac{du'_i}{dt}, \quad \frac{du_i}{dw} < \frac{du'_i}{dw}, \quad \frac{d^2u_i}{dw^2} < \frac{d^2u'_i}{dw^2}, \quad \ldots.$$

Let $u'^0_i$ be the value of $u'_i$ for t=0; if $u_i \ll u'_i$, for positive values of w we will have

$$|u_i| < u'^0_i.$$

However, $u'^0_i$ is developable in powers of $\alpha$: we can therefore assign it an upper limit independent of $\alpha$ for small values of $\alpha$ since it tends toward a finite limit when $\alpha$ tends toward 0.

The case is the same by virtue of the inequalities of $|u_i|$ which we have just established.

We would likewise demonstrate that the same is also true for the derivatives

$$\left|\frac{du_i}{dt}\right|, \quad \left|\frac{du_i}{dw}\right|, \quad \left|\frac{d^2 u_i}{dw_2}\right|.$$

Q. E. D.

Reduction to the Canonical Form

113. Let us observe that equations (14) and likewise equations (21) can be put in the canonical form. /368

In fact, if we set, as at the beginning of article 110,

$$x_i = \varphi_i(t) + \xi_i, \qquad y_i = \psi_i(t) + \eta_i,$$

the canonical equations of motion

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}$$

will become

$$\frac{d\xi_i}{dt} = \frac{dF^*}{d\eta_i}, \qquad \frac{d\eta_i}{dt} = -\frac{dF^*}{d\xi_i},$$

F* being defined in the following manner.

When in F we replace $x_i$ and $y_i$ by $\varphi_i+\xi_i$ and $\psi_i+\eta_i$, this function F can be developed in powers of the $\xi$ and $\eta$, the coefficients being periodic functions of t. Then let F' be the sum of the terms of degree 0 and 1 with respect to the $\xi$ and $\eta$; we will set

$$F^* = F - F'.$$

If we designate by $\delta\xi_i$ and $\delta\eta_i$ arbitrary virtual increases of $\xi_i$ and $\eta_i$

and by $\delta F^*$ the corresponding increase of $F^*$, these equations can be written

$$\Sigma(d\xi_i\,\delta\eta_i - d\eta_i\,\delta\xi_i) = \delta F^*\,dt.$$

What does this equation become when $\theta_i$ are taken as new variables?

Adopting a notation analogous to that of article 70, we will set

$$(U, U') = \Sigma(S_i T'_i - S'_i T_i)$$

and we will so define $(U,U')$, $(U',U'')$, ... Article 70 teaches us that all these quantities are zero with the exception of $(U, U')$ and $(U'', U''')$, which are constants. These constants must be divisible by $\alpha$, but they can otherwise be arbitrary, since $S_i$, $T_i$, $S'_i$, $T'_i$, ... are determined only to within a constant factor. /369

We will therefore be able to set

$$(U, U') = (U'', U''') = \alpha.$$

If we observe that, on the other hand,

$$d\xi_i = \theta_1 dS_i + \theta_2 dS'_i + \theta_3 dS''_i + \theta_4 dS'''_i + S_i d\theta_1 + S'_i d\theta_2 + S''_i d\theta_3 + S'''_i d\theta_4$$
$$\delta\xi_i = S_i\delta\theta_1 + S'_i\delta\theta_2 + S''_i\delta\theta_3 + S'''_i\delta\theta_4, \qquad \ldots.$$

We conclude that

$$\alpha(d\theta_1\delta\theta_2 - d\theta_2\delta\theta_1 + d\theta_3\delta\theta_4 - d\theta_4\delta\theta_3) = (\delta F^* + \delta\Omega)\,dt,$$

$\delta\Omega$ designating a homogeneous and linear expression with respect to the $\theta_i$ as well as with respect to the $\delta\theta_i$; the coefficients of this bilinear function are in addition periodic functions of t.

I say that $\delta\Omega$ is an exact differential and, in fact, equations (14) give us

$$\alpha(d\theta_1\delta\theta_2 - d\theta_2\delta\theta_1 + d\theta_3\delta\theta_4 - d\theta_4\delta\theta_3) = \alpha^2(\delta G + \delta G')\,dt$$

where $\delta G$ is the exact differential of a function

$$G = \theta_1\theta_3 + \frac{\theta_2^2}{2}$$

and where

$$\delta G' = \theta_1\delta\theta_2 - \theta_2\delta\theta_1 + \theta_3\delta\theta_4 - \theta_4\delta\theta_3.$$

I say that $\delta F^* + \delta\Omega = \alpha^2(\delta G + \delta G')$ is an exact differential; to be convinced of this it is sufficient to observe that, in this expression, the terms of first degree with respect to $\theta_i$ reducing to $\delta G$ are an exact differential and that the same must be true for those whose degree is more than 1, since $\delta F^*$ is an exact differential and $\delta\Omega$ contains only terms of the first degree.

We can therefore set

$$\delta F'' + \delta\Omega = \alpha^2 \delta\Phi,$$

whence

$$\Phi = G + \frac{F''}{\alpha^2},$$

F'' designating the total of the terms of F which are of a degree greater than the second with respect to $\xi_i$ and $\eta_i$. /53

We can therefore write

$$\frac{d\theta_1}{dt} = \alpha \frac{d\Phi}{d\theta_2}, \quad \frac{d\theta_2}{dt} = -\alpha \frac{d\Phi}{d\theta_1}.$$

If we recall that θ depend on t, not only directly but also through the intermediary of w, we will write these equations in the form

$$\frac{d\theta_1}{dt} + \alpha w \frac{d\theta_1}{dw} = \alpha \frac{d\Phi}{d\theta_2}, \quad \frac{d\theta_2}{dt} + \alpha w \frac{d\theta_2}{dw} = -\alpha \frac{d\Phi}{d\theta_1}, \tag{14a}$$

to which it is necessary to adjoin two analogous equations which we may deduce from the first by changing $\theta_1$ and $\theta_2$ into $\theta_3$ and $\theta_4$.

These are the equations (14) put into the canonical form.

It is a matter of doing so also for equations (21).

If in Φ we replace $\theta_i$ by their values (17), this function becomes developable in increasing powers of α and $u_i$; if then we designate by $\alpha^{2p}\Phi'$ the sum of the terms of at least degree 2p with respect to α, our equations become

$$\frac{du_1}{dt} + \alpha w \frac{du_1}{dw} = \alpha \frac{d\Phi'}{du_2}, \quad \frac{du_2}{dt} + \alpha w \frac{du_2}{dw} = -\alpha \frac{d\Phi'}{du_1} \tag{21a}$$

with two other analogous equations.

These are equations (21) reduced to the canonical form.

## Form of Functions $V_i$

114. Let us consider the function

$$F(x_1, x_2, y_1, y_2)$$

and let us here replace $x_i$ with

$$x_i^0 + \alpha x_i^1 + \alpha^2 x_i^2 + \dots + \alpha^{p+1} x_i^{p+1} + \alpha^{p+1} v_i, \tag{22}$$

and $y_i$ with

$$n_i t + y_i^0 + \alpha y_i^1 + \alpha^2 y_i^2 + \dots + \alpha^p y_i^p + \alpha^p v_i. \tag{22a}$$

The letters /371

$$\left.\begin{array}{l} x_i^0, \; x_i^1, \; x_i^2, \; \dots, \; x_i^{p+1}, \\ n_i, \; y_i^0, \; y_i^1, \; \dots, \; y_i^p \end{array}\right\}. \tag{23}$$

have the same significance as in article 108. The only difference is that here we have only two degrees of freedom and the parameter with respect to which we develop and which plays the role of $\mu$ is equal to $\alpha^2$; the quantities (23) are thus known functions of v and w. As for $\alpha^{p+1} v_i$ and $\alpha^p v'_i$, these are arbitrary complementary terms. I propose to inquire as to the condition for F to be developable in powers of $\alpha$, $v_i$ and $v'_i$.

For brevity let us set

$$\begin{aligned} \alpha x_i^1 + \alpha^2 x_i^2 + \dots + \alpha^{p+1} x_i^{p+1} + \alpha^{p+1} v_i &= x'_i, \\ \alpha y_i^1 + \alpha^2 y_i^2 + \dots + \alpha^p y_i^p \quad + \alpha^{p+1} v'_i &= y'_i. \end{aligned}$$

The necessary and sufficient condition for

$$F(x_i^0 + x'_i, \; n_i t + y_i^0 + y'_i)$$

to be developable in increasing powers of $x'_i$ and $y'_i$ and, consequently, those of $\alpha$, $v_i$ and $v'_i$ will obviously be that the point

$$x_i = x_i^0, \qquad y_i = n_i t + y_i^0$$

not be a singular point for F.

$n_i$, $x_i^0$, are constants; $y_i^0$ are functions of w defined by equations (8) of article 108. However, in the majority of applications it will occur that if we give $x_i^0$ and $n_i$ the constant values which correspond to a periodic solution, F will remain holomorphic whatever real values may be attributed to the $y_i^0$.

For example, let us take the problem of article 9 and let us assume that $x_1 = L$, $x_2 = G$ define the form of the ellipse described by the infinitesimal mass, while $y_1 = l$, $y_2 = g - t$ define the position of the perihelion of this ellipse and that of the mass on its orbit.

In order for F to cease being holomorphic, it would be necessary that this infinitesimal mass encounter one of the two other masses; now, if the ellipse does not intersect the circumference described by the second mass, as will be the case in almost all applications, this encounter can never occur, whatever real values are attributed to $l$ and g-t. /372

This will be true if we take a greater number of degrees of freedom and study the Problem of Three Bodies in all its generality.

Then the variables $x_i$ define the form of the ellipses and the mutual inclination of their planes, the variables $y_i$ define the position of the nodes, of the perihelions and of the masses themselves. It will then occur in most cases that if we give the variables $x_i$ the values $x_i^0$ which correspond to a periodic solution and to the limit $\mu=0$, these two ellipses may intersect each other only if they are rotated in their plane. The function F will therefore cease being holomorphic, whatever the real values attributed to the $y_i$ may be.

We are thus led to assume that for $x_i=x_i^0$, F is holomorphic for all real values of $y_i$. The cases where this would not occur have no importance from the point of view of application. This is additionally the hypothesis which we have always made until now.

If in F the $x_i$ and $y_i$ are replaced by expressions (22), F can be developed in powers of $\alpha$, $v_i$ and $v_i'$, and this development, whose coefficients are functions of t and w, remains convergent for all values of t and w. The radii of convergence are continuous functions of t and w which do not vanish for any real value of these variables as much with respect to $\alpha$ as $v_i$ and $v_i'$.

If we observe that $x_i$, $\theta_i$, $u_i$, $\xi_i$, $v_i$, ... are connected among themselves by the relationships

$$x_i = \varphi_i(t) + \alpha\zeta_i, \qquad y_i = \psi_i(t) + \eta_i$$

and by relationships (13a), (17) and (22), we will conclude that F and consequently $\Phi'$ are developable in powers of $\alpha$ and $u_i$, that the coefficients of development and the radii of convergence are continuous functions of t and w, and that these radii of convergence vanish for no real values of t and w.

From this fact, and from what we already know on the subject of the functions $V_i'$ (which are nothing other than derivatives of $\Phi'$), we may conclude the following: /373

We can find two real and positive numbers M and $\beta$ independent of t and w and large enough for us to have (setting $s=u_1+u_2+u_3+u_4$ for brevity)

$$V_i < M\varpi^2 + M\varpi s + \frac{M\alpha^p s^2}{1-\beta\alpha-\beta\alpha^p s} \qquad (\arg \alpha,\ u_1,\ u_2,\ u_3,\ u_4),$$

for all real values of t and for all values of w between 0 and an arbitrary upper limit W. This will hold true however large W may be, but the numbers M and $\beta$ must be chosen much greater than W itself.

## Fundamental Lemma

115. Let us now establish the following lemma:

Let $\varphi(x,t,w)$, $\varphi'(x,t,w)$ be two functions of x, t and w which are developable in powers of x and such that for all values of t and w we have what must be considered

$$\varphi < \varphi' \quad (\arg x).$$

Let us consider the two following equations

$$\frac{dx}{dt} + \alpha w \frac{dx}{dw} = \varphi(x, t, w) \tag{1}$$

and

$$\frac{dx'}{dt} + \alpha w \frac{dx'}{dw} = \varphi'(x', t, w). \tag{1a}$$

Let us consider a particular solution of each of these two equations, chosen such that for $w=w_0$ ($w_0$ being an arbitrary positive value of w), we have

$$|x| < x'.$$

I say that for all values of w larger than $w_0$, we will still have /374

$$|x| < x'. \tag{2}$$

Let us change variables, setting

$$t = \frac{1}{\alpha}\log w + \tau.$$

Representing by round ($\partial$) the partial derivatives with respect to the variables $\tau$ and w, we will then have

$$\frac{\partial x}{\partial w} = \frac{dx}{dw} + \frac{1}{\alpha w}\frac{dx}{dt}.$$

Our equations will therefore become

$$\alpha w \frac{\partial x}{\partial w} = \varphi, \qquad \alpha w \frac{\partial x'}{\partial w} = \varphi',$$

if for a certain system of values of the variables

$$w = w_1, \qquad \tau = \tau_1,$$

inequality (2) is satisfied; likewise we will have

$$|\varphi| < \varphi',$$

$$\left|\frac{\partial x}{\partial w}\right| < \frac{\partial x'}{\partial w},$$

such that inequality (2) will be satisfied also for

$$w = w_1 + dw, \qquad \tau = \tau_1,$$

since we will have

$$|x| < x' \left|\frac{\partial x}{\partial w}\, dw\right| < \frac{\partial x'}{\partial w}\, dw$$

and consequently

$$\left|x + \frac{\partial x}{\partial w}\, dw\right| < |x| + \left|\frac{\partial x}{\partial w}\, dw\right| < x' + \frac{\partial x'}{\partial w}\, dw.$$

Therefore it will be sufficient for this to hold also when we have

$$w = w_0, \qquad \tau = \tau_1$$

for it to hold when have /375

$$w > w_0, \qquad \tau = \tau_1.$$

However, we have assumed that they are so whatever t, and consequently τ, may be for $w=w_0$; they will therefore also be so whatever τ, and consequently t, may be for $w>w_0$.

Q. E. D.

We would demonstrate absolutely in the same way a somewhat more general lemma:

Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi'_1, \varphi'_2, \ldots, \varphi'_n$ be functions of $x_1, x_2, \ldots, x_n$, t and w developable in powers of x and such that for all considered values of t and w we have

$$\varphi_1 < \varphi'_1, \quad \varphi_2 < \varphi'_2, \quad \ldots, \quad \varphi_n < \varphi'_n \qquad (\arg x_1, x_2, \ldots, x_n).$$

Let us consider the equations

$$\frac{dx_i}{dt} + \alpha w \frac{dx_i}{dw} = \varphi_i(x_1, x_2, \ldots, x_n, t, w) \tag{3}$$

and

$$\frac{dx'_i}{dt} + \alpha w \frac{dx'_i}{dw} = \varphi'_i(x'_1, x'_2, \ldots, x'_n, t, w) \qquad (i = 1, 2, \ldots, n). \tag{3a}$$

Let us assume that we have, whatever t may be for $w=w_0$,

$$|x_i| < x'_i;$$

this will hold whatever t may be for $w>w_0$.

Let us now make some more particular hypotheses on the subject of the functions $\varphi_i$ and $\varphi'_i$.

Let us assume that:

1) these functions are periodic with respect to t and of period $2\pi$;

2) for small values of w, they are developable in increasing powers of w; in addition, this may not hold true for all considered values of w: it is sufficient that it be so for small values of this variable;

3) these functions are developable in integral powers of the parameter $\alpha$ and are divisible by $\alpha$: in addition we must have /376

$$\varphi_i < \varphi'_i \quad (\arg x_1, x_2, \ldots, x_n, \alpha);$$

4) if we call $\varphi_i^0$ and $\varphi'^0_i$ what $\varphi_i$ and $\varphi'_i$ become when all x vanish, these quantities $\varphi_i^0$ and $\varphi'_0$ are divisible by $w^2$.

If all these hypotheses are realized, the theories of the preceding articles show us that there exist particular solutions of equations (3) and (3a) of the following form

$$\begin{cases} x_i = A_{i,2} w^2 + A_{i,3} w^3 + \ldots, \\ x'_i = A'_{i,2} w^2 + A'_{i,3} w^3 + \ldots, \end{cases} \tag{4}$$

$A_{i,n}$ and $A'_{i,n}$ being functions of t and $\alpha$ periodic with respect to t and developable in increasing powers of $\alpha$.

Equations (3) [or (3a), which are of the same form] can in fact be reduced to the form of equations (2) of article 104.

Let us in fact return to equations (2) of article 104; these are written

$$\frac{d\xi_i}{dt} = \Xi_i;$$

the $\Xi_i$, being developable in powers of $\xi_i$ and of a very small parameter, are in addition functions of t: they vanish with $\xi_i$.

The $\xi_i$ depend on t not only directly, but through the intermediary of the exponentials $A_1 e^{\alpha_1 t}, \; A_2 e^{\alpha_2 t}, \; \ldots, \; A_n e^{\alpha_n t}$.

Here we assume that all coefficients $A_1$, $A_2$, ..., $A_n$ are zero with the exception of one; we will therefore have to concern ourselves with only one exponential $w=Ae^{\alpha t}$. The $\xi_i$ will then depend on t first directly, then through the intermediary of w. If we therefore represent the partial derivatives by d and the total derivatives by $\partial$, it will follow that

$$\frac{\partial \xi_i}{\partial t} = \frac{d\xi_i}{dt} + \alpha w \frac{d\xi_i}{dw},$$

and our equations will become /377

$$\frac{d\xi_i}{dt} + \alpha w \frac{d\xi_i}{dw} = \Xi_i. \tag{5}$$

The sole difference in form between equations (3) and equations (5) is then that the second members of equations (3) depend on w and do not vanish for

$$x_1 = x_2 = \ldots = x_n = 0.$$

However, it is easy to make this difference in form disappear. To do so it is sufficient to join to equations (3) the following equation:

$$\frac{dx_{n+1}}{dt} + \alpha w \frac{dx_{n+1}}{dw} = \alpha x_{n+1},$$

which admits $x_{n+1}=w$ as a solution, and to replace w by $x_{n+1}$ in the functions $\varphi_i$. Then these functions $\varphi_i$ no longer contain w and vanish for

$$x_1 = x_2 = \ldots = x_{n+1} = 0.$$

We can therefore apply to equations (3) and (3a) the results of article 104 and conclude that these equations admit solutions of form (4).

Calculation of coefficients $A_{i,2}$, $A_{i,3}$, ... is performed very easily by recurrence through application of the processes of article 104.

Let us therefore assume that we thus find

$$|A_{i,2}| < A'_{i,2}$$

whatever t may be.

From this we will conclude that

$$\lim \left| \frac{x_i}{w^2} \right| < \lim \frac{x_i}{w^2} \quad (\text{for } w = 0)$$

and consequently that we can find a value $w_0$ of w sufficiently small for us to have

$$|x_i| < x'_i$$

for all real values of t and for all values of w smaller than $w_0$ and greater than 0.

We will then have, by virtue of the lemma demonstrated earlier, /378

$$|x_i| < x'_i$$

for all real values of t and for all positive values of w.

### Analogy of the Series of Article 108 With That of Stirling

116. Let us apply the preceding lemma to equations (21), which we will write

$$\frac{du_i}{dt} + \alpha w \frac{du_i}{dw} = z U'_i. \tag{21}$$

According to what we saw at the end of article 114, we can find two positive numbers M and $\beta$ such that for all real values of t and for all values of w between 0 and W (this will remain true however large W may be), we have

$$U'_i < u_k + M w^2 + M w s + \frac{M a^p s^2}{1 - \beta z - \beta z^p s} \quad (\arg a,\ u_1,\ u_2,\ u_3,\ u_4),$$

$$s = u_1 + u_2 + u_3 + u_4.$$

As for the index k of $u_k$, it is equal to i for i=1 or 2 and to 4 for i=2 or 3. Let us then set

$$u_k + M w^2 + M w s \frac{M a^p s^2}{1 + \beta z - \beta a^p s} = \Phi(w,\ u_1,\ u_2,\ u_3,\ u_4)$$

and let us compare to equations (21) the equations

$$\frac{du'_i}{dt} + \alpha w \frac{du'_i}{dw} = z \Phi(w,\ u'_1,\ u'_2,\ u'_3,\ u'_4). \tag{21a}$$

Among the particular solutions of equations (21) and (21a) we will choose those which are divisible by $w^2$ (these are really those which we earlier called $u_i$).

It is clear that we will still be able to take M sufficiently large so that

$$\left|\lim \frac{u_i}{w^2}\right| < \lim \frac{u'_i}{w^2}.$$

From this we will then conclude that /379

$$|u_i| < u'_i$$

for

$$o < w < W.$$

Let us now attempt to integrate equations (21a). I first observe that, $\Phi$ not depending on t, $u'_i$ will no longer depend on it and we will have

$$u'_1 = u'_2 = u'_3 = u'_4 = \frac{s'}{4},$$

$$w\frac{ds'}{dw} = \frac{s'}{4} + Mw^2 + Ms'w + \frac{M\alpha^p s'^2}{1 - \beta\sigma - \beta\alpha^p s'^2}.$$

This last equation admits an integral

$$s' = \varphi(w, \alpha)$$

developable in powers of w and $\alpha$ and divisible by $w^2$; when $\alpha$ tends toward 0, s' manifestly tends toward the integral of the equation

$$w\frac{ds'}{dw} = \frac{s'}{4} + Mw^2 + Ms'w.$$

This linear equation is very easily integrated; we find

$$\lim s' = Mw^{\frac{1}{4}}e^{Mw}\int_0^w e^{-Mw}w^{\frac{3}{4}}dw \text{ (for } \alpha=0).$$

From this formula I wish to extract only one thing, that is if

$$o < w < W.$$

s' and consequently $u_1$, $u_2$, $u_3$ and $u_4$ tend toward a finite limit when $\alpha$ tends toward 0.

$$\theta_i^0 + \alpha\theta_i^1 + \alpha^2\theta_i^2 + \ldots$$

From this it results that the series

$$\frac{\theta_i - \theta_i^0 - \alpha\theta_i^1 - \alpha^2\theta_i^2 - \ldots - \alpha^{p-1}\theta_i^{p-1}}{\alpha^{p-1}}$$

represents the function $\theta_i$ asymptotically (that is to say in the manner of the Stirling series) or, in other words, that the expression /380

$$\alpha(\theta_i^p + u_i)$$

tends toward 0 with $\alpha$. In fact, this expression is equal to

$$\alpha(\theta_i^p + u_i)$$

and we have just seen that $\theta_i^p + u_i$ remains finite when $\alpha$ tends toward 0.

117. But this is not all; I say that $\frac{du_i}{dw}$ remains finite when $\alpha$ tends toward 0.

We have, in effect,

$$\frac{d}{dt}\left(\frac{du_i}{dw}\right) + \alpha w \frac{d}{dw}\left(\frac{du_i}{dw}\right) + \alpha\left(\frac{du_i}{dw}\right) = \alpha \sum_k \frac{dU'_i}{du_k}\frac{du_k}{dw} + \alpha \frac{dU'_i}{dw}$$

$\frac{dU'_i}{du_k}$ and $\frac{dU'_i}{dw}$ are functions of t, w, $\alpha$ and $u_i$; however, according to what we have just seen, we can assign the $u_i$ upper limits; we will therefore be able to assign them likewise to the $\frac{dU'_i}{du_k}$ and $\frac{dU'_i}{dw}$. Let us assume, for example, that we have

$$\left|\frac{dU'_i}{du_k}\right| < A, \quad \left|\frac{dU'_i}{dw}\right| < B \text{ for } w<w),$$

A and B being two positive numbers.

On the other hand, we know that we can assign a limit to $\frac{du_i}{dw}$ for $w=w_1$, if $w_1$ is less than the quantity which we have called $w_0$ at the end of article 112.

Let us assume, for example, that we have

$$\left|\frac{du_i}{dw}\right| < u'_0 \text{ for } w=w_1,$$

$u'_0$ being a positive number. Then let u' be a function defined as follows

$$\frac{du'}{dt} + \alpha w \frac{du'}{dw} = \alpha u'(4A + W) + \alpha B,$$

$$u' = u'_0 \quad \text{for} \quad w = w_1.$$

We will have manifestly

$$\left|\frac{du_i}{dw}\right| < u' .$$

Now we see without difficulty that u' depends only on w and satisfies the equation /381

$$w \frac{du'}{dw} = u'(4A + W) + B.$$

Therefore u' is finite; therefore $\frac{du_i}{dw}$ remains finite when $\alpha$ tends toward 0.

Therefore we have asymptotically (extending this word to the same sense as above)

$$\frac{d\theta_i}{dw} = \frac{d\theta_i^0}{dw} + \alpha \frac{d\theta_i^1}{dw} + \alpha^2 \frac{d\theta_i^2}{dw} + \dots$$

We would demonstrate likewise that we have asymptotically

$$\frac{d\theta_i}{dt} = \frac{d\theta_i^0}{dt} + \alpha \frac{d\theta_i^1}{dt} + \alpha^2 \frac{d\theta_i^2}{dt} + \ldots,$$

$$\frac{d^2\theta_i}{dw^2} = \frac{d^2\theta_i^0}{dw^2} + \alpha \frac{d^2\theta_i^1}{dw^2} + \alpha^2 \frac{d^2\theta_i^2}{dw^2} + \ldots.$$

Here, therefore, is our final conclusion:

The series

$$x_i^0 + \sqrt{\mu} x_i^1 + \mu x_i^2 + \ldots, \quad n_i t + y_i^0 + \sqrt{\mu} y_i^1 + \mu y_i^2 + \ldots$$

defined in this paragraph are divergent, but they enjoy the same property as the Stirling series, so that we have asymptotically

$$x_i = x_i^0 + \sqrt{\mu} x_i^1 + \mu x_i^2 + \ldots,$$

$$y_i = n_i t + y_i^0 + \sqrt{\mu} y_i^1 + \mu y_i^2 + \ldots.$$

In addition, if D is a symbol of differentiation, that is to say if we set

$$\mathrm{D}f = \frac{d^{\lambda_0+\lambda_1+\ldots+\lambda_k} f}{dt^{\lambda_0}\, dw_1^{\lambda_1}\, dw_2^{\lambda_2} \ldots dw_k^{\lambda_k}}$$

we will again have asymptotically

$$\mathrm{D}x_i = \mathrm{D}x_i^0 + \sqrt{\mu}\,\mathrm{D}x_i^1 + \mu\,\mathrm{D}x_i^2 + \ldots,$$

$$\mathrm{D}y_i = \mathrm{D}(n_i t + y_i^0) + \sqrt{\mu}\,\mathrm{D}y_i^1 + \mu\,\mathrm{D}y_i^2 + \ldots.$$

Concerning the study of the series analogous of those of Stirling, I will return to Section 1 of a memoir which I published in Acta mathematica (Vol. VIII, p. 295). /332

It is in addition clear that the same reasonings would subsist when we have more than 2 degrees of freedom, and consequently n-1 variables $w_1, w_2, \ldots, w_{n-1}$ instead of only one.

End of Volume One

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