History of Modern Physics and Astronomy Volume 13

New Methods of Celestial Mechanics

2. Approximations by Series

HENRI POINCARÉ

Edited and introduced by

DANIEL L. GOROFF

Originally published as
Les Methodes nouvelles de la Mécanique céleste

AIP
American Institute of Physics

New Methods of Celestial Mechanics

Parts 1, 2, and 3
by Henri Poincaré

originally published as *Les Méthodes nouvelles de la Mécanique céleste*

History of Modern Physics and Astronomy

TITLES IN SERIES

VOLUME 1
Alsos
by Samuel A. Goudsmit

VOLUME 2
Project Y: The Los Alamos Story
Part I: Toward Trinity
by David Hawkins
Part II: Beyond Trinity
by Edith C. Truslow and Ralph Carlisle Smith

VOLUME 3
American Physics in Transition: A History of Conceptual Change in the Late Nineteenth Century
by Albert E. Moyer

VOLUME 4
The Question of the Atom: From the Karlsruhe Congress to the First Solvay Conference, 1860–1911
by Mary Jo Nye

VOLUME 5
Physics For a New Century: Papers Presented at the 1904 St. Louis Congress

VOLUME 6
Basic Bethe: Seminal Articles on Nuclear Physics, 1936–1937
by Hans A. Bethe, Robert F. Bacher, and M. Stanley Livingston

VOLUME 7
History of the Theories of Aether and Electricity: Volumes I and II
by Sir Edmund Whittaker

VOLUME 8
Radar in World War II
by Henry Guerlac

VOLUME 9
Atoms in the Family: My Life with Enrico Fermi
by Laura Fermi

VOLUME 10
Quantum Physics in America: The Years Through 1935
by Katherine R. Sopka

VOLUME 11
The Theory of Heat Radiation
by Max Planck

VOLUME 12
The Conceptual Development of Quantum Mechanics
by Max Jammer

VOLUME 13
New Methods of Celestial Mechanics: Parts 1–3
by Henri Poincaré

INTRODUCTORY NOTE

With the publication of the English translation of *Les Méthodes nouvelles de la Mécanique céleste* in 1992, The History of Modern Physics 1800–1950 Series title changed to History of Modern Physics and Astronomy, reflecting the inclusion of books on the history of astronomy. The History of Modern Physics and Astronomy Series is an evolution and continuation of the original series and incorporates the twelve volumes published in the original series.

History of Modern Physics and Astronomy retains the original aim of bringing to modern readers a variety of important works related to the history of physics since 1800 that are not readily available elsewhere. In addition, works specifically about the history of astronomy are included both in relation to the internal development of astronomy and to the physics of the period in which they were written, emphasizing the close bonds between physics and astronomy.

The books in the Series are all noteworthy additions to the literature of the history of physics and astronomy. They have been selected for their merit, distinction, and uniqueness. Each book in the Series is prefaced by an introductory essay by a scholar who places the work in its historical context, thus making it more valuable as a reference work.

We believe that these books will be of interest not only to the advanced scholar in the history of physics and astronomy but also to a much broader, less specialized group of readers who may wish to understand major scientific disciplines that have become central forces in society and an integral part of our twentieth-century culture. Taken in its entirety, the Series will bring to the reader a comprehensive picture of these major disciplines not readily achieved in any one work. Taken individually, the works selected will surely be enjoyed and valued in themselves.

History of Modern Physics and Astronomy *Volume* 13

New Methods of Celestial Mechanics

1. Periodic and asymptotic solutions
2. Approximations by series
3. Integral invariants and asymptotic properties of certain solutions

HENRI POINCARÉ

Edited and Introduced by

DANIEL L. GOROFF
Harvard University

AIP

American Institute of Physics

Appendix: Translation of excerpts from commentaries in Henri Poincaré, *Izhrannye Trudy* (Collected Works), volumes I and II, Nauka 1971 and 1973. Used with permission.

Library of Congress Cataloging-in-Publication Data

Poincaré, Henri, 1854-1912.
[Méthodes nouvelles de la mécanique céleste. English]
New methods of celestial mechanics / Henri Poincaré; edited and introduced by Daniel L. Goroff.
p. cm. (History of modern physics and astronomy; v. 13)
Translation of: Les méthodes nouvelles de la mécanique céleste. Originally published in 1892–1899.
Contents: 1. Periodic and asymptotic solutions
2. Approximations by series 3. Integral invariants and asymptotic properties of certain solutions.
ISBN 1-56396-114-8 (v. 1). ISBN 1-56396-115-6 (v. 2). ISBN 1-56396-116-4 (v. 3). ISBN 1-56396-117-2 (set)
1. Mechanics, Celestial. I. Title. II. Series.
QB351.P7513 1991
521 dc20 89-14884
CIP
r92

CONTENTS
PART 1

PERIODIC AND ASYMPTOTIC SOLUTIONS

CHAPTER 5
NONEXISTENCE OF UNIFORM INTEGRALS 191

CHAPTER 6
APPROXIMATE DEVELOPMENT OF THE PERTURBATIVE FUNCTION 220

CHAPTER 7
ASYMPTOTIC SOLUTIONS 275

CONTENTS
PART 2

APPROXIMATIONS BY SERIES

CHAPTER 11
APPLICATION TO THE THREE-BODY PROBLEM 355

CHAPTER 12
APPLICATION TO ORBITS 378

CHAPTER 13
DIVERGENCE OF THE LINDSTEDT SERIES 395

CHAPTER 14
DIRECT CALCULATION OF THE SERIES 410

CHAPTER 15
OTHER METHODS OF DIRECT CALCULATION 450

CHAPTER 16
GYLDÉN METHODS 490

CHAPTER 17
CASE OF LINEAR EQUATIONS 511

CHAPTER 18
CASE OF NONLINEAR EQUATIONS 557

CONTENTS
PART 3

INTEGRAL INVARIANTS AND ASYMPTOTIC PROPERTIES OF CERTAIN SOLUTIONS

CHAPTER 24
USE OF INTEGRAL INVARIANTS 784

CHAPTER 25
INTEGRAL INVARIANTS AND ASYMPTOTIC SOLUTIONS 801

CHAPTER 26
POISSON STABILITY 847

CHAPTER 27
THEORY OF CONSEQUENTS 877

Preface to Part 2 of the French Edition

The methods to be discussed in this second volume have been elaborated by numerous contemporary astronomers; however, the methods developed by Gyldén, which range among the most perfect known, will be given the largest coverage.

All these methods have one characteristic in common: The scientists who conceived these methods attempted to expand the stellar coordinates in series all of whose terms are periodic and to thus cause vanishing of the so-called secular terms, encountered in the older methods of successive approximations in which the time came out from under the sine and cosine symbols. However, in return, these scientists did not concern themselves with checking whether their series were convergent in the sense in which the mathematician defines this term.

Thus despite the fact that the results derived in my first volume were established with the entire rigorousness to which mathematicians are accustomed, the results to be discussed here are valid only within a certain approximation which is relatively greater the smaller the masses become. It is extremely difficult to measure exactly, in each individual case, the error involved; however, an upper limit, can be defined, albeit quite roughly.

The terms of these series first decrease rapidly and then start increasing; however, since astronomers usually stop with the first terms of the series long before these terms have ceased to decrease, the approximation is sufficient for practical purposes. The divergence of these series is inconvenient only if they are intended for rigorously establishing certain results, such as, for example, the stability of the solar system.

In Chap. 8 I attempt to explain the cause of this discrepancy between mathematician and astronomer, including the reason for the fact that some series called divergent by mathematicians may be of use to astronomers and the manner in which the conventional rules of calculus may become applicable to these series. The somewhat protracted methods that lead up to this latter result have nevertheless the advantage of demonstrating a way for determining an upper limit of the error; in all other respects, Chap. 8 can be combined with the discussion at the end of Chap. 7.

In the following chapters, the simplest of the new methods, namely, those due to Newcomb and Lindstedt will be expounded. We will show a way to overcome certain difficulties encountered in attempting to apply these methods to the most general case of the three-body problem.

These difficulties mainly consist of two: First, so as to render the Lindstedt method applicable, either in its original form or in the form modified by me, it is necessary that, in first approximation, the mean motions are not linked by any linear relation with integral coefficients. However, in the three-body problem, the mean motions that must be taken into consideration are not only those of two planets but also those of the perihelions and nodes. However, in first approximation, i.e., in the Keplerian motion, the perihelions and nodes are fixed; thus their mean motions are zero and the above-stipulated condition, i.e., the absence of any linear relation with integral coefficients, is not satisfied. After having explained a process by which the following approximations can be arranged to avoid this drawback, a second difficulty will be discussed which is created as soon as the eccentricities become extremely small. It will be shown that this difficulty is artificial and that it can be avoided by starting, instead of from the circles to which the Keplerian ellipses are reduced as soon as the eccentricities become zero, from the orbits described by our planets in the case of periodic solutions of the first kind, investigated in Chap. 3.

In the following parts, the first methods by Gyldén will be discussed. Based on principles which are not without analogy to those discussed here, they permit overcoming exactly the same obstacles; in addition, many detail difficulties are overcome by artifices that are as elegant as they are ingenious.

A few sections will be devoted to integration processes applicable to certain differential equations which Gyldén has considered; specifically, one of these equations which seems of special interest and which numerous other mathematicians have also considered will be discussed in greater detail.

In studying these methods, we will frequently deviate considerably from the mode of exposition by their authors. It would have been useless merely to repeat what they had done so excellently already. In addition, I have no great interest to arrange these methods in a form most convenient for numerical calculators; rather, an attempt will be made to explain the basic essence of these methods so as to facilitate a comparison.

When the reader has come this far, he will be fully aware that there are always means for eliminating the so-called secular terms that had been more or less artificially introduced into the older calculation methods. However, calculators frequently encounter a much more serious obstacle, namely, the occurrence of small divisors as soon as the mean motions come close to being commensurable. The procedures discussed in Part 1 of this volume then become inapplicable, and it will be necessary to use either the Delaunay method or the Bohlin method which latter is closely correlated with the former and which will be covered in a full chapter. However, even this method is not yet perfect since it introduces, if not small divisors then at least large multipliers, which might render the approximation insufficient in certain cases. This left one more step to be taken, which has been performed by the latest method of Gyldén, whose discussion will terminate this particular volume. Despite the fact that also these methods are not yet perfect in the eyes of a pure mathematician, they are the most perfected ones known to date.

Review of Notations

To save the reader the bother of having to refer frequently to the first volume, I shall briefly recall here the meaning of certain notations that I defined in Vol. 1, and which I will use in this volume.

I first recall that body m_2 is attracted to body m_1, and body m_3 is attracted to the center of mass of bodies m_1 and m_2. I set (see no. 11)

$$\beta\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \beta'\mu = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3}$$

such that μ is a very small quantity and that β and β' are finite.

The function F is the total energy of the system divided by μ; it is developable in a series of powers of μ.

I now define the osculating elements of the first planet, that is of body m_2 in its motion relative to body m_1.

I call (see no. 8) the semimajor axis a, the eccentricity e, the inclination i, and I set

$$L = \sqrt{a}, \quad \beta L = \Lambda, \quad G = \sqrt{a(1 - e^2)}, \quad \Theta = G \cos i\,.$$

I call the mean anomaly l, the mean longitude λ, the longitude of the node θ, and that of the perihelion $g + \theta$, which I will also designate by $\bar{\omega}$.

I set (see no. 12)

$$\xi = \sqrt{2\beta(L - G)} \cos \bar{\omega}, \quad \eta = -\sqrt{2\beta(L - G)} \sin \bar{\omega}\,,$$

$$p = \sqrt{2\beta(L - G)} \cos \theta, \quad q = -\sqrt{2\beta(L - G)} \sin \theta\,.$$

These are the meanings of the letters

$$\beta, L, \Lambda, G, \Theta, l, \lambda, g, \theta, \bar{\omega}, \xi, \eta, p, q,$$

which refer to the movement of the first planet. The same letters carrying accents, β', L', . . . , have the same meaning in referring to the motion of the second planet, that is, for the relative motion of m_3 about the center of mass of m_1 and m_2.

CHAPTER 8

Formal Calculus

Various Meanings of the Term "Convergence"

118. Between mathematicians and astronomers some misunderstanding exists with respect to the meaning of the term "convergence." Mathematicians who are mainly concerned with perfect rigorousness of the calculation and frequently are indifferent to enormous length of some calculation which they consider useful, without actually thinking of ever performing it efficiently, stipulate that a series is convergent if the sum of the terms tends to a predetermined limit even if the first terms decrease very slowly. Conversely, astronomers are in the habit of saying that a series converges whenever the first twenty terms, for example, decrease rapidly even if the following terms might increase indefinitely.

Thus to take a simple example, let us consider two series that have the following general term:

$$\frac{1000^n}{1\cdot 2\cdot 3\cdots n} \quad \text{and} \quad \frac{1\cdot 2\cdot 3\cdots n}{1000^n}.$$

Pure mathematicians would say that the first series converges and even that it converges rapidly since the millionth term is much smaller than the 999 999th; however, they will consider the second series to be divergent since the general term is able to grow beyond all bounds.

Conversely, astronomers will consider the first series to be divergent since the first thousand terms increase; they will call the second series convergent since the first thousand terms decrease and since this decrease is rapid at first.

Both rules are legitimate; the first for theoretical research and the second for numerical applications. Both must prevail, but in two entirely separate domains of which the boundaries must be accurately defined.

Astronomers do not always know these boundaries accurately but rarely exceed them; the approximation with which they are satisfied usually keeps them far on this side of the boundary. In addition, their instinct guides them and, if they are wrong, a check on the actual observation promptly reveals their error.

Nevertheless, it seems useful to bring some greater precision to this question, which will be attempted here despite the fact that its very nature makes this rather difficult. So as to avoid any confusion, let us say now, unless stated differently, that the term "convergence" will always be used here in the sense of the pure mathematician.

Series Analogous to the Stirling Series

119. The first example, which clearly shows the legitimacy of certain expansions, is the classic example of the Stirling series. Cauchy has demonstrated that the terms of this series first decrease and then increase so that the series diverges. However, if the series is terminated at the smallest term, the Eulerian function will be represented with an approximation that is better the greater the argument.

Since then, numerous analogous facts have been demonstrated and I myself have studied earlier,[1] an important class of series that exhibit the same properties as the Stirling formula.

Let us cite still another example which has some interesting properties and which might be useful in what follows.

Let w_0 be a positive number smaller than unity.

The series

$$\varphi(w,\mu) = \sum \frac{w^n}{1+n\mu} \tag{8.1}$$

converges for all values of w and of μ, such that

$$|w| < w_0, \quad \mu \geqslant 0 .$$

In addition, the convergence is absolute and uniform.

On the other hand, we have

$$\frac{w^n}{1+n\mu} = \sum_p w^n(-1)^p n^p \mu^p .$$

One could thus be tempted to equate $\varphi(w,\mu)$ to the double-entry series

$$\sum_n \sum_p w^n(-1)^p n^p \mu^p .$$

However, this series does not converge absolutely.

Arranging the terms in ascending powers of μ, we obtain

$$A_0 - A_1\mu + A_2\mu^2 - A_3\mu^3 + \cdots , \tag{8.2}$$

where

$$A_0 = \sum w^n, \quad A_1 = \sum nw^n, \quad A_2 = \sum n^2 w^n, \quad A_3 = \sum n^3 w^n, \quad \ldots .$$

Series (8.2), arranged in ascending powers of μ, diverges. Assuming that w is positive and real so as to fix our ideas, we obtain

$$k^k w^k < A_k < \frac{k!}{(1-w)^k}.$$

It is obvious that the series

$$\sum (-kw\mu)^k$$

diverges and that this is *a fortiori* true for series (8.2). However, on considering the series

$$\sum \frac{(-\mu)^k k!}{(1-w)^k},$$

it will be found that, if $\mu/(1-w)$ is very small, the first terms decrease rapidly despite the fact that the following terms increase beyond all bounds.

Does this series (8.2) approximately represent the function $\varphi(w,\mu)$?

To answer this question, let us set

$$\varphi_p(w,\mu) = A_0 - A_1\mu + A_2\mu^2 - \cdots \pm A_p\mu^p.$$

We state that

$$\lim \frac{\varphi - \varphi_p}{\mu^p} = 0 \quad \text{for } \mu = 0.$$

In fact, this yields

$$\frac{\varphi - \varphi_p}{\mu^p} = (-1)^{p+1}\mu \sum \frac{n^{p+1}w^n}{1+n\mu}.$$

It is easy to see that the series

$$\sum \frac{n^{p+1}w^n}{1+n\mu}$$

converges uniformly; thus, for $\mu = 0$, we have

$$\lim \sum \frac{n^{p+1}w^n}{1+n\mu} = \sum n^{p+1}w^n = \text{finite quantity}$$

and, consequently,

$$\lim \frac{\varphi - \varphi_p}{\mu^p} = 0. \qquad \text{Q.E.D.}$$

Calculation of the Series

120. This leads to a relation of an entirely new nature which is able to exist between a function of x and μ, denoted by $\varphi(x,\mu)$, and a divergent series arranged in powers of μ

$$f_0 + \mu f_1 + \mu^2 f_2 + \cdots + \mu^p f_p + \cdots, \tag{8.3}$$

where the coefficients $f_0, f_1, \ldots$ can be functions of x only, independent of μ (which would happen in the above example), or else depending simultaneously on x and μ.

Let us put

$$\varphi_p = f_0 + \mu f_1 + \mu^2 f_2 + \cdots + \mu^p f_p \,.$$

If we have

$$\lim \frac{\varphi - \varphi_p}{\mu^p} = 0 \quad \text{for } \mu = 0\,,$$

we state that series (8.3) asymptotically represents the function φ and write

$$\varphi(x,\mu) \equiv f_0 + \mu f_1 + \mu^2 f_2 + \cdots \,. \tag{8.4}$$

The relations of the form of Eq. (8.4) will be designated as asymptotic equalities.

It is obvious that, if μ is very small, the difference $\varphi - \varphi_p$ will also be very small and, despite the fact that series (8.4) is divergent, the sum of its $p + 1$ first terms will, very approximately, represent the function φ.

Astronomers would say that this series is convergent and represents the function φ.

Astronomers have always continued to look for series that formally satisfy the proposed differential equations, without much concerning themselves with the convergence of these series. This procedure, at first glance, seems entirely illegitimate but nevertheless will frequently lead to success.

To explain this fact, it is necessary to study the question in more detail, and this I now propose to do.

Let us introduce some new definitions.

Let us consider a system of differential equations

$$\frac{dx_i}{dt} = X_i \quad (i = 1,2, \ldots, n)\,. \tag{8.5}$$

We assume that X_i is a uniform function of t, of $x_1, x_2, \ldots, x_n$, and of a parameter μ and that the series can be expanded in ascending powers of μ.

Let us now consider n divergent series which will be written as follows:

$$S_1 = f_{01} + \mu f_{11} + \mu^2 f_{12} + \cdots,$$
$$S_2 = f_{02} + \mu f_{12} + \mu^2 f_{22} + \cdots,$$
$$\vdots$$
$$S_n = f_{0n} + \mu f_{1n} + \mu^2 f_{2n} + \cdots.$$

Let us suppose that the quantities f_{ik} are known functions of t and μ and, in addition, that these functions can be expanded in convergent series in ascending powers of μ.

Let φ_{pk} be the sum of the $p+1$ first terms of the series S_k. We state that the series $S_1, S_2, \ldots, S_n$ formally satisfy the differential equations (8.5) if, on substituting

$$\varphi_{p1}, \quad \varphi_{p2}, \quad \ldots, \quad \varphi_{pn},$$

for

$$x_1, \quad x_2, \quad \ldots, \quad x_n,$$

the difference $dx_i/dt - X_i$ becomes divisible by μ^{p+1}.

After having made this definition, we propose to establish the following. Let us consider a particular solution of Eqs. (8.5), namely, a solution which is such that

$$x_1 = x_2 = \cdots = x_n = 0$$

for $t = 0$.

Let

$$x_1 = \theta_1(t,\mu), \quad x_2 = \theta_2(t,\mu), \quad \ldots, \quad x_n = \theta_n(t,\mu).$$

We will assume that the functions f_{ik} all vanish for $t = 0$.

We state that the following asymptotic equalities will be obtained:

$$\theta_1(t,\mu) \equiv S_1, \quad \theta_2(t,\mu) \equiv S_2, \quad \ldots, \quad \theta_n(t,\mu) \equiv S_n. \qquad (8.6)$$

In fact, let us put

$$x_1 = \varphi_{p1} + \mu^{p+1}\xi_1, \; x_2 = \varphi_{p2} + \mu^{p+1}\xi_2, \; \ldots, \; x_n = \varphi_{pn} + \mu^{p+1}\xi_n.$$

Substituting these values of x into Eqs. (8.5), these equations become

$$\mu^{p+1}\frac{d\xi_i}{dt} = X_i - \frac{d\varphi_{p.i}}{dt}.$$

After the substitution, X_i will become expandable in ascending powers of μ, of

$$\mu^{p+1}\xi_1, \quad \mu^{p+1}\xi_2, \quad \ldots, \quad \mu^{p+1}\xi_n,$$

where the coefficients of the expansion are known functions of time.

An exception to this exists only if the particular solution

$$x_1 = \theta_1(t,0), \quad x_2 = \theta_2(t,0), \quad \ldots, \quad x_n = \theta_n(t,0), \quad \mu = 0$$

would pass through one of the singular points of the differential equation for one of the values of t to be considered here (as in Chap. 2, no. 27, we will denote by this the systems of values of $x_1, x_2, \ldots, x_n, t$, and μ for which the terms X_i cease being holomorphous functions).

As demonstrated in no. 27, it is thus possible to obtain two positive numbers M and α such that

$$X_i - \frac{d\varphi_{p\,i}}{dt} \ll \frac{M}{1-\alpha(\mu+\mu^{p+1}\xi_1+\cdots+\mu^{p+1}\xi_n)} \arg(\mu,\xi_k) .$$

However, by hypothesis, the series $S_1, S_2, \ldots, S_n$ formally satisfy Eqs. (8.5). This means that, if we set

$$\xi_1 = \xi_2 = \cdots = \xi_n = 0 ,$$

whence

$$x_i = \varphi_{p\,i} ,$$

then the differences $X_i - d\varphi_{p\,i}/dt$ become divisible by μ^{p+1}. Thus we have

$$X_i - \frac{d\varphi_{p\,i}}{dt} \ll \frac{M\mu_\alpha^{p+1}(\alpha^p+\xi_1+\xi_2+\cdots+\xi_n)}{1-\alpha(\mu+\mu^{p+1}\xi_1+\cdots+\mu^{p+1}\xi_n)} . \tag{8.7}$$

If, for abbreviation, we denote by $\mu^{p+1}Z$ the second term of the inequality (8.7) and if we put

$$X_i - \frac{d\varphi_{p\,i}}{dt} = \mu^{p+1} Y_i ,$$

then Eqs. (8.5) become

$$\frac{d\xi_i}{dt} = Y_i , \tag{8.8}$$

with the condition

$$Y_i \ll Z .$$

Let us now consider the particular solution of Eqs. (8.8) which is such that

$$\xi_1 = \xi_2 = \cdots = \xi_n = 0$$

for $t = 0$. This solution can be written as follows:

$$\xi_i = \frac{\theta_i(t,\mu) - \varphi_{p\,i}}{\mu^{p+1}} .$$

For demonstrating the asymptotic equalities (8.6), it is thus sufficient to establish that ξ_i is finite. For this, in turn, it is sufficient to compare Eqs. (8.8) with the equations

$$\frac{d\xi_i}{dt} = Z \,. \tag{8.8a}$$

So long as the solution of Eq. (8.8a) is finite, this will be also the case for the solution of Eq. (8.8). However, Eqs. (8.8a) are easy to integrate. Putting

$$\xi_1 + \xi_2 + \cdots + \xi_n = \sigma \,,$$

we obtain, for the particular solution under consideration,

$$\xi_1 = \xi_2 = \cdots = \xi_n = \frac{\sigma}{n}$$

and

$$\frac{d\sigma}{dt} = \frac{M\alpha(\alpha^p + \sigma)}{1 - \alpha\mu - \sigma\mu^{p+1}} \,.$$

It is easy to integrate this latter equation and to find that σ is finite as well as that σ tends to a finite limit as soon as μ tends to zero.

This is then also true for the ξ_i. Q.E.D.

This theorem justifies the procedure used by astronomers, provided that μ is sufficiently small. Possibly, this could have been established in a simpler form; however, the above demonstration yields a simple means for finding an upper bound for the error term.

121. It now remains to see to what extent the conventional rules of calculus are applicable to the formal calculus:

For this, let us consider two simultaneous equations

$$\frac{dx}{dt} = X, \quad \frac{dy}{dt} = Y \,, \tag{8.9}$$

where X and Y are uniform functions of x, y, t, and μ, which can be expanded in powers of μ.

Let us change variables by putting

$$x = \psi_1(\xi,\eta) \,,$$
$$y = \psi_2(\xi,\eta) \,,$$

where ψ_1 and ψ_2 are functions of ξ, η, t, and μ.

The differential equations then become

$$\frac{d\xi}{dt} = X', \quad \frac{d\eta}{dt} = Y' \,, \tag{8.10}$$

where

$$X = \frac{dx}{d\xi} X' + \frac{dx}{d\eta} Y' ,$$

$$Y = \frac{dy}{d\xi} X' + \frac{dy}{d\eta} Y' ,$$

where X' and Y' can be expanded in ascending powers of μ, unless $(dx/d\xi)(dy/d\eta) - (dx/d\eta)(dy/d\xi)$ is divisible by μ which we do not assume here.

After this, let

$$S = f_0 + \mu f_1 + \mu^2 f_2 + \cdots ,$$

$$S' = f_0' + \mu f_1' + \mu^2 f_2' + \cdots$$

be two divergent series where f_i and f_i' are functions of t and μ' which can be expanded in convergent series in ascending powers of μ'.

Let us assume that these series S and S' formally satisfy Eqs. (8.10) when one substitutes these series for ξ and η and sets $\mu' = \mu$.

Let us now substitute, in the two equations

$$x = \psi_1(\xi,\eta), \quad y = \psi_2(\xi,\eta) ,$$

the series S and S' for ξ and η and let us then expand

$$\psi_1(S,S'), \quad \psi_2(S,S') ,$$

in ascending powers of μ. Despite the fact that the series S and S' are assumed to be divergent, the expansion will be performed by the conventional rules of calculus. Below, we will show what we mean by this.

Let S_p and S_p' be the sum of the $p+1$ first terms of S and S'. Let us assume that the $p+1$ first terms of the expansion of $\psi_1(S,S')$ and of $\psi_2(S,S')$ are to be calculated.

For these $p+1$ first terms, it is necessary to take the $p+1$ first terms of the expansion of

$$\psi_1(S_p,S_p') \text{ and } \psi_2(S_p,S_p') .$$

This will yield two divergent series that can be written as follows:

$$\begin{aligned} \psi_1(S,S') &= F_0 + \mu F_1 + \mu^2 F_2 + \cdots , \\ \psi_2(S,S') &= F_0' + \mu F_1' + \mu^2 F_2' + \cdots , \end{aligned} \tag{8.11}$$

and these series have the same form as the series S and S'.

We state that these two series formally satisfy Eqs. (8.9) if they are substituted for x and y and if one then sets $\mu' = \mu$.

In fact, if we put

$$x = \psi_1(S_p,S_p'), \quad y = \psi_2(S_p,S_p') ,$$

then the difference of the two sides of Eqs. (8.9) becomes divisible by μ^{p+1}.

On the other hand, denoting by Σ_p and Σ'_p the sum of the $p+1$ first terms of the series (8.11), the differences

$$\Sigma_p - \psi_1(S_p,S'_p), \quad \Sigma'_p - \psi_2(S_p,S'_p),$$

will be divisible by μ^{p+1}.

It is easy to conclude from this that, if we set

$$x = \Sigma_p, \quad y = \Sigma'_p,$$

then the difference of the two sides of Eqs. (8.9) becomes divisible by μ^{p+1}. Q.E.D.

Let now

$$\frac{dx}{dt} = X, \tag{8.12}$$

be a unique equation where X is a function of x, t, and μ.

By putting

$$\frac{dx}{dt} = y,$$

we obtain

$$\frac{dy}{dt} = \frac{dX}{dt} + y\frac{dX}{dx}. \tag{8.13}$$

Let

$$S = f_0 + \mu f_1 + \mu^2 f_2 + \cdots,$$

be a divergent series that formally satisfies Eq. (8.12).

Let us form the series

$$S' = f'_0 + \mu f'_1 + \mu^2 f'_2 + \cdots,$$

obtained by differentiating each term with respect to t.

We state that the two series S and S' formally satisfy the two equations (8.12) and (8.13).

In fact, let S_p and S'_p be the sum of the $p+1$ first terms of S and of S'; this will yield

$$S'_p = \frac{dS_p}{dt}.$$

Let us put

$$X = X(x,t),$$

$$\frac{dX}{dt} + y\frac{dX}{dx} = Y(x,y,t).$$

We state that the difference

$$Y(S_p,S'_p,t)-\frac{dS'_p}{dt}$$

is divisible by μ^{p+1}.

By hypothesis, the difference

$$U=X(S_p,t)-\frac{dS_p}{dt}$$

is divisible by μ^{p+1}; thus this must also be so for its derivative

$$\frac{dU}{dt}=Y(S_p,S'_p,t)-\frac{dS'_p}{dt}. \qquad \text{Q.E.D.}$$

Consequently, the conventional rules of calculus are applicable to the formal calculus.

The most interesting question for what follows is to know whether the Jacobi theorems, discussed in nos. 3 and 4, are applicable to the formal calculus.

This question must be answered in the affirmative; we will demonstrate this in no. 125 on a specific example, but the proof itself can be extended without change to the general case.

122. Earlier (Ref. 1, p. 295), we demonstrated certain properties of asymptotic equalities: One can add two asymptotic equalities; one can also multiply two asymptotic equalities.

Let now

$$S=f_0+\mu f_1+\mu^2 f_2+\cdots$$

be a divergent series where f_i are functions of t.

Let

$$\varphi(t,\mu)\equiv S$$

be an asymptotic equality.

Let us assume that $f_0=0$ so that, for $\mu=0$, we have

$$S=0, \quad \varphi(t,0)=0.$$

Now, let it be a function of z, holomorphic at $z=0$.

Let us substitute S for z in $F(z)$ and then expand $F(S)$ in powers of μ by the conventional rules of calculus, as had been explained in the preceding number. This yields the asymptotic equality

$$F[\varphi(t,\mu)]\equiv F(S).$$

It is unnecessary to reproduce the proof here. The reader may find it in the cited paper, but the proof is so easy that it can be done more rapidly by the reader himself.

Let now

$$\varphi(t,\mu) \equiv f_0 + \mu f_1 + \mu^2 f_2 + \cdots ,$$

be an asymptotic equality where the terms f depend on t and μ. We will assume that this equality takes place uniformly. This means that the expression

$$\frac{\varphi - \varphi_p}{\mu^p} ,$$

where φ_p denotes the sum of the $p+1$ first terms of the series, tends uniformly to zero irrespective of the value of t, as soon as μ tends to zero. This means that one can derive a number ϵ independent of t, depending only on μ and vanishing with μ such that

$$|\varphi - \varphi_p| < \mu^p \epsilon .$$

This will yield

$$\left| \int_{t_0}^{t_1} (\varphi - \varphi_p) dt \right| < \mu^p \epsilon (t_1 - t_0) ,$$

which shows that one has the asymptotic equality

$$\int \varphi dt \equiv \int f_0 dt + \mu \int f_1 dt + \mu^2 \int f_2 dt + \cdots .$$

Thus we may integrate an asymptotic equality. On the other hand, one does not generally have the right to differentiate such an equality. Nevertheless, one case exists in which the above principles permit us to do so.

Let $\varphi(t,\mu)$ be a solution of a differential equation and let S be a series that formally satisfies this equation.

Asymptotically, we then have

$$\varphi(t,\mu) \equiv S .$$

Let S' be the series obtained by differentiating each term of S.

According to the preceding number, this series formally satisfies the differential equation that is satisfied by the derivative $d\varphi/dt$.

This will yield the asymptotic equality

$$\frac{d\varphi}{dt} \equiv S' .$$

The reader's indulgence is asked for such a lengthy explanation of these simple points; however, it seemed important to define the nature of the misunderstanding mentioned above. In addition, before starting a study of the methods of successive approximations used in celestial mechanics, methods that are divergent from the viewpoint of the mathematician, we wish to explain why their use may be of service to astronomers.

CHAPTER 9

Methods of Newcomb and Lindstedt

Historical Background

123. Lindstedt[2] proposed a method of integration by successive approximations of the equation

$$\frac{d^2x}{dt^2} + n^2x = \mu\varphi(x,t) , \tag{9.1}$$

where $\varphi(x,t)$ is a function expanded in ascending powers of x whose coefficients are periodic functions of time t.

He was even able to demonstrate that the method is applicable to the following equations:

$$\frac{d^2x}{dt^2} + n_1^2x = \mu\varphi(x,y,t) ,$$

$$\frac{d^2y}{dt^2} + n_2^2y = \mu\psi(x,y,t) ,$$

which are more general than Eq. (9.1) and which reduce to a particular case of the equations of dynamics, provided that one has

$$\frac{d\varphi}{dy} = \frac{d\psi}{dx} .$$

Equation (9.1) is of extreme importance in celestial mechanics, since Gyldén has arrived at these equations several times during his excellent research.

Lindstedt did not prove the convergence of the expansions performed in this manner; in fact, they actually are divergent. However, it has been shown in the preceding chapter that they might nevertheless be of interest and of considerable usefulness.

However, another and more serious difficulty exists here. It is easy to see that the method is applicable in the first approximations but it is questionable whether it might still be valid in the next approximations. Lindstedt was unable to establish this rigorously and even harbored some doubts on this subject. These doubts had no basis in fact and his excellent method is still legitimate. I proved this first by using integral invariants[3] and, later, without using these invariants.[4] The second of these proofs will be discussed in this chapter. This led to a modification of the Lindstedt method which is applicable directly to the most general case of the equations of dynamics.

Several special cases, however, are not covered by this, including the general case of the three-body problem.

Because of its importance, the latter case attracted also the attention of Lindstedt. This astronomer[5] has described a procedure for applying his method to this case.

Unfortunately, the same difficulties as those mentioned above still remain; these include not only the divergence of the expansions, which can be disregarded here for the reasons discussed in the preceding chapter, but even the very feasibility of the expansions and thus the legitimacy of the method itself.

I believe that I have been able to erase these doubts; Chap. 11 will be devoted to this topic.

Thus for explaining the manner of applying the Lindstedt method to the three-body problem, we will adopt a mode of presentation which will neither be that of its inventor, nor that suitable for calculation of the various terms of the series but rather that most suitable for demonstrating the legitimacy of the method.

Lindstedt was preceded in his trend of work by Newcomb[6] who was the first to derive series representing the motion of planets and containing only sines and cosines. His method, which will be discussed later in the text, is based on the variation of arbitrary constants.

124. Despite the fact that, among the methods recently introduced to celestial mechanics, those by Lindstedt are chronologically not the first, I believe nevertheless that they constitute the most suitable basis for starting a discussion of these novel methods of successive approximations. Actually, it is difficult to separate the discussion from the Newcomb equations which were first in chronological order; in addition, the Lindstedt methods are the least complicated of all and best adapted to the simplest cases. These methods only fail in the presence of very small divisors, in which case the more perfected methods by Gyldén must be used. My manner of discussing the Lindstedt theory will differ considerably from that given by this astronomer; in addition, I will apply the method to many more cases although the obtained series will be identical to his series, as will be shown below.

In addition, his results will be supplemented on a large number of points, with an attempt to extend them to as many problems as possible.

Discussion of the Method

125. Let us return to the equations of no. 13:

$$\frac{dx_i}{dt} = -\frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1,2, \ldots ,n), \tag{9.2}$$

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots .$$

The problem consists here in formally satisfying Eqs. (9.2) by series of the following form:

$$\begin{aligned} x_i &= x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots + \mu^n x_i^n + \cdots , \\ y_i &= y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + \cdots + \mu^n y_i^n + \cdots , \end{aligned} \tag{9.3}$$

where the quantities x_i^k and y_i^k themselves have the form

$$x_i^k = \sum A \cos ht + \sum B \sin ht + C ,$$

$$y_i^k = \sum A' \cos ht + \sum B' \sin ht + C't + D' ,$$

where A, B, C, A', B', C', and D' are independent coefficients of μ and of the time t but can be functions of a certain number of constants of integration; the terms h are coefficients depending on μ and expanded in powers of this parameter.

When stating that series (9.3) formally satisfy Eqs. (9.2), we mean the following.

In Eqs. (9.2), let us substitute series (9.3) terminated at the $p + 1$ term, i.e., let us set

$$x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots + \mu^p x_i^p ,$$

$$y_i = y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + \cdots + \mu^p y_i^p ,$$

I shall say that the series (9.3) formally satisfy Eqs. (9.2) if, after the substitution, the difference of the two sides of these equations becomes divisible by μ^p.

For determining series (9.3), we will use a procedure totally different from that applied by Lindstedt himself.

Thus let us attempt to derive a series of the form

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \cdots + \mu^p S_p + \cdots \tag{9.4}$$

whose coefficients S_k themselves are assumed to be series of the form

$$S_k = \alpha_{k\,1} y_1 + \alpha_{k\,2} y_2 + \cdots + \alpha_{k\,n} y_n + \varphi_k ,$$

where $\alpha_{k\,i}$ are constant coefficients while φ_k is a periodic function of y_1, $y_2, \ldots , y_n$, with period 2π with respect to these n variables.

We will attempt to determine series (9.4) in such a manner as to formally satisfy the partial differential equation

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, y_1, y_2, \ldots, y_n\right) = \text{const.} \tag{9.5}$$

Since the constant on the right-hand side (which is nothing else but the *vis viva* or kinetic energy constant) can depend on μ, we will put it equal to

$$C_0 + \mu C_1 + \mu^2 C_2 + \cdots .$$

Setting $\mu = 0$ in Eq. (9.5) and recalling that F_0 does not depend on y, we obtain

$$F_0\left(\frac{dS_0}{dy_1}, \frac{dS_0}{dy_2}, \ldots, \frac{dS_0}{dy_n}\right) = C_0 . \tag{9.6}$$

This equation can be satisfied by setting

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n ,$$

$$\frac{dS_0}{dy_1} = x_1^0, \quad \frac{dS_0}{dy_2} = x_2^0, \quad \ldots, \quad \frac{dS_0}{dy_n} = x_n^0 ,$$

where x_i^0 are constants that can be arbitrarily selected since the constant C_0 itself is arbitrary.

As in the preceding chapters, we will put

$$n_i^0 = -\frac{dF}{dx_i^0} .$$

By then equating the coefficients of similar powers of μ in the two members of Eq. (9.4), a series of equations will be obtained that permits determining, by recurrence, $S_1, S_2, \ldots, S_p, \ldots$.

These equations have the form

$$n_1^0 \frac{dS_p}{dy_1} + n_2^0 \frac{dS_p}{dy_2} + \cdots + n_n^0 \frac{dS_p}{dy_n} = \Phi_p + C_p . \tag{9.7}$$

Here, Φ_p is an integral polynomial with respect to the quantities

$$\frac{dS_k}{dy_i} \quad (k = 1,2, \ldots, p-1; i = 1,2, \ldots, n)$$

while the coefficients of this polynomial are periodic functions of x_1^0, x_2^0, $\ldots$, x_n^0 and of $y_1, y_2, \ldots, y_n$ with period 2π with respect to the y.

We state that the function S_p can be derived from Eq. (9.7) in such a manner that $dS_p/dy_1, dS_p/dy_2, \ldots, dS_p/dy_n$ are periodic, with period 2π with respect to y.

Let us assume that this is true for the derivatives of $S_1, S_2, \ldots, S_{p-1}$ with respect to y.

Then, Φ_p will be a periodic function of $y_1, y_2, \ldots, y_n$ so that we can write

$$\Phi_p = A + \sum B \cos(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n)$$

$$+ \sum C \sin(m_1 y_1 + \cdots + m_n y_n),$$

where the numbers $m_1, m_2, \ldots, m_n$ are integers whereas the terms A, B, and C are constant coefficients independent of y.

Then, we can write

$$\begin{aligned} S_p = {} & \alpha_{p1} y_1 + \alpha_{p2} y_2 + \cdots + \alpha_{pn} y_n \\ & + \sum \frac{B \sin(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n)}{m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0} \\ & + \sum \frac{C \cos(m_1 y_1 + \cdots + m_n y_n)}{m_1 n_1^0 + \cdots + m_n n_n^0}. \end{aligned}$$

The terms α_{p1} are constants that can be arbitrarily selected since they are subject only to the condition

$$\alpha_{p1} n_1^0 + \alpha_{p2} n_2^0 + \cdots + \alpha_{pn} n_n^0 = A + C_p$$

and since the condition C_p is arbitrary.

This method will fail only if there exist integers $m_1, m_2, \ldots, m_n$ such that

$$\sum m_i n_i^0 = 0.$$

We will assume that this is not the case.

It should be mentioned that the functions S_p defined in this manner contain arbitrary constants; they depend first on

$$x_1^0, \quad x_2^0, \quad \ldots, \quad x_n^0,$$

then on

$$\alpha_{11}, \quad \alpha_{12}, \quad \ldots, \quad \alpha_{1n},$$

then on

$$\begin{matrix} \alpha_{21}, & \alpha_{22}, & \ldots, & \alpha_{2n}, \\ \ldots, & \ldots, & \ldots, & \ldots, \end{matrix}$$

and then on

$$\begin{matrix} \alpha_{p1}, & \alpha_{p2}, & \ldots, & \alpha_{pn}, \\ \ldots, & \ldots, & \ldots, & \ldots . \end{matrix}$$

We will retain here only n arbitrary constants. Consequently, we will continue to consider the terms x_i^0 as arbitrary, selecting the quantities α_{ik} in any, but some definite, manner. For example, it could be agreed to select the α_{ik} terms in such a manner that

$$0 = C_1 = C_2 = \cdots = C_p = \cdots .$$

However, we prefer to assume that all α_{ik} are zero. The constants C_1, C_2, $\ldots$, C_p, $\ldots$ are then not zero. In general, they depend, just as C_0, on x_1^0, $x_2^0, \ldots, x_n^0$.

After this, let

$$\Sigma_p = S_0 + \mu S_1 + \mu^2 S_2 + \cdots + \mu^p S_p .$$

Let us put

$$\frac{d\,\Sigma_p}{dy_i} = x_i, \quad \frac{d\,\Sigma_p}{dx_i^0} = w_i . \tag{9.8}$$

If we do not change variables by using as new variables the quantities x_i^0 and w_i instead of x_i and y_i [the new variables would be linked to the old variables by the relations (9.8)], then the theorem in no. 4 shows that the equations will remain canonical and that we will have[R1]

$$\frac{dx_i^0}{dt} = \frac{dF}{dw_i}, \quad \frac{dw_i}{dt} = -\frac{dF}{dx_i^0} \quad (i = 1,2, \ldots ,n) .$$

Let us see now what will be the form of F when expressing it as a function of the new variables x_i^0 and w_i^0. By hypothesis, the series S formally satisfies Eq. (9.5). This reduces to stating that we will have

$$\begin{aligned} F(x_i,y_i) &= F\left(\frac{d\,\Sigma_p}{dy_i},y_i\right) \\ &= C_0 + \mu C_1 + \mu^2 C_2 + \cdots + \mu^p C_p + \mu^{p+1}\Phi_p , \end{aligned}$$

where Φ_p is a function of x_i^0, of w_i, and of μ which can be expanded in powers of μ. As for the quantities C_0, $C_1, \ldots, C_p$, we have seen that they are functions of x_i^0.

We will then put

$$\nu_i^p = -\frac{dC_0}{dx_i^0} - \mu\frac{dC_1}{dx_i^0} - \mu^2\frac{dC_2}{dx_i^0} - \ldots - \mu^p\frac{dC_p}{dx_i^0} .$$

Thus, for $\mu = 0$, the term ν_i^p reduces to n_i^0.

This yields

$$\frac{dx_i^0}{dt} = \mu^{p+1}\frac{d\Phi_p}{dw_i}, \quad \frac{dw_i}{dt} = \nu_i^p - \mu^{p+1}\frac{d\Phi_p}{dx_i^0} .$$

Neglecting all quantities of the order μ^{p+1}, these equations will furnish

$$x_i^0 = \text{const.}, \quad w_i = \nu_i^p t + \text{const}\,.$$

This result can be expressed by stating that the Jacobi theorem of no. 3 is applicable to the formal calculus when using the notations given in Chap. 8.

Let us put

$$n_i = -\frac{dC_0}{dx_i^0} - \mu\frac{dC_1}{dx_i^0} - \mu^2\frac{dC^2}{dx_i^0} - \cdots \textit{ ad inf.}$$

This constitutes a series arranged in powers of μ, which may be divergent. However, this is immaterial here since we are using the viewpoint given in the preceding chapter, namely, the formal viewpoint.

Let us then put

$$w_i = n_i t + \varpi_i\,,$$

where the terms ϖ_i are considered to be constants of integration. Let us then consider the equations

$$\frac{dS}{dy_i} = x_i, \quad \frac{dS}{dx_i^0} = w_i\,. \tag{9.9}$$

From Eqs. (9.9), the quantities x_i and y_i can be derived in the form of series arranged in powers of μ, whose coefficients are functions of x_i^0 and of w_i. These series can be either convergent or divergent, but this is immaterial.

If, in these series, the terms w_i are replaced by $n_i t + \varpi_i$ and if the terms x_i^0 are considered as constants, the series will formally satisfy Eqs. (9.2).

Let Eq. (9.3) be the series. Let us now see what the form of x_i^k and of y_i^k will be. For $\mu = 0$, the series S reduces to

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n\,,$$

from which it follows that

$$x_i = x_i^0, \quad y_i = w_i\,.$$

Thus the first term of the expansion of x_i is a constant and the first term of the expansion of y_i (i.e., y_i^0) reduces to

$$w_i = n_i t + \varpi_i\,.$$

If, instead of deriving the terms x_i and y_i from Eqs. (9.9), they had been derived from Eqs. (9.8), then the $p+1$ first terms would have been the same since the difference $S - \Sigma_p$ is of the order of μ^{p+1}.

For determining the quantities

$$x_i^k, \quad y_i^k \quad (i = 1,2, \ldots, n;\ k = 0,1,2, \ldots, p)\,,$$

let us consider Eqs. (9.8) which will be written in the form

$$x_i = x_i^0 + \frac{d(\Sigma_p - S_0)}{dy_i}, \quad y_i = w_i + \frac{d(\Sigma_p - S_0)}{dx_i^0}. \tag{9.8a}$$

From Eqs. (9.8a), we can derive the quantities x_i and y_i in series arranged in powers of μ and convergent if μ is sufficiently small. For this, it is sufficient to apply the theorem of no. 30 since $\Sigma_p - S_0$ represents a completely defined function and no longer is a simple formal expression.

We have assumed that the quantities $\alpha_{k\,i}$ are zero. It results from this that the S_k $(k>0)$ and thus also $\Sigma_p - S_0$ are periodic functions, with period 2π with respect to y_i.

Thus, if in Eqs. (9.8a) the term y_i is changed into $y_i + 2k_i\pi$ and the term w_i into $w_i + 2k_i\pi$ ($k_1, k_2, \ldots, k_n$ being integers), then these equations will not change. Consequently, the values of x_i and of $y_i - w_i$, derived from these equations, are periodic, with period 2π with respect to w.

Thus in the series (9.3), the quantities x_i^k and y_i^k are periodic functions, with period 2π with respect to w_i.

Various Forms of Series

126. The existence of the series (9.9) having been demonstrated in this manner, it becomes possible to form these series without going over the auxiliary expression S.

However, we will first demonstrate that it is possible to formally satisfy Eqs. (9.2) of the preceding number by an infinity of other series of the same form as the series (9.3).

(i) The function S of the preceding number is determined by Eq. (9.5) to within a constant or, expressed differently, since the quantities $x_1^0, x_2^0, \ldots, x_n^0$ are considered constants, to within an arbitrary function of $x_1^0, x_2^0, \ldots$, and x_n^0.

Thus if a function S satisfies Eq. (9.5), this will be true also for the function

$$S' = S + R,$$

where R is a function of $x_1^0, x_2^0, \ldots, x_n^0$ and μ that can be expanded in ascending powers of μ.

Let us now replace Eqs. (9.9) by the expressions

$$x_i = \frac{dS'}{dy_i} = \frac{dS}{dy_i}, \quad w_i = \frac{dS'}{dx_i^0} = \frac{dS}{dx_i^0} + \frac{dR}{dx_i^0}. \tag{9.9a}$$

It can be assumed that R is divisible by μ; in that case, the quantities x_i and y_i can be derived from Eqs. (9.9a) in the form of series (9.3a) having the same configuration as series (9.3).

We then have

$$x_i = x_i^0 + \mu x_i'^1 + \mu^2 x_i'^2 + \cdots ,$$
$$y_i = w_i + \mu y_i'^1 + \mu^2 y_i'^2 + \cdots , \tag{9.3a}$$

where $x_i'^k$ and $y_i'^k$, just as the terms x_i^k and y_i^k, are periodic functions of w.

A comparison of Eqs. (9.9a) and Eqs. (9.9) shows that series (9.3a) are obtained from series (9.3) on substituting there w_i by $w_i + dR/dx_i^0$.

(ii) More generally, let

$$\omega_1, \quad \omega_2, \quad \ldots, \quad \omega_n$$

be n functions of $x_1^0, x_2^0, \ldots, x_n^0$ and μ that can be expanded in powers of μ.

If, in series (9.3), the terms $w_1, w_2, \ldots, w_n$ are changed into

$$w_1 + \mu\omega_1, \quad w_2 + \mu\omega_2, \quad \ldots, \quad w_n + \mu\omega_n ,$$

then these series will retain the same form. In fact, we have

$$x_i = x_i^0 + \mu\varphi_i(w_k, \mu) ,$$
$$y_i = w_i + \mu\psi_i(w_k, \mu) , \tag{9.3b}$$

where φ_i and ψ_i can be expanded in powers of μ and are periodic with respect to w.

On changing w_i into $w_i + \mu w_i$, we obtain

$$x_i = x_i^0 + \mu\varphi_i(w_k + \mu\omega_k, \mu) ,$$
$$y_i = w_i + \mu[\omega_i + \psi_i(w_k + \mu\omega_k, \mu)] . \tag{9.3c}$$

It is obvious that $\varphi_i(w_k + \mu\omega_k, \mu)$ and $\omega_i + \psi_i(w_k + \mu\omega_k, \mu)$ can still be expanded in powers of μ and are periodic with respect to w.

In addition, series (9.3c) formally satisfy Eqs. (9.2). In fact, series (9.3b) satisfy these equations if we set there

$$w_i = n_i t + \overline{\omega}_i ,$$

irrespective of the values attributed to the integration constants $\overline{\omega}_i$.

However, the terms ω_i are functions of x_i^0 that are constants; thus they also are constants. Therefore, changing w_i into $w_i + \mu\omega_i$ reduces to replacing the integration constants $\overline{\omega}_i$ by different constants $\overline{\omega}_i + \mu\omega_i$ which, according to the above remarks, will not prevent our series from still satisfying the differential equations (9.2).

Thus series (9.3c) formally satisfy Eqs. (9.2).

However, these cannot be derived from equations analogous to Eqs. (9.9) and (9.9a), unless

$$\mu(\omega_1 dx_1^0 + \omega_2 dx_2^0 + \cdots + \omega_n dx_n^0)$$

is the exact differential of a function $x_1^0, x_2^0, \ldots, x_n^0$ which then is nothing else but the function R which we had considered above.

(iii) In series (9.3b), let us change

$$x_1^0, \quad x_2^0, \quad \ldots, \quad x_n^0$$

into

$$x_1^0 + \mu v_1, \quad x_2^0 + \mu v_2, \quad \ldots, \quad x_n^0 + \mu v_n ,$$

where $v_1, v_2, \ldots, v_n$ are functions of $x_1^0, x_2^0, \ldots, x_n^0$ and μ, which can be expanded in powers of μ.

If the quantities x_i^0 are considered as constants, then the terms v_i will also be constants.

When we change the value of the integration constants in this manner, series (9.3b) will retain the same form and will still formally satisfy Eqs. (9.2).

To recapitulate, let us write the series (9.3b) in the form

$$\begin{aligned} x_i &= x_i^0 + \mu\varphi_i(w_k, x_k^0, \mu) , \\ y_i &= w_i + \mu\psi_i(w_k, x_k^0, \mu) , \end{aligned} \tag{9.3d}$$

by thus proving that $x_i - x_i^0$ and $y_i - w_i$ depend not only on w_k and on μ but also on x_k^0.

Let then

$$\omega_1, \quad \omega_2, \quad \ldots, \quad \omega_n ; \quad v_1, \quad v_2, \quad \ldots, \quad v_n$$

be $2n$ functions of x_i^0 and of μ, which can be expanded in powers of μ.

Let us form the series

$$\begin{aligned} x_i &= x_i^0 + \mu[v_i + \varphi_i(w_k + \mu\overline{\omega}_k, x_k^0 + \mu v_k, \mu)] , \\ y_i &= w_i + \mu[\omega_i + \psi_i(w_k + \mu\omega_k, x_k^0 + \mu v_k, \mu)]. \end{aligned} \tag{9.3e}$$

These series formally satisfy Eqs. (9.2) no matter what the functions ω_i and v_i might be.

In addition, since the functions

$$\varphi_i(w_k, x_k^0, \mu) \quad \text{and} \quad \psi_i(w_k, x_k^0, \mu) ,$$

are periodic with respect to w, this will be true also for the functions

$$v_i + \varphi_i(w_k + \mu\omega_k, x_k^0 + \mu v_k, \mu) \quad \text{and} \quad \omega_i + \psi_i(w_k + \mu\omega_k, x_k^0 + \mu v_k, \mu) .$$

A second remark should be made here.

Let us put

$$\begin{aligned} v_i + \varphi_i(w_k + \mu\omega_k, x_k^0 + \mu v_k, \mu) &= \varphi_i'(w_k, x_k^0, \mu) , \\ \omega_i + \psi_i(w_k + \mu\omega_k, x_k^0 + \mu v_k, \mu) &= \psi_i'(w_k, x_k^0, \mu) . \end{aligned}$$

The functions φ_i, ψ_i, φ_i', and ψ_i' are periodic functions of w; we will consider the mean values of these periodic functions and denote them, respectively, by

$$\varphi_i^0(x_k^0,\mu), \quad \psi_i^0(x_k^0,\mu), \quad \varphi_i'^0(x_k^0,\mu), \quad \psi_i'^0(x_k^0,\mu) .$$

After this, we propose to demonstrate the following.

Let θ_i and η_i $(i = 1, 2, \ldots, n)$ be $2n$ entirely arbitrary functions of x_i^0, $x_2^0, \ldots, x_n^0$ and μ, subject solely to being expandable in powers of μ.

We state that, no matter what the functions θ_i and η_i might be, it is always possible to choose the functions v_i and ω_i in such a manner that

$$\varphi_i'^0 = \theta_i, \quad \psi_i'^0 = \eta_i .$$

In fact, for this it is sufficient to define the quantities v_i and ω_i by the equations

$$v_i + \varphi_i^0(x_k^0 + \mu v_k,\mu) = \theta_i(x_k^0,\mu) ,$$

$$\omega_i + \psi_i^0(x_k^0 + \mu v_k,\mu) = \eta_i(x_k^0,\mu) .$$

However, it is always possible to derive, from these equations, the quantities v_i and ω_i in the form of series arranged in powers of μ whose coefficients are functions of x_k^0.

Writing series (9.3e) in the form

$$x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots ,$$

$$y_i = w_i + \mu y_i^1 + \mu^2 y_i^2 + \cdots ,$$

the quantities x_i^k and y_i^k will be periodic functions of w. According to the above remark, it is always possible to arrange matters such that the mean values of these periodic functions x_i^k and y_i^k will be any desired functions of $x_1^0, x_2^0, \ldots, x_n^0$.

Direct Calculation of the Series

127. Let us now pass to a direct calculation of series (9.3e). For this, let us assume that, for example, in dF/dy_i which is a function of x_i, y_i, and μ, these variables are replaced by their expansions

$$x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots ,$$

$$w_i + \mu y_i^1 + \mu^2 y_i^2 + \cdots ,$$

so that this dF/dy_i will become a function of x_i^0, x_i^k, w_i, y_i^k, and μ. This function will be periodic with respect to w_i and can be expanded in powers of μ, x_i^k, and y_i^k (if $k > 1$). The function will depend on x_i^0 in any manner.

Let us then write

$$\frac{dF}{dy_i} = X_i^0 + \mu X_i^1 + \mu^2 X_i^2 + \cdots + \mu^k X_i^k + \cdots , \tag{9.10}$$

where X_i^k are functions of w_i, x_i^k, y_i^k, and x_i^0 periodic with respect to w_i.

Similarly, we have

$$-\frac{dF}{dx_i} = Y_i^0 + \mu Y_i^1 + \mu^2 Y_i^2 + \cdots + \mu^k Y_i^k + \cdots , \qquad (9.11)$$

where Y_i^k are functions of the same form as the quantities X_i^k.

Recalling that dF_0/dy_i is zero and that dF_0/dx_i does not depend on y_i, it is easy to conclude that X_i^k depends only

$$\text{on } x_i^0, \quad \text{on } x_i^1, \quad \ldots, \quad x_i^{k-1},$$
$$\text{on } w_i, \quad \text{on } y_i^1, \quad \ldots, \quad y_i^{k-1}.$$

Conversely, Y_i^k depends on the same quantities and, in addition, on x_i^k but is independent of y_i^k. Aside from this, X_i^0 is zero and Y_i^0 reduces to n_i^0.

On the other hand, we will assume that

$$w_i = n_i t + \overline{\omega}_i ,$$

from which it follows that

$$\frac{dw_i}{dt} = n_i .$$

We now assume that n_i can be expanded in powers of μ, so that we can write

$$n_i = n_i^0 + \mu n_i^1 + \mu^2 n_i^2 + \cdots . \qquad (9.12)$$

Our differential equations are then written as

$$\sum_k n_k \frac{dx_i}{dw_k} = \frac{dF}{dy_i}, \quad \sum_k n_k \frac{dy_i}{dw_k} = -\frac{dF}{dx_i}. \qquad (9.13)$$

In fact, we have

$$\frac{dx_i}{dt} = \sum \frac{dx_i}{dw_k}\frac{dw_k}{dt} = \sum n_k \frac{dx_i}{dw_k}.$$

In Eqs. (9.13), let us replace dF/dy_i, $-dF/dx_i$, and n_k by their expansions (9.10), (9.11), and (9.12) and let us then equate the like powers of μ.

On putting, for abbreviation,

$$\sum_{k=1}^{k=n} \sum_{q=1}^{q=p-1} n_k^q \frac{dx_i^{p-q}}{dw_k} = -Z_i^p \quad (\text{if } p>1) \quad Z_i^1 = 0 .$$

$$\sum_{k=1}^{k=n} \sum_{q=1}^{q=p-1} n_k^q \frac{dy_i^{p-q}}{dw_k} = -T_i^p \quad (\text{if } p>1) \quad T_i^1 = 0 .$$

On equating the coefficients of μ^p $(p>1)$, we obtain

$$\sum_k n_k^0 \frac{dx_i^p}{dw_k} = X_i^p + Z_i^p - \sum_k n_k^p \frac{dx_i^0}{dw_k},$$
$$\sum_k n_p^0 \frac{dy_i^p}{dw_k} = Y_i^p + T_i^p - \sum_k n_k^p \frac{dy_i^0}{dw_k}. \tag{9.14}$$

Equating the terms independent of μ, it simply follows that

$$\sum n_k^0 \frac{dx_i^0}{dw_k} = 0, \quad \sum n_k^0 \frac{dy_i^0}{dw_k} = n_i^0,$$

which are equations that, as we know already, can be satisfied by setting

$$x_i^0 = \text{const.}, \quad y_i^0 = w_i .$$

Then, Eqs. (9.14) reduce to

$$\sum n_k^0 \frac{dx_i^p}{dw_k} = X_i^p + Z_i^p, \quad \sum n_k^0 \frac{dy_i^p}{dw_k} = Y_i^p + T_i^p - n_i^p . \tag{9.15}$$

Let us now see how one can use Eqs. (9.15) for determining, by recurrence, the functions

$$x_i^p \quad \text{and} \quad y_i^p,$$

in such a manner that these functions will be periodic with respect to w and that their mean values will be any desired functions of x_k^0.

We have seen in the two preceding numbers that this determination is entirely possible.

Let us assume that we have calculated

$$x_i^1, \quad x_i^2, \quad \ldots, \quad x_i^{p-1}, \quad y_i^1, \quad y_i^2, \quad \ldots, \quad y_i^{p-1}, \tag{9.16}$$

and that we wish to calculate x_i^p and y_i^p by means of Eqs. (9.15).

Since X_i^p and Z_i^p depend only on the variables (9.16), the second term of the first equation of system (9.15) is a known function of w, periodic with respect to these variables.

Let

$$X_i^p + Z_i^p = \sum A \cos(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h)$$

be this function from which, by integrating Eq. (9.15), we obtain

$$x_i^p = \sum \frac{A \sin(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h)}{m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0} + K_i^p .$$

Thus x_i^p is a periodic function of w; the only exception to this occurs in two cases: if the quantities n_i^0 satisfy a linear relation with integral coefficients

$$\sum m_i n_i^0 = 0 ,$$

(but we have assumed the opposite) or else if the periodic function $X_i^p + Z_i^p$ has a mean value differing from zero. It is not easy to demonstrate directly that this is not the case; however, since we know in advance that x_i^p must be a periodic function of w, we can be certain that the mean value of $X_i^p + Z_i^p$ is zero. This is the reason for the fact that we have started discussing the Lindstedt method based on the considerations of the two preceding numbers, instead of immediately applying the calculation of the present number.

As to the constant K_i^p, this can be arbitrarily equated to any desired function of x_k^0, according to what we have shown in the preceding number.

Then, the quantity y_i^p remains to be calculated by means of the second equation of system (9.15). It will be demonstrated that, as in the case of x_i^p, the quantity y_i^p is obtained in the form of a periodic function of w, under the condition that the mean portion of

$$Y_i^p + T_i^p - n_i^p$$

be zero. However, the constant n_i^p has remained arbitrary and it is obvious that it can be always selected in such a manner as to cancel this mean-square value.

Thus there is no limit in the calculation of the various terms of series (9.3e).

This leaves many arbitrary quantities, which a skilled calculator could easily use for abbreviating his computations. In fact, it is possible to arbitrarily select the mean values of x_i^p and y_i^p.

Among the possible choices, we should mention the following without, however, recommending them specifically. The constants K_i^p can be selected such that

$$n_i^p = 0, \quad n_i = n_i^0 .$$

This method is applicable each time that one can choose the quantities n_i^0 in such a manner that no linear relation with integral coefficients exists between them, and thus each time that one can arbitrarily choose the ratios of these n quantities.

This happens, for example, in the specific case of the three-body problem defined in no. 9. In this case, we actually have

$$F_0 = \frac{1}{2x_1^2} + x_2 ,$$

whence

$$n_1^0 = \frac{1}{x_1^3}, \quad n_1^0 = -1 .$$

It is obvious that we can select x_1 in such a manner that the ratio n_1^0/n_2^0 will have any desired value.

This also happens with the following equation which is introduced when applying the Gyldén methods and which had been particularly studied by Lindstedt:

$$\frac{d^2y}{dx^2} + n^2y = \mu\varphi'(y,x) , \tag{9.17}$$

where φ' is a function expanded in powers of y and periodic in x.

Let us mention first that φ' must always be considered as the derivative, with respect to y, of a function φ of the same form. Then, as demonstrated in no. 2, we can replace the above equation by the expressions

$$-F = \frac{q^2}{2} + \frac{n^2y^2}{2} - \mu\varphi(y,x) + p ,$$

$$\frac{dx}{dt} = -\frac{dF}{dp} = 1, \quad \frac{dy}{dt} = -\frac{dF}{dq} = q, \quad \frac{dp}{dt} = \frac{dF}{dx} ,$$

$$\frac{dq}{dt} = \frac{dF}{dy} = -n^2y + \mu\varphi'(y,x) .$$

Let us then put

$$y = \rho \sin y_1, \quad q = n\rho \cos y_1, \quad \frac{n\rho^2}{2} = x_1, \quad p = x_2, \quad x = y_2 ,$$

after which our equations will become

$$-F = nx_1 + x_2 - \mu\varphi\left(\sqrt{\frac{2x_1}{n}} \sin y_1, y_2\right) ,$$

$$\frac{dx_1}{dt} = \frac{dF}{dy_1}, \quad \frac{dx_2}{dt} = +\frac{dF}{dy_2}, \quad \frac{dy_1}{dt} = -\frac{dF}{dx_1}, \quad \frac{dy_2}{dt} = -\frac{dF}{dx_2} .$$

The canonical form of the equations will not be altered, in view of the statements in no. 6.

Setting $\mu = 0$, we have here

$$F_0 = -nx_1 - x_2 ,$$

whence

$$n_1^0 = -\frac{dF_0}{dx_1} = -n, \quad n_2^0 = -\frac{dF_0}{dx_2} = 1 .$$

Thus, if n is irrational, then no linear relation with integral coefficients exists between n_1^0 and n_2^0, and the method becomes applicable.

The method is also applicable to the general case of the three-body problem if the three bodies move in a common plane and attract each other in accordance with any law other than the Newtonian law; however, the method is no longer applicable (unless considerable modifications, to be discussed below, are made) as soon as the law of universal gravitation is Newton's law of gravitation.

Actually, in this case (and returning to the notations given in no. 125), F_0 no longer contains x_3 so that n_3^0 is zero. From this it follows that between the quantities n_i^0 a linear relation with integral coefficients exists, namely,

$$n_3^0 = 0 .$$

The direct calculation, as discussed in this number, closely resembles the original Lindstedt method. It offers a considerable advantage over the indirect procedures discussed in the two preceding numbers since it directly yields the values of x_i and y_i as a function of w and thus of the time and consequently can be used for calculation of the ephemerides. However, these indirect procedures were necessary in view of the fact that, without them, it would have been impossible to prove the legitimacy of the direct calculation (which can be achieved only if the mean value of $X_i^p + Z_i^p$ is zero) or, at least, it would have been impossible to do so without making use of the integral invariants which we will discuss in a later chapter.

From still another viewpoint, knowing these indirect procedures will be of some use. We have shown in the Introduction that it is sometimes advantageous to use an integral or an invariant relation (to use the language of nos. 1 and 19) instead of a solution. In addition, the calculation of the function S may serve as verification in the direct calculation.

128. The above-defined constant K_i^p can be chosen such that $[Y_i^p + T_i^p]$, i.e., the mean value of $Y_i^p + T_i^p$, becomes zero and that, consequently,

$$n_i^p = 0, \quad n_i = n_i^0 .$$

In fact, we have

$$Y_i^p = -\frac{d^2F_0}{dx_i dx_1} x_1^p - \frac{d^2F_0}{dx_i dx_2} x_2^p - \cdots - \frac{d^2F_0}{dx_i dx_n} x_n^p + U_i^p ,$$

where U_i^p depends only on x_i^k and y_i^k ($i = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots, p-1$). Equating the mean values, we obtain

$$[Y_i^p + T_i^p] = -\sum \frac{d^2F_0}{dx_i dx_k} K_k^p + [U_i^p + T_i^p] .$$

The functions U_i^p and T_i^p are completely known. Thus this is true also for $[U_i^p + T_i^p]$ so that it is sufficient, for canceling the terms n_i^p, to choose the constants K_k^p in such a manner as to satisfy the n linear equations

$$\frac{d^2F_0}{dx_i dx_1} K_1^p + \frac{d^2F_0}{dx_i dx_2} K_2^p + \cdots + \frac{d^2F_0}{dx_i dx_n} K_n^p = [U_i^p + T_i^p] . \tag{9.18}$$

To make this possible,[R2] it is necessary and sufficient that the Hessian of F_0 be not zero. However, it is exactly zero in the case of Eq. (9.17), i.e., in the specific case always used by Lindstedt. This is the reason for the fact that this astronomer had not been aware of the possibility of setting

$$n_i = n_i^0 .$$

This Hessian is again zero in the specific case of the three-body problem defined in no. 9; however, we have demonstrated in no. 43 that this difficulty can be overcome by a simple artifice.

Comparison with the Newcomb Method

129. To arrive at series of the same form as those discussed in this chapter, Newcomb used the variational method of arbitrary constants. To demonstrate that the result could not differ from that obtained by us in the preceding numbers, we will present this method in the following form.

Let us return to the partial differential equation

$$F\left(\frac{dS}{dy_i}, y_i\right) = \text{const.}, \tag{9.19}$$

which is Eq. (9.5) of no. 125.

Let S' be a function of $y_1, y_2, \ldots, y_n$ and of n constants $x_1^0, x_2^0, \ldots, x_n^0$ approximately satisfying Eq. (9.19), such that we will obtain

$$F\left(\frac{dS'}{dy_i}, y_i\right) = \varphi(x_i^0, y_i) = \varphi_0(x_i^0) + \epsilon\varphi_1(x_i^0, y_i) ,$$

where φ_0 depends only on the constants x_i^0 and where ϵ is very small. Then, we will have an approximate solution of the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{9.20}$$

by setting

$$\frac{dS'}{dy_i} = x_i, \quad \frac{dS'}{dx_i^0} = y_i^0 = n_i t + \varpi_i, \quad n_i = -\frac{d\varphi_0}{dx_i^0}, \tag{9.21}$$

and by regarding the terms x_i^0 and ϖ_i as arbitrary constants.

Let us now assume that the approximation is to be continued further by applying the Lagrangian method. In that case, the quantities x_i^0 and ϖ_i are no longer considered constants but new unknown functions. According to the theorem of no. 4, we must derive the new equations in the following manner. Let us replace the terms y_i by their values as a function of x_i^0 and of y_i^0 derived from Eqs. (9.21); this yields

$$\varphi(x_i^0,y_i) = \psi(x_i^0,y_i^0)\,,$$

furnishing the canonical equations

$$\frac{dx_i^0}{dt} = \frac{d\psi}{dy_i^0}\,, \quad \frac{dy_i^0}{dt} = -\frac{d\psi}{dx_i^0}\,. \tag{9.22}$$

As variables, we used y_i^0 instead of $\overline{\omega}_i$ (which comes to the same) so as to better demonstrate the canonical form of the equations.

An integration of Eqs. (9.22) can be reduced to that of the partial differential equation

$$\psi\left(\frac{dS}{dy_i^0},y_i^0\right) = \text{const.} \tag{9.23}$$

Let S'' be a function of y_i^0 and of n new constants x_i^1 satisfying this equation. If we put

$$\frac{dS''}{dy_i^0} = x_i^0, \quad \frac{dS''}{dx_i^1} = y_i^1\,, \tag{9.24}$$

we will satisfy Eqs. (9.22) by equating x_i^1 to constants and y_i^1 to linear functions of time.

If S'' is only an approximate integral of Eq. (9.23), we will thus have only approximate solutions of Eqs. (9.22).

This constitutes the variational method of constants; however, this is not entirely the method used by us in no. 125. Retaining Eq. (9.19), after having derived its approximate solution, we will attempt to search for a still closer solution. Let S''' be the solution which will depend on y_i and on n constants x_i^1. If we then put

$$\frac{dS'''}{dy_i} = x_i, \quad \frac{dS'''}{dx_i^1} = y_i^1\,, \tag{9.25}$$

the terms x_i^1 will be constants and the quantities y_i^1 will be linear functions of time, being exact if S''' is an exact solution of Eq. (9.19) and being approximate if S''' is only an approximate solution. Is it possible to select S''' in such a manner that Eqs. (9.25) become equivalent to Eqs. (9.21) and (9.24)? Equations (9.21) and (9.24) can be written as follows:

$$dS' = \sum x_i dy_i + \sum y_i^0 dx_i^0\,,$$

$$dS'' = \sum x_i^0 dy_i^0 + \sum y_i^1 dx_i^1\,,$$

and Eqs. (9.25):

$$dS''' = \sum x_i dy_i + \sum y_i^1 dx_i^1 .$$

Consequently, it is sufficient to use

$$S''' = S' + S'' - \sum x_i^0 y_i^0 .$$

The method given in no. 125 thus does not differ basically from the Newcomb method and has no other advantage over the latter but that of avoiding excessive changes of variables.

It should be added that we have selected the constants of integration in a specific manner so as to retain the canonical form of the equations. Newcomb has not restricted himself to this, which is the general habit of astronomers in applying the Lagrangean method. The equations in which the Lagrangeans occur thus assume an apparently much more complex form. However, this difference is not essential.

CHAPTER 10
Application to the Study of Secular Variations

Discussion of the Question

130. The principles discussed in the preceding chapter can be applied to the study of certain equations frequently used by astronomers.

Let

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{10.1}$$

be our canonical equations and let

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots .$$

Let us now assume that our conjugate variables x_i and y_i are the Keplerian variables of no. 11, that F_0 depends solely on βL and $\beta' L'$, i.e., on the two major axes, and that—neglecting $\mu^2 F_2$ and the following terms—μF_1 will represent the perturbing function.

Then F_1 can be expanded in sines and cosines of multiples of the two mean anomalies l and l'. We will denote by R the mean value of this periodic function of l and of l'.

Frequently, for studying the secular variations of the elements of two planets, the periodic terms in F_1 were neglected, thus reducing this function to its mean value R. In that case, our equations become

$$\frac{dx_i}{dt} = \frac{dF_0}{dy_i} + \mu \frac{dR}{dy_i}; \quad \frac{dy_i}{dt} = -\frac{dF_0}{dx_i} - \mu \frac{dR}{dx_i}. \tag{10.1a}$$

However, is it really certain that, in operating in this manner, exactly the coefficients of the secular terms of x_i and y_i are obtained, namely, the coefficients of terms whose period increases infinitely when the masses tend to zero? This question must obviously be answered in the negative. However, the approximation is generally sufficiently large and astronomers, with complete justification, have always been content with this. This is the reason for the interest centered on a study of these equations (10.1a).

Since F_0 and R do not depend on l and l', we immediately have

$$\frac{d(\beta L)}{dt} = \frac{d(\beta' L')}{dt} = 0,$$

such that L and L' can be considered as constants. Therefore, we will be content to consider the four pairs of conjugate variables

$$\begin{matrix} \beta G, & \beta\Theta, & \beta' G', & \beta'\Theta', \\ g, & \theta, & g', & \theta', \end{matrix}$$

(notations of no. 11) which, for the time being, will be called

$$\begin{matrix} x_1, & x_2, & x_3, & x_4, \\ y_1, & y_2, & y_3, & y_4. \end{matrix}$$

Then, F_0 depends on none of these eight variables and Eqs. (10.1a) become

$$\frac{dx_i}{\mu\, dt} = \frac{dR}{dy_i}, \quad \frac{dy_i}{\mu\, dt} = -\frac{dR}{dx_i} \quad (i = 1,2,3,4). \qquad (10.1b)$$

The function R depends only on our eight variables x_i and y_i since it is independent of l and of l' and since L and L' will be considered constants from here on. Equations (10.1b) thus have the canonical form.

After x_i and y_i have been determined by Eqs. (10.1b), the quantities l and l' will be calculated from the equations

$$\frac{dl}{dt} = -\mu \frac{dR}{\beta\, dL}, \quad \frac{dl'}{dt} = -\mu \frac{dR}{\beta' dL'},$$

which can be integrated by simple quadrature, since l and l' do not enter the right-hand side.

The founders of celestial mechanics have made use of these equations by reducing R to its first terms, i.e., to terms of the second order with respect to the eccentricities and inclinations. The equations are then linear and have constant coefficients. Since then, LeVerrier and Cellérier have considered the fourth-order terms and have found that they do not alter the stability.

However, the principles of the preceding chapter, as demonstrated below, permit generalizing this result and demonstrating that the result is still valid (naturally, from the viewpoint of formal calculus) no matter how far the approximation is continued.

New Change of Variables

131. If the variables (1.29) of no. 12 are used, then R can be expanded in powers of $\xi, \xi', \eta, \eta', p, p', q$, and q'; as we have seen, there are no terms of odd degree with respect to these quantities

$$\xi, \quad \xi', \quad \eta, \quad \eta', \quad p, \quad p', \quad q, \quad q'. \tag{10.2}$$

[R3]

Thus we can write

$$R(\xi,\xi',\eta,\eta',p,p',q,q') = R_0 + R_2 + R_4 + R_6 + \cdots,$$

where R_k comprises all terms of the k th degree with respect to the quantities (10.2). It is now a question of integrating the canonical equations

$$\frac{d\xi}{dt} = \frac{dR}{d\eta}, \quad \frac{d\eta}{dt} = -\frac{dR}{d\xi} \quad \cdots.$$

However, we still have a change of variables to perform so as to bring our equations into the most convenient form.

Let us first assume that the terms of an order higher than the second are neglected with respect to the quantities (10.2) and that one writes

$$R = R_0 + R_2.$$

Here, R_0 is a constant while R_2 is a homogeneous polynomial and of the second degree with respect to the variables (10.2). Thus if the canonical equations

$$\frac{d\xi}{dt} = \frac{dR_2}{d\eta}, \quad \frac{d\xi'}{dt} = \frac{dR_2}{d\eta'}, \quad \frac{dp}{dt} = \frac{dR_2}{dq}, \quad \frac{dp'}{dt} = \frac{dR_2}{dq'},$$

$$\frac{d\eta}{dt} = -\frac{dR_2}{d\xi}, \quad \frac{d\eta'}{dt} = -\frac{dR_2}{d\xi'}, \quad \frac{dq}{dt} = -\frac{dR_2}{dp}, \quad \frac{dq'}{dt} = -\frac{dR_2}{dp'}, \tag{10.3}$$

are formed, these equations will be linear with respect to the variables (10.2).

Let us assume that instead of expanding R in powers of the variables (10.2) we will expand it in powers of the eccentricities and inclinations, thus leading to the series

$$R = R_0^* + R_2^* + R_4^* + \cdots,$$

where R_k^* represents the ensemble of the terms of degree k with respect to the eccentricities and inclinations.

In accordance with our statements in no. 12, the variables (10.2) can be expanded in powers of the eccentricities and inclinations, such that, by terminating each of these series at the first term, we obtain

$$\begin{aligned} \xi &= \sqrt{\Lambda} e \cos \bar{\omega}, & \xi' &= \sqrt{\Lambda'} e' \cos \bar{\omega}', \\ \eta &= -\sqrt{\Lambda} e \sin \bar{\omega}, & \eta' &= -\sqrt{\Lambda'} e' \sin \bar{\omega}', \\ p &= \sqrt{\Lambda} i \cos \theta, & p' &= \sqrt{\Lambda'} i' \cos \theta', \\ q &= -\sqrt{\Lambda} i \sin \theta, & q' &= -\sqrt{\Lambda'} i' \sin \theta' \end{aligned} \tag{10.4}$$

(for abbreviation, we have set $\Lambda = \beta L$, $\Lambda' = \beta' L'$, as was done in no. 12).

From this, it follows that

$$R_0 = R_0^*$$

and that, for obtaining R_2^*, it is sufficient to substitute, in R_2, the variables (10.2) by their approximate value (10.4).

Inversely, R_2 will be obtained on replacing, in R_2^*, the quantities

$$e \cos \overline{\omega},\ e' \cos \overline{\omega}',\ e \sin \overline{\omega},\ e' \sin \overline{\omega}',\ i \cos \theta,\ i' \cos \theta',\ i \sin \theta,\ i' \sin \theta'$$

by

$$\frac{\xi}{\sqrt{\Lambda}},\quad \frac{\xi'}{\sqrt{\Lambda'}},\quad \frac{-\eta}{\sqrt{\Lambda}},\quad \frac{-\eta'}{\sqrt{\Lambda'}},\quad \frac{p}{\sqrt{\Lambda}},\quad \frac{p'}{\sqrt{\Lambda'}},\quad \frac{-q}{\sqrt{\Lambda}},\quad \frac{-q'}{\sqrt{\Lambda'}}.$$

However, the expansion of R_2^* is well known; in fact, R_2^* is nothing else but the ensemble of the secular terms of the perturbing function which are of degree two with respect to the eccentricities and inclinations.

From this, we can draw two conclusions:

(i) The linear equations (10.3) can be derived, through a simple change of variables, from equations (A) and (C) of Laplace's celestial mechanics,[7] which are useful for calculating the secular variations of the eccentricities and perihelions and of the inclinations and nodes.

(ii) The function R_2 is of a particular form and can be written as

$$R_2 = R_2'(\xi,\xi') + R_2'(\eta,\eta') + R_2''(p,p') + R_2''(q,q')\ .$$

It is thus the sum of the four quadratic forms, where the first depends solely on ξ and ξ', the second is formed with η and η' as is the first with ξ and ξ', while the third depends solely on p and p', and the fourth is formed with q and q' as is the third with p and p'.

After this, we will perform a linear change of variables, arranging matters so that the canonical form of the equations remains unchanged.

For this, let us put

$$V = +\xi(\sigma_1 \cos\varphi + \sigma_2 \sin\varphi) + \xi'(-\sigma_1 \sin\varphi + \sigma_2 \cos\varphi)$$
$$+ p(\sigma_2 \cos\varphi' + \sigma_4 \sin\varphi') + p'(-\sigma_3 \sin\varphi' + \sigma_4 \cos\varphi')\ ,$$

where φ and φ' are two angles depending on Λ and Λ'.

Let us next put

$$\eta = \frac{dV}{d\xi} = \sigma_1 \cos\varphi + \sigma_2 \sin\varphi,\quad \eta' = \frac{dV}{d\xi'} = -\sigma_1 \sin\varphi + \sigma_2 \cos\varphi,$$

$$q = \frac{dV}{dp} = \sigma_3 \cos\varphi' + \sigma_4 \sin\varphi',\ q' = \frac{dV}{dp'} = -\sigma_3 \sin\varphi' + \sigma_4 \cos\varphi'.$$

This furnishes relations that will define the new variables σ_i as a function of the old variables.

Let us next introduce four new variables τ_1, τ_2, τ_3, τ_4 defined by the relations

$$\tau_i = \frac{dV}{d\sigma_i},$$

whence

$$\xi = \tau_1 \cos\varphi + \tau_2 \sin\varphi, \qquad \xi' = -\tau_3 \sin\varphi + \tau_2 \cos\varphi,$$
$$p = \tau_3 \cos\varphi' + \tau_4 \sin\varphi', \quad p' = -\tau_3 \sin\varphi' + \tau_4 \cos\varphi'.$$

According to the theorem in no. 4, the canonical form of the equations will not be altered if the old variables

$$\xi, \quad \xi', \quad p, \quad p',$$
$$\eta, \quad \eta', \quad q, \quad q'$$

are replaced by the new variables

$$\tau_1, \quad \tau_2, \quad \tau_3, \quad \tau_4,$$
$$\sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4.$$

It remains to demonstrate the manner of selecting the angles φ and φ' as a function of Λ and Λ'.

The angle φ will be selected in such a manner that the quadratic form

$$R_2'(\xi,\xi') = R_2'(\tau_1 \cos\varphi + \tau_2 \sin\varphi, \ -\tau_1 \sin\varphi + \tau_2 \cos\varphi)$$

is reduced to a sum of two squares

$$A_1\tau_1^2 + A_2\tau_2^2 .$$

Similarly, we will have

$$R_2'(\eta,\eta') = R_2'(\sigma_1 \cos\varphi + \sigma_2 \sin\varphi, \ -\sigma_1 \sin\varphi + \sigma_2 \cos\varphi)$$
$$= A_1\sigma_1^2 + A_2\sigma_2^2 .$$

The angle φ' is chosen in the same manner, such that

$$R_2''(p,p') = A_3\tau_3^2 + A_4\tau_4^2 ,$$
$$R_2''(q,q') = A_3\sigma_3^2 + A_4\sigma_4^2 ;$$

yielding

$$R_2 = A_1(\sigma_1^2 + \tau_1^2) + A_2(\sigma_2^2 + \tau_2^2) + A_3(\sigma_3^2 + \tau_3^2) + A_4(\sigma_4^2 + \tau_4^2) .$$

Let us note that A_1, A_2, A_3, A_4 depend on Λ and Λ'.

The relation between the variables ξ, η, σ, and τ, which can be written

$$\begin{aligned} \xi &= \tau_1 \cos\varphi + \tau_2 \sin\varphi, \\ \xi' &= -\tau_1 \sin\varphi + \tau_2 \cos\varphi , \\ \eta &= \sigma_1 \cos\varphi + \sigma_2 \sin\varphi, \\ \eta' &= -\sigma_1 \sin\varphi + \sigma_2 \cos\varphi , \end{aligned} \qquad (10.5)$$

is an orthogonal linear substitution. Due to this fact, as explained in no. 5, the canonical form of these equations is retained. Thus the problem reduces to searching for the angles φ and φ', i.e., to the choice of the orthogonal substitution (10.5); however, this reduces again to an integration of the Laplace equations (A) and (C) mentioned above. The numerical calculation may thus be extremely protracted, but it has already been performed with respect to the solar system.

Similar results are obtained in the case in which, instead of three bodies, $n+1$ bodies are considered.

The function R_2 will be the sum of four quadratic forms; however, each of these four forms, instead of depending solely on two variables, will contain n variables.

We will then have n variables analogous to ξ, n variables analogous to η, n variables analogous to p, and n variables analogous to q. All this again reduces to determining an orthogonal linear substitution which, when applied to the variables ξ, transforms the first of these four quadratic forms into a sum of n squares.

Let us return, however, to the three-body problem.

Let us make a last change of variables, by putting

$$\tau_i = \sqrt{2\rho_i}\cos\omega_i, \quad \sigma_i = \sqrt{2\rho_i}\sin\omega_i,$$

which, according to no. 6, does not change the canonical form of the equations.

Then, R can be expanded in powers of $\sqrt{\rho_i}$ and is periodic with respect to ω_i; in addition, we have

$$R_2 = 2A_1\rho_1 + 2A_2\rho_2 + 2A_3\rho_3 + 2A_4\rho_4,$$

i.e., R_2 does not depend on ω_i.

Application of the Method of Chap. 9

132. After these various changes of variables, our equations will assume the following form:

$$\frac{d\rho_i}{dt} = -\mu\frac{dR}{d\omega_i}, \quad \frac{d\omega_i}{dt} = \mu\frac{dR}{d\rho_i}. \tag{10.6}$$

To apply the methods of the preceding chapter to these equations, it is necessary to have R expandable in ascending powers of a very small parameter. For this, we can no longer make use of μ since all terms on the right-hand side are of the same degree (of degree 1) with respect to μ.

Fortunately, the quantities ρ_i are of the order of the square of the eccentricities, and the inclinations themselves are very small.

To return to the case treated in the preceding chapter, we then only have to put

$$\rho_i = \epsilon \rho'_i \, ,$$

where ϵ is a very small constant and the quantities ρ'_i are finite. This yields

$$\frac{d\rho'_i}{dt} = -\frac{\mu}{\epsilon}\frac{dR}{d\omega_i}, \quad \frac{d\omega_i}{dt} = \frac{\mu}{\epsilon}\frac{dR}{d\rho'_i}, \tag{10.6a}$$
$$R = R_0 + R_2 + R_4 + \cdots .$$

Here, R_{2p} will be homogeneous and of degree p with respect to ρ_i, so that, on replacing ρ_i with $\epsilon\rho'_i$, we obtain

$$R = R_0 + \epsilon R'_2 + \epsilon^2 R'_4 + \cdots ,$$

where R'_{2p} is obtained on replacing ρ_i by ρ'_i in R_{2p}.

Therefore, since R_0 is reduced to a constant, our equations will become

$$\begin{aligned} \frac{d\rho'_i}{dt} &= -\mu\frac{dR'_2}{d\omega_i} - \mu\epsilon\frac{dR'_4}{d\omega_i} - \mu\epsilon^2\frac{dR'_6}{d\omega_i} - \cdots , \\ \frac{d\omega_i}{dt} &= \mu\frac{dR'_2}{d\rho'_i} + \mu\epsilon\frac{dR'_4}{d\rho'_i} + \mu\epsilon^2\frac{dR'_6}{d\rho'_i} + \cdots . \end{aligned} \tag{10.7}$$

It can be seen that the equations have retained the canonical form. Then, the function F reduces to

$$\mu(R'_2 + \epsilon R'_4 + \epsilon^2 R'_6 + \cdots) .$$

It is obvious that this has been expanded in powers of ϵ. The function is periodic with respect to the variables of the second series ω_i. Finally, the first term R'_2 does not depend on these variables ω_i. Thus, we have the conditions under which the results of the preceding chapter are applicable.

The only hypothesis that we have to make is that no linear relation with integral coefficients exist between the four constants A_1, A_2, A_3, A_4. The probability for this relation to exist is zero; nevertheless, one might ask whether there is a simple relation of this form that is sufficiently close to being satisfied that the series only converge extremely slowly. It is known that LeVerrier has discussed this particular question but he was obliged to leave it open with respect to the minor planets since their masses are relatively unknown and since the coefficients A depend on these masses.

It is obvious that all the above statement apply without change to the case of more than three bodies.

Thus it is possible to formally satisfy the equations that define the secular variations by trigonometric series of the form derived by Newcomb and Lindstedt. Then, $e \cos \varpi$, $e \sin \varpi$, $i \cos \theta$, and $i \sin \theta$ are expressed by series whose terms are periodic with respect to t. This result might have been considered by Laplace or by Lagrange as completely establishing the stability of the solar system. We are somewhat more painstaking today, since the convergence of the expansions has not been proved; the result is no less important for that.

Finally, let us note that in the case in which three bodies are involved and in which these bodies move in one plane, our canonical equations (10.7) can be reduced to the case of only one degree of freedom; thus they can be integrated by simple quadrature.

There is no need to mention specifically that an integration of Eqs. (10.7) is equivalent to an integration of the partial differential equation

$$R\left(\frac{dT}{d\omega_i}, \omega_i\right) = \text{const.},$$

where T is an unknown function while ω_i are independent variables, and whose left-hand side is the function R where ρ_i has been replaced by $dT/d\omega_i$.[R4]

CHAPTER 11
Application to the Three-Body Problem

Difficulty of the Problem

133. In the case of the three-body problem, a special difficulty arises which renders an application of the methods of Chap. 9 even more complex.

In fact, F_0 no longer depends on the six variables of the first series

$$\beta L, \quad \beta' L', \quad \beta G, \quad \beta' G', \quad \beta\Theta, \quad \beta'\Theta',$$

but only on two of these, namely,

$$\beta L \quad \text{and} \quad \beta' L'.$$

Among the quantities which we denoted by

$$n_i^0 = -\frac{dF_0}{dx_i};$$

there are thus four that are zero, namely,

$$-\frac{dF_0}{d\beta G}, \quad -\frac{dF_0}{d\beta' G'}, \quad -\frac{dF_0}{d\beta\Theta}, \quad -\frac{dF_0}{d\beta'\Theta'}.$$

The condition necessary for having the conclusions of the chapter remain valid, namely, that no linear relation with integral coefficients exist between the terms n_i^0, is thus not satisfied.

This difficulty does not arise, at least not if the three bodies move in one and the same plane, with any other law of attraction than the Newtonian law. Actually, these equations

$$\frac{dF_0}{dG} = \frac{dF_0}{dG'} = \frac{dF_0}{d\Theta} = \frac{dF_0}{d\Theta'} = 0$$

have an obvious significance. They mean that, in the Keplerian motion, the perihelions and the nodes are fixed. In fact, we have the following equations:

$$\frac{dg}{dt} = -\frac{dF}{\beta\, dG}, \quad \frac{d\theta}{dt} = -\frac{dF}{\beta' dG'}.$$

In the Keplerian motion, F is reduced to F_0 while g and θ are constants.

However, in the case of the two-body problem and with a law different from the Newtonian law, the nodes are still fixed but the perihelions no longer are fixed. Therefore, if the motion takes place in a plane and if the nodes can be disregarded, the method of Chap. 9 is applicable without modification.

Extension of the Method of Chap. 9 to Certain Singular Cases

134. Let us first examine the case in which F_0 does not contain all variables $x_1, x_2, \ldots, x_n$.

To fix our ideas, let us assume that there are three degrees of freedom and that F_0 contains two of the variables of the first series, x_1 and x_2, but does not contain the third, x_3.

This yields

$$n_3^0 = 0 .$$

We still assume

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots .$$

Here, F_1 is a function of $x_1, x_2, x_3, y_1, y_2, y_3$, periodic with respect to y_1, y_2, and y_3.

For the moment, let us consider F_1 as a function of y_1 and y_2 only. This is a periodic function of these two variables. We will denote by R the mean value of this periodic function, which also depends on x_1, x_2, x_3, and y_3.

Let us first consider the case in which R depends only on x_1, x_2, and x_3 but is independent of y_3.

Let us search for a function

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \cdots ,$$

having the same form as the function S considered in no. 125 and formally satisfying the equation

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \frac{dS}{dy_3}, y_1, y_2, y_3\right) = C , \tag{11.1}$$

with C being a constant which can be written in the form

$$C = C_0 + \mu C_1 + \mu^2 C_2 + \cdots ,$$

where C_0, C_1, C_2, ... are arbitrary constants.

Let us first put

$$\frac{dS_0}{dy_1} = x_1^0, \quad \frac{dS_0}{dy_2} = x_2^0 .$$

The constants x_1^0 and x_2^0 will be connected by the relation

$$F_0(x_1^0, x_2^0) = C_0 .$$

However, since the constant C_0 is arbitrary, x_1^0 and x_2^0 will also be arbitrary.

We will then put

$$\frac{dF_0}{dx_1^0} = -n_1^0, \quad \frac{dF_0}{dx_2^0} = -n_2^0 .$$

This will yield

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + [S_0] ,$$

where $[S_0]$ is an arbitrary function of y_3 which remains to be determined. On equating the coefficients of μ in Eq. (11.1), we obtain, as in no. 125,

$$n_1^0 \frac{dS_1}{dy_1} + n_2^0 \frac{dS_1}{dy_2} = F_1\left(x_1^0, x_2^0, \frac{d\,[S_0]}{dy_3}, y_1, y_2, y_3\right) - C_1 . \quad (11.2)$$

No matter what the arbitrary function $[S_0]$ might be, the right-hand side of Eq. (11.2) will be a periodic function of y_1 and y_2, so that the mean-square value of this function will be

$$R\left(x_1^0, x_2^0, \frac{d\,[S_0]}{dy_3}, y_3\right) - C_1 .$$

We postulate that the function S_1 has the following form:

$$\alpha_{11} y_1 + \alpha_{12} y_2 + \alpha_{13} y_3 + \text{periodic function of } y_1, y_2, \text{and } y_3 .$$

To make this possible, it is necessary and sufficient, as indicated in no. 125, that the mean value of the right-hand side of Eq. (11.2) reduces to a constant which will be denoted by $C_1' - C_1$. Then, for determining the arbitrary function $[S_0]$, we will have the equation

$$R\left(x_1^0, x_2^0, \frac{d\,[S_0]}{dy_3}, y_3\right) = C_1' . \quad (11.3)$$

We had assumed above that R did not depend on y_3; thus for satisfying Eq. (11.3), it is sufficient to take

$$[S_0] = x_3^0 ,$$

where x_3^0 is a constant which can still be considered as arbitrary since the constant C_1' is arbitrary itself. This will yield

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + x_3^0 y_3 .$$

On equating, in Eq. (11.2), the mean values of the two sides, we obtain

$$n_1^0 \alpha_{11} + n_2^0 \alpha_{12} = C_1' - C_1 .$$

Since C_1 is arbitrary, we will set $C_1 = C_1'$ which will allow us to set, as in no. 125,

$$\alpha_{11} = \alpha_{12} = 0 .$$

Then, Eq. (11.2) permits determining S_1 to within an arbitrary function of y_3.

After this, let us imagine that the functions

$$S_0, \quad S_1, \quad S_2, \quad \ldots, \quad S_{p-2},$$

have been completely determined and that S_{p-1} had been calculated to within an arbitrary function of y_3; we assume that the completion of the determination of S_{p-1} and the calculation of S_p to within an arbitrary function of y_3 are proposed.

On equating the coefficients of μ^p on the two sides of Eq. (11.1), we obtain

$$n_1^0 \frac{dS_p}{dy_1} + n_2^0 \frac{dS_p}{dy_2} = \frac{dF_1}{dx_3^0} \frac{dS_{p-1}}{dy_3} + \Phi_p - C_p , \qquad (11.4)$$

where Φ_p is a function which depends only on y and on the derivatives of $S_0, S_1, S_2, \ldots, S_{p-2}$ as well as on dS_{p-1}/dy_1 and on dS_{p-1}/dy_2. The functions $S_0, S_1, S_2, \ldots, S_{p-2}$ are known. We also know S_{p-1} to within an arbitrary function of y_3; consequently, we know dS_{p-1}/dy_1 and dS_{p-1}/dy_2. Thus Φ_p can be considered as being a known function of y, and this function will be periodic.

Since we have a periodic function U of y_1, y_2, and y_3, we will designate by $[U]$ the mean value of U considered, for the moment, as being solely a function of y_1 and y_2. It results from this that $[U]$ is also a function of y_3.

It can be demonstrated, as done above, that the mean value of the right-hand side of Eq. (11.4) must reduce to a constant $C_p' - C_p$ from which it follows that

$$\left[\frac{dF_1}{dx_3^0} \frac{dS_{p-1}}{dy_3}\right] + [\Phi_p] = C_p' ,$$

$$\left[\frac{dF_1}{dx_3^0} \frac{d[S_{p-1}]}{dy_3}\right] + \left[\frac{dF_1}{dx_3^0} \frac{d(S_{p-1} - [S_{p-1}])}{dy_3}\right] + [\Phi_p] = C_p' .$$

Since $[S_{p-1}]$ does not depend on y_1, y_2, we obtain

$$\left[\frac{dF_1}{dx_3^0} \frac{d[S_{p-1}]}{dy_3}\right] = \frac{d[S_{p-1}]}{dy_3} \left[\frac{dF_1}{dx_3^0}\right] = \frac{dR}{dx_3^0} \frac{d[S_{p-1}]}{dy_3} ,$$

whence

$$\frac{dR}{dx_3^0} \frac{d[S_{p-1}]}{dy_3} = C_p' - [\Phi_p] - \left[\frac{dF_1}{dx_3^0} \frac{d(S_{p-1} - [S_{p-1}])}{dy_3}\right] . \qquad (11.5)$$

Knowing S_{p-1} to within an arbitrary function of y_3, we also know

$$S_{p-1} - [S_{p-1}] \, .$$

Thus the second member of Eq. (11.5) is fully known. On the other hand, R is a known function of x_1, x_2, and x_3 where these variables are replaced by the known constants x_1^0, x_2^0, and x_3^0. Thus dR/dx_3^0 is known, and $d[S_{p-1}]/dy_3$ can be derived from Eq. (11.5) while $[S_{p-1}]$ can be obtained by integration.

So that $[S_{p-1}]$ be a periodic function of y_3, it is necessary that the mean value of the right-hand side of Eq. (11.5) be zero; one can always make use of the arbitrary constant C'_p to have this be the case.

This completes the determination of S_{p-1}. Equation (11.4) will then permit determining S_p to within an arbitrary function of y_3. So that the value of S_p, derived from Eq. (11.4), be periodic in y_1 and y_2, it is necessary that the mean value of the right-hand side be zero. However, this mean value is $C'_p - C_p$ so that, since the constant C_p remains arbitrary, we can set

$$C_p = C'_p \, .$$

Thus it is always possible to determine the functions S_p by recurrence. The conclusions of no. 125 consequently remain valid. The only difference is that the expansion of n_3 in powers of μ, instead of starting with a completely known term, will start with a term in μ.

Let us now assume that there are four degrees of freedom and eight variables $x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4$, and that F_0 depends only on x_1 and x_2 while R depends on x_1, x_2, x_3, x_4.

The same conclusions are applicable, provided that

(a) no linear relation with integral coefficients exists between dF_0/dx_1^0 and dF_0/dx_2^0 (i.e., between n_1^0 and n_2^0);

(b) no linear relation with integral coefficients exists between dR/dx_3^0 and dR/dx_4^0.

In fact, the equation analogous to Eq. (11.5) and used for determining $[S_{p-1}]$ will then be written

$$\frac{dR}{dx_3^0}\frac{d[S_{p-1}]}{dy_3} + \frac{dR}{dx_4^0}\frac{d[S_{p-1}]}{dy_4}$$
$$= (\text{known periodic function of } y_3 \text{ and } y_4 \text{ whose mean value can be made zero}) \, . \tag{11.5a}$$

So that $[S_{p-1}]$ can be derived from this as a periodic function of y_3 and y_4, it is necessary and sufficient that no linear relation with integral coefficients exist between dR/dx_3^0 and dR/dx_4^0.

135. Up to now, we assumed that R depends only on the variables of the first series x_1, x_2, x_3, x_4 (assuming, as at the end of the preceding number, that four degrees of freedom exist and that F_0 depends only on x_1 and x_2).

Let us now suppose that R depends not only on x_1, x_2, x_3, x_4 but also on y_3 and y_4.

If we replace x_1 and x_2 by the constants ξ_1 and ξ_2 while replacing x_3 and x_4 by dT/dy_3 and dT/dy_4, and if we equate R to a constant C_1', we will obtain the following equation:

$$R\left(\xi_1, \xi_2, \frac{dT}{dy_3}, \frac{dT}{dy_4}, y_3, y_4\right) = C_1', \tag{11.6}$$

which defines a function T of the two variables y_3 and y_4.

Let us assume that we have found a function T satisfying this equation. In addition, let us assume that this function depends also on the two constants ξ_1 and ξ_2 as well as on two new integration constants denoted by ξ_3 and ξ_4.

Then, the function

$$U = \xi_1 y_1 + \xi_2 y_2 + T$$

will satisfy the equation

$$R\left(\frac{dU}{dy_1}, \frac{dU}{dy_2}, \frac{dU}{dy_3}, \frac{dU}{dy_4}, y_3, y_4\right) = C_1'.$$

In addition, the relations

$$x_i = \frac{dU}{dy_i}, \quad \eta_i = \frac{dU}{d\xi_i} \quad (i = 1, 2, 3, 4)$$

will define a change of variables, where the old variables are x_i and y_i while the new variables are ξ_i and η_i.

According to the statements in no. 4, this change of variables will not alter the canonical form of the equations.

It is immediately obvious that

$$x_1 = \xi_1, \quad x_2 = \xi_2,$$

and, consequently, that F_0 after the change of variables will depend only on ξ_1 and ξ_2.

If it is assumed (which we will do) that the function U is such that x_3, x_4, $y_1 - \eta_1$, $y_2 - \eta_2$, $y_3 - \eta_3$ (or y_3), $y_4 - \eta_4$ (or y_4) are functions of ξ_i and of η_i, periodic with respect to η_i, then the function F—after change of variables—will be periodic with respect to η_i.

We have called R the mean value of F_1, considered as a periodic function of y_1 and y_2. We now state that, after the change of variables, we consider F_1 as a periodic function of η_1 and η_2 with its mean value still being R.

By definition, we have

$$4\pi^2 R = \int_0^{2\pi}\int_0^{2\pi} F_1\, dy_1\, dy_2\,,$$

and we propose to demonstrate that

$$4\pi^2 R = \int_0^{2\pi}\int_0^{2\pi} F_1\, d\eta_1\, d\eta_2\,.$$

In fact, we have

$$\int\int F_1\, d\eta_1\, d\eta_2 = \int\int F_1\left(\frac{d\eta_1}{dy_1}\frac{d\eta_2}{dy_2} - \frac{d\eta_1}{dy_2}\frac{d\eta_2}{dy_1}\right) dy_1\, dy_2\,.$$

However, in the relations

$$x_1 = \xi_1, \quad x_2 = \xi_2, \quad x_i = \frac{dU}{dy_i}, \quad \eta_i = \frac{dU}{d\xi_i} \quad (i = 3, 4)\,,$$

y_1 and y_2, η_1 and η_2 do not enter. This shows that, if the new variables are expressed as a function of the old variables, then ξ_1, ξ_2, ξ_3, ξ_4, η_3, and η_4 will be independent of both y_1 and y_2.

Thus if, in T, we replace ξ_3 and ξ_4 by their values as a function of x_1, x_2, x_3, x_4, y_3, and y_4, we will have

$$\frac{dT}{dy_1} = \frac{dT}{dy_2} = 0, \quad \frac{d^2T}{d\xi_i dy_1} = \frac{d^2T}{d\xi_i dy_2} = 0\,,$$

whence

$$\frac{d\eta_1}{dy_1} = 1 + \frac{d^2T}{d\xi_1 dy_1} = 1, \quad \frac{d\eta_1}{dy_2} = \frac{d^2T}{d\xi_2 dy_1} = 0$$

and, similarly,

$$\frac{d\eta_2}{dy_1} = 0, \quad \frac{d\eta_2}{dy_2} = 1.$$

Consequently, we have

$$\int\int F_1\, d\eta_1\, d\eta_2 = \int\int F_1\, dy_1\, dy_2\,.$$

In addition, the quantity C_1' which must be a constant can depend only on the integration constants, i.e., on ξ_i such that R will depend only [because of Eq. (11.6)] on ξ_1, ξ_2, ξ_3, and ξ_4.

This reduces to the case treated in the preceding number so that we must conclude that the canonical equations

$$\frac{d\xi_i}{dt} = \frac{dF}{d\eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{dF}{d\xi_i}$$

can be formally satisfied by series of the form:

$$\xi_i = \xi_i^0 + \mu\xi_i^1 + \cdots + \mu^p\xi_i^p + \cdots,$$

$$\eta_i = w_i + \mu\eta_i^1 + \cdots + \mu^p\eta_i^p + \cdots,$$

$$w_i = n_i t + \overline{w}_i, \quad n_i = n_i^0 + \mu n_i^1 + \cdots + \mu^p n_i^p + \cdots,$$

where the terms ξ_i^0 are constants while ξ_i^k and η_i^k are periodic functions of w depending, in addition, on n integration constants ξ_i^0. In addition, $\overline{w}_i$ are n other constants of integration and the quantities η_i^p still depend on the constants ξ_i^0.

On returning to the original variables, it can be demonstrated that the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}$$

can be formally satisfied by series of the form

$$x_i = x_i^0 + \mu x_i^1 + \cdots + \mu^p x_i^p + \cdots,$$

$$y_i = \epsilon_i w_i + y_i^0 + \mu y_i^1 + \cdots + \mu^p y_i^p + \cdots;$$

where x_i^k and y_i^k are periodic functions of w.

As regards the coefficient ϵ_i, it can be equal to either zero or to one. In any case, it is always equal to one for $i = 1$ or 2; it is equal to one or zero for $i = 3$ depending on whether $y_3 - \eta_3$ or y_3 is periodic with respect to η_i. Similarly, the coefficient is equal to one or zero for $i = 4$ depending on whether $y_4 - \eta_4$ or y_4 is periodic with respect to η_i.

All this reduces to an integration of the partial differential equation (11.6) or, what comes to the same thing, to an integration of the canonical equations:

$$\frac{dx_i}{dt} = \frac{dR}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dR}{dx_i}.$$

Application to the Three-Body Problem

136. Let us apply the preceding statements to the three-body problem. We have brought the equations of this problem to the form

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{11.7}$$

with

$$F = F_0 + \mu F_1,$$

where μ is a very small parameter and μF_1 is the perturbing function. Our variables are those of no. 11:

$$\begin{array}{cccccc} \Lambda=\beta L, & \Lambda'=\beta' L', & \beta G, & \beta\Theta, & \beta' G', & \beta'\Theta', \\ l, & l', & g, & \theta, & g', & \theta', \end{array} \tag{11.8}$$

or else those of no. 12:

$$\begin{array}{cccccc} \Lambda, & \Lambda', & \xi, & \xi', & p, & p', \\ \lambda, & \lambda', & \eta, & \eta', & q, & q', \end{array} \tag{11.9}$$

where F_0 depends only on Λ and Λ' while F_1 depends on 12 variables but is periodic with respect to l and l'. Thus if one considers F_1 as a periodic function of l and l' and if R is designated as the mean value of this function, then R will be nothing else but the function which had been designated as such in the preceding chapter. This function depends on 10 variables, namely, on the 12 variables of Eq. (11.8) with the exception of l and l' or else on the 12 variables of Eq. (11.9) with the exception of λ and λ'. If the variables of Eq. (11.8) are adopted, the function will be periodic with respect to g, g', θ, and θ'.

Consequently, the method of nos. 134 and 135 will be applicable to Eqs. (11.7) and will permit its formal integration, provided that it is possible to integrate the equations

$$\frac{dx_i}{dt}=\frac{dR}{dy_i}, \quad \frac{dy_i}{dt}=-\frac{dR}{dy_i}, \tag{11.10}$$

where the variables x_i and y_i are the last four pairs of conjugate variables (11.8) or the last four pairs of conjugate variables (11.9) and where Λ and Λ' are considered constants.

However, Eqs. (11.10) are exactly those that we were able to formally integrate in the preceding chapter. Thus it is easy to show that the Lindstedt method can be applied to the general case of the three-body problem.

Application of this method, which I have just briefly discussed, will be the object of the following numbers.

Change of Variables

137. We will make a change of variables analogous to those performed in no. 131.

For this, let us put

$$(\Lambda=\beta L, \quad \Lambda'=\beta' L'),$$

$$V=\Lambda\lambda_1+\Lambda'\lambda_1'+\xi(\sigma_1\cos\varphi+\sigma_2\sin\varphi)+\xi(-\sigma_1\sin\varphi+\sigma_2\cos\varphi)$$
$$+p(\sigma_3\cos\varphi+\sigma_4\sin\varphi')+p'(-\sigma_3\sin\varphi'+\sigma_4\cos\varphi'),$$

where φ and φ' are the angles defined in no. 131.

Then, as in that number, we will set

$$\lambda = \frac{dV}{d\Lambda}, \quad \eta = \frac{dV}{d\xi}, \quad \eta' = \frac{V}{\xi'}, \quad q = \frac{dV}{dp}, \quad q' = \frac{V}{p'}, \quad \tau_i = \frac{dV}{d\sigma_i}$$

showing that the canonical form of the equations is not altered if the old variables

$$\begin{array}{cccccc} \Lambda, & \Lambda', & \xi, & \xi', & p, & p', \\ \lambda, & \lambda', & \eta, & \eta', & q, & q', \end{array}$$

are replaced by the new variables

$$\begin{array}{cccccc} \Lambda, & \Lambda', & \tau_1, & \tau_2, & \tau_3, & \tau_4, \\ \lambda_1, & \lambda_1', & \sigma_1, & \sigma_2, & \sigma_3, & \sigma_4. \end{array}$$

The variables τ_i and σ_i have already been determined as a function of ξ, η, p, q, and Λ in the preceding chapter.

It remains to determine the form of the relation linking the new variables λ_1 and λ_1' with λ and λ'.

We obtain

$$\lambda = \lambda_1 + \psi, \quad \lambda' = \lambda_1' + \psi',$$

where ψ and ψ' are quadratic forms of σ and of τ whose coefficients depend on Λ and Λ' and which are written as

$$\psi = \frac{d\varphi}{d\Lambda}(\sigma_2\tau_1 - \sigma_1\tau_2) + \frac{d\varphi'}{d\Lambda}(\sigma_4\tau_3 - \sigma_3\tau_4),$$

$$\psi' = \frac{d\varphi}{d\Lambda'}(\sigma_2\tau_1 - \sigma_1\tau_2) + \frac{d\varphi'}{d\Lambda'}(\sigma_4\tau_3 - \sigma_3\tau_4),$$

where φ and φ' are the angles denoted the same way in no. 131.

Reasoning as in no. 135, this shows that any function periodic in λ and λ', after change in variables, will still be periodic in λ_1 and λ_1' and that the mean values will be the same in both cases.

From this, several conclusions can be drawn as to the form of the function F.

The function F depends in an arbitrary manner on Λ and Λ'; however, it is periodic in λ_1 and λ_1'. In addition, the function can be expanded in powers of σ and τ.

It should be added here that this function must not change when one changes λ_1 and λ_1' into $\lambda_1 + \pi$ and $\lambda_1' + \pi$ and when σ and τ change sign at the same time. To ascertain this, it is sufficient to refer to our statements in no. 12 and to note that, when

$$\lambda_1, \quad \lambda_1', \quad \sigma_i, \quad \tau_i$$

change into

$$\lambda_1 + \pi, \quad \lambda_1' + \pi, \quad -\sigma_i, \quad -\tau_i,$$

the quantities λ and λ' will also change into $\lambda+\pi$ and $\lambda'+\pi$ and that the variables ξ, etc. will change sign.

Finally, we will make a last change of variables by putting, as in no. 131,

$$\tau_i=\sqrt{2\rho_i}\cos\omega_i,\quad \sigma_i=\sqrt{2\rho_i}\sin\omega_i\ ,$$

which, in virtue of the remark made in no. 6, will not alter the canonical form.

Then, F can be expanded in powers of $\sqrt{\rho_i}$, yielding

$$R_2=2A_1\rho_1+2A_2\rho_2+2A_3\rho_3+2A_4\rho_4\,.$$

Case of Plane Orbits

138. After this change of variables, the equations of motion assume the following form.

The two series of conjugate variables are

$$\begin{array}{ccc}\Lambda, & \Lambda', & \rho_i, \\ \lambda_1, & L_1', & \omega_i,\end{array}$$

and we have

$$F=F_0+\mu F_1+\mu^2F_2+\cdots,$$

where F_0 depends only on Λ and Λ', while $F_1, F_2, \ldots$, which are periodic with respect to λ_1 and λ_1', can be expanded in powers of[R5]

$$\cos\omega_i\sqrt{\rho_i},\quad \sin\omega_i\sqrt{\rho_i}\,.$$

In addition, these functions will not change on increasing λ_1, λ_1', and ω_i by the same amount. Thus, these functions depend only on the differences

$$\lambda_1-\omega_i,\quad \lambda_1'-\omega_i,\quad \omega_k-\omega_i\,.$$

If, in R, the term ρ_i is replaced by $dT/d\omega_i$ and if R is equated to a constant, while considering Λ and Λ' as given constants, we obtain a partial differential equation of the form

$$R\left(\frac{dT}{d\omega_i},\omega_i\right)=C. \tag{11.11}$$

According to what we have seen in nos. 134 and 135, it is sufficient to know how to integrate this equation in order to be able to form series expandable in ascending powers of μ and formally satisfying the equations of motion

$$\frac{d\Lambda}{dt}=\frac{dF}{d\lambda_1}, \quad \frac{d\Lambda'}{dt}=\frac{dF}{d\lambda_1'}, \quad \frac{d\rho_i}{dt}=\frac{dF}{d\omega_i},$$
$$\frac{d\lambda_1}{dt}=-\frac{dF}{d\Lambda}, \quad \frac{d\lambda_1'}{dt}=-\frac{dF}{d\Lambda'}, \quad \frac{d\omega_i}{dt}=-\frac{dF}{d\rho_i}. \tag{11.12}$$

There exists a specific case in which the integration of Eq. (11.11) is relatively easy, namely, the case in which we investigate the motion of only three bodies moving in the same plane.

In this case, the number of quantities ρ_i reduces to two so that, considering Λ and Λ' as constants, R will depend only on the four variables ρ_1, ρ_2, ω_1, and ω_2. Further: We have demonstrated above that F depended only on the differences $\lambda_1-\omega_i$, $\lambda_1'-\omega_i$, $\omega_k-\omega_i$. Thus R will depend only on the three variables ρ_1, ρ_2, and $\omega_1-\omega_2$ so that Eq. (11.11) can be written as

$$R\left(\frac{dT}{d\omega_1}, \frac{dT}{d\omega_2}, \omega_1-\omega_2\right)=C.$$

If we put $\omega_1-\omega_2=\varphi$ and if we use ω_1 and φ as new variables, then the equation is transformed into

$$R\left(\frac{\partial T}{\partial\omega_1}+\frac{\partial T}{\partial\varphi}, \frac{\partial T}{\partial\varphi}, \varphi\right)=C.$$

If we give an arbitrary constant value to $\partial T/\partial\omega_1$ which will be denoted by h, then the equation will contain only $\partial T/\partial\varphi$ and φ. From this, $\partial T/\partial\varphi$ can be derived as a function of φ, of the constant C, and of h, yielding

$$\frac{\partial T}{\partial\varphi}=f(\varphi, C, h),$$

whence

$$T=h\omega_1+\int f\,d\varphi.$$

Let us see what form this function T will have.

It should be noted that R can be expanded in powers of $dT/d\omega_1$ and of $dT/d\omega_2$, that the zero-degree term reduces to a constant denoted by H, and that the first-degree terms reduce to

$$2A_1\frac{dT}{d\omega_1}+2A_2\frac{dT}{d\omega_2}.$$

We will next put (introducing two new integration constants Ω_1 and Ω_2 instead of C and h)

$$C=H+2A_1\Omega_1+2A_2\Omega_2.$$
$$h=\Omega_1+\Omega_2.$$

For determining $dT/d\omega_1$ and $dT/d\omega_2$, we will then have two simultaneous equations

$$R - H = A_1\Omega_1 + A_2\Omega_2 ,$$
$$\frac{dT}{d\omega_1} + \frac{dT}{d\omega_2} = \Omega_1 + \Omega_1 .$$

The functional determinant of the two first terms with respect to $dT/d\omega_1$ and $dT/d\omega_2$ reduces, for $dT/d\omega_1 = dT/d\omega_2 = 0$, to $A_1 - A_2$. Thus this determinant is not zero.

Consequently, in accordance with the theorem in no. 30, $dT/d\omega_1$ and $dT/d\omega_2$ can be derived from these equations in the form of series arranged in ascending powers of Ω. The terms of zero degree will be zero while the terms of first degree will be reduced, respectively, to Ω_1 and Ω_2; the terms of higher degree will have periodic functions of $\omega_1 - \omega_2$ as coefficients.

The function U of no. 135 can then be written as

$$U = \Lambda\lambda_1 + \Lambda'\lambda'_1 + T ,$$

yielding

$$T = V_1\omega_1 + V_2\omega_2 + T' ,$$

where V_1 and V_2 are two constants depending on Ω_1 and Ω_2, while T' is a periodic function of ω_1 and ω_2.

Let us now make the change in variables defined in no. 4 by taking as new variables of the first series

$$\Lambda, \ \Lambda', \ V_1, \text{ and } V_2$$

connected with the old variables Λ, Λ', λ_1, λ'_1, ρ_i, and ω_i by the relations

$$\Lambda = \frac{dU}{d\lambda_1}, \quad \Lambda' = \frac{dU}{d\lambda'_1}, \quad \rho_i = \frac{dU}{d\omega_i} = \frac{dT}{d\omega_i} . \tag{11.13}$$

The conjugate variables, which will be denoted by

$$\lambda_2, \ \lambda'_2, \ v_1, \text{ and } v_2,$$

are then defined by the equations

$$\lambda_2 = \frac{dU}{d\Lambda} = \lambda_1 + \frac{dT}{d\Lambda}, \quad \lambda'_2 = \frac{dU}{d\Lambda'} = \lambda'_1 + \frac{dT}{d\Lambda'},$$
$$v_i = \frac{dU}{dV_i} = \frac{dT}{dV_i} . \tag{11.14}$$

We will suppose that in T, which depended on the constants Λ, Λ', Ω_1, and Ω_2, these constants have been replaced as functions of Λ, Λ', V_1, and V_2. It is this sense that $dT/d\Lambda$, $dT/d\Lambda'$, dT/dV_i must be understood.

So as to make the conclusions of no. 135 applicable, it is necessary that the old variables

$$\Lambda, \quad \Lambda', \quad \lambda_1, \quad \lambda_1', \quad \rho_i ,$$

as well as the variables

$$\omega_i - v_i ,$$

be uniform functions of the new variables

$$\Lambda, \quad \Lambda', \quad \lambda_2, \quad \lambda_2', \quad V_i \quad v_i$$

and that these functions be periodic with respect to v_1 and v_2.

Let us first search for the expressions of ω_1 and ω_2 as functions of the new variables; for this, we have the following two equations:

$$v_i = \frac{dT}{dV_1} = \omega_1 + \frac{dT'}{dV_1}, \quad v_2 = \frac{dT}{dV_2} = \omega_2 + \frac{dT'}{dV_2}. \tag{11.15}$$

First, we have to ask whether the values of ω_1 and of ω_2, derived from these equations, will be uniform functions of the new variables. Were they to cease to be such, it would be necessary that the functional determinant of the right-hand sides with respect to ω_1 and ω_2 would vanish, i.e., that one would have

$$\frac{\partial\left(\frac{dT}{dV_1}, \frac{dT}{dV_2}\right)}{\partial(\omega_1,\omega_2)} = 1 + \frac{d^2T'}{dV_1 d\omega_1} + \frac{d^2T'}{dV_2 d\omega_2} + \frac{d^2T'}{dV_1 d\omega_1}\frac{d^2T'}{dV_2 d\omega_2} - \frac{d^2T'}{dV_2 d\omega_1}\frac{d^2T'}{dV_1 d\omega_2} = 0 .$$

For abbreviation, we will write this equation in the form:

$$1 + J = 0 .$$

We note first that ρ_1 and ρ_2, in the applications, will be very small quantities, of the order of the squares of the eccentricities.

These quantities ρ_i are connected with Ω_i by the following relations:

$$\rho_i = \frac{dT}{d\omega_i} .$$

However, $dT/d\omega_i$ can be expanded in ascending powers of Ω_i. This expansion contains no completely known term, and the terms of first degree reduce to Ω_i.

From this, based on the theorem of no. 30, we can derive

$$\Omega_1 = f_1(\rho_1,\rho_2), \quad \Omega_2 = f_2(\rho_1,\rho_2) ,$$

where f_1 and f_2 are series that can be expanded in powers of ρ_1 and ρ_2 and whose coefficients depend otherwise in any manner on

$$\Lambda, \ \Lambda', \ \omega_1, \text{ and } \omega_2 .$$

The terms of zero degree will be zero and those of first degree will reduce, respectively, to ρ_1 and to ρ_2.

From this it follows that Ω_1 and Ω_2, just like ρ_1 and ρ_2, will be of the order of the squares of the eccentricities.

According to the definition of V_1 and V_2, these quantities can be expanded in powers of Ω_1 and Ω_2, since the coefficients of the expansion depend in an arbitrary manner on Λ and Λ'. These series will contain no zero-degree terms and the first-degree terms will reduce, respectively, to Ω_1 and Ω_2.

From this it follows that

(a) The quantities V_1 and V_2 are of the order of the squares of the eccentricities.

(b) The terms Ω_1 and Ω_2 can be inversely expanded in powers of V_1 and V_2, yielding

$$\Omega_1 = V_1 + \varphi_1(V_1,V_2), \quad \Omega_2 = V_2 + \varphi_2(V_1,V_2)\,,$$

where φ_1 and φ_2 contain only terms of the second degree, at least with respect to V_1 and V_2.

(c) The quantity T' can be expanded in ascending powers of V_1 and V_2 and then will contain only terms of the second degree, at least with respect to these two quantities.

(d) Since the expansion of dT'/dV_1 and of dT'/dV_2 in ascending powers of V_1 and V_2 starts with first-degree terms, these two derivatives are of the same order of magnitude as the squares of the eccentricities.

(e) This is exactly the same for the second derivatives $d^2T'/dV_k d\omega_i$ and, consequently, for J. Since J is very small, $1 + J$ cannot be zero.

Consequently, we must conclude that ω_1 and ω_2 and thus also $\omega_1 - v_1$, $\omega_2 - v_2$ are uniform functions of the new variables.

It should be added that $v_1 - \omega_1$, $v_2 - \omega_2$ are periodic functions of v_1 and of v_2. If, in effect, we increase v_1 and ω_1 by $2K_1\pi$, and v_2 and ω_2 by $2K_2\pi$ (with K_1 and K_2 being integers), then Eqs. (11.15) will still be satisfied since T' is periodic in ω_1 and ω_2 and since $v_1 - \omega_1$, $v_2 - \omega_2$ do not change.

Substituting these values of ω_1 and ω_2 into Eqs. (11.13) and (11.14), it will be found that the old variables

$$\Lambda, \quad \Lambda', \quad \lambda_1, \quad \lambda_1', \quad \rho_i$$

are uniform functions of the new variables, periodic with respect to v_i.

Thus we now have conditions under which the results of no. 135 become applicable.

Let us express the function F by means of the new variables. Let us note first that F_0 remains expressed as a function of only Λ and Λ'. In addition, F is periodic with respect to the variables of the second series λ_2, λ_2', v_1, and v_2.

The mean value of F_1, considered as a periodic function of λ_2 and λ'_2, reduces to R. On the other hand, by virtue of Eq. (11.11), R reduces to the constant C or else to

$$H + 2A_1\Omega_1 + 2A_2\Omega_2\,,$$

or to

$$H + 2A_1V_1 + 2A_2V_2 + 2A_1\varphi_1(V_1,V_2) + 2A_2\varphi_2(V_1,V_2)\,.$$

Thus R depends only on Λ, Λ', V_1, and V_2 and does not depend on the variables of the second series.

This returns us to the case studied in no. 134.

We now state that F does not change when λ_2, λ'_2, v_1, and v_2 are increased by the same amount. In fact, we know already that F does not change when λ_1, λ'_1, ω_1, and ω_2 are increased by the same amount and that T' depends only on the difference $\omega_1 - \omega_2$.

Then Eqs. (11.14) and (11.15) demonstrate that, as soon as $\lambda_1, \lambda'_1, \omega_1$, and ω_2 are increased by the same amount ϵ, the quantities $\lambda_2, \lambda'_2, v_1$, and v_2 will also increase by the same amount ϵ. Thus when these four new variables increase by ϵ, the quantity F does not change.

The manner in which F depends on V_1 and on V_2 is rather complex since F, before the change of variables, contained the radicals $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$.

Let

$$F(\Lambda,\Lambda',\lambda_2,\lambda'_2,V_1,V_2,v_1,v_2)$$

what becomes of the function F after change of variables. Next, the equation

$$F\left(\frac{dS}{d\lambda_2},\frac{dS}{d\lambda'_2},\lambda_2,\lambda'_2,\frac{dS}{dv_1},\frac{dS}{dv_2},v_1,v_2\right)$$
$$= C_0 + C_1\mu + C_2\mu^2 + \cdots \qquad (11.16)$$

must be integrated.

We will formally satisfy this equation by setting

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \cdots$$

and

$$S_0 = \Lambda_0\lambda_2 + \Lambda'_0\lambda'_2 + V_1^0 v_1 + V_2^0 v_2\,,$$

where Λ_0, Λ'_0, V_1^0, and V_2^0 must be our four constants of integration. For this, as demonstrated above, it is only necessary to apply the method given in no. 134.

Study of a Particular Integral

139. A remarkable particular integral is obtained by assuming that the two last constants V_1^0 and V_2^0 are zero.

For this, it is sufficient to set, in Eq. (11.16)

$$\frac{dS}{dv_1} = \frac{dS}{dv_2} = 0 .$$

It then happens that the left-hand side of this equation depends neither on v_1 nor on v_2.

In fact, before the last change of variables made by us, F had been expandable in powers of

$$\sqrt{\rho_i} \cos \omega_i \quad \text{and} \quad \sqrt{\rho_i} \sin \omega_i ,$$

and, otherwise, depended only on the other variables

$$\Lambda, \quad \Lambda', \quad \lambda, \quad \text{and} \quad \lambda_1' .$$

Thus if we set

$$\rho_1 = \rho_2 = 0 ,$$

then F will depend only on Λ, Λ', λ_1, and λ_1'.

If, on the other hand, we set

$$V_1 = V_2 = 0 ,$$

then T' which can be expanded in ascending powers of V and which contains only terms of the second degree at least with respect to these quantities will vanish, which is true also for its first-order derivatives. Similarly, Ω_1 and Ω_2 will vanish, resulting in

$$\rho_i = \frac{dT}{d\omega_i} = \Omega_i + \frac{dT'}{d\omega_i} = 0 ,$$

$$\lambda_2 = \lambda_1 + \frac{dT}{d\Lambda} = \lambda_1 + \frac{dT'}{d\Lambda} = \lambda_1 .$$

In the same way

$$\lambda_2' = \lambda_1' .$$

From this it results that ρ_1 and ρ_2 vanish and that, on the other hand, λ_2 and λ_2' reduce to λ_1 and λ_1'. We state that this means that F depends only on the four variables

$$\Lambda, \quad \Lambda, \quad \lambda_2, \quad \text{and} \quad \lambda_2' .$$

Thus we can set

$$\frac{dS}{dv_1} = \frac{dS}{dv_2} = 0 .$$

The left-hand side of Eq. (11.16) contains only λ_2, λ'_2, $dS/d\lambda_2$, and $dS/d\lambda'_2$.

This equation is then easy to integrate. To do this, it is only necessary to apply the procedures given in no. 125. However, there is more to be said here.

The integral is no longer purely formal, and the series expanded in powers of μ, at which one arrives, is convergent.

In fact, F depends only on the difference $\lambda_2 - \lambda'_2$ since we have shown that F cannot change when $\lambda_2, \lambda'_2, v_1$, and v_2 increase by the same amount and that here F has ceased to depend on v_1 and v_2.

From this, it follows that the two equations

$$F\left(\frac{dS}{d\lambda_2}, \frac{dS}{d\lambda'_2}, \lambda_2, \lambda'_2\right) = C, \quad \frac{dS}{d\lambda_2} + \frac{dS}{d\lambda'_2} = C',$$

(where C and C' are any two constants) are compatible. From this we derive $dS/d\lambda_2$ and $dS/d\lambda'_2$ and thus also S in the form of series developed in powers of μ.

The resulting integral depends on two arbitrary constants C and C'. However, these two constants can be expressed by means of two of the four originally chosen constants, namely, by Λ_0 and Λ'_0, since the two other constants V_1^0 and V_2^0 are, by hypothesis, zero.

We will denote this particular integral of Eq. (11.16) by

$$\Sigma\ (\lambda_2, \lambda'_2, \Lambda_0, \Lambda'_0)\ . \tag{11.17}$$

If the constants C and C' are properly chosen (cf. no. 125), then Σ will have the following form:

$$\Lambda_0 \lambda_2 + \Lambda'_0 \lambda'_2 + \text{periodic function of } \lambda_2 - \lambda'_2\ .$$

A discussion of this particular integral Σ does not lead, in contrast to what one would be tempted to believe, to simple particular solutions of the three-body problem.

Form of the Expansions

140. Since the existence of the function S has been thus demonstrated, the following result can be derived, reasoning as in no. 125.

The following series exist:

$$
\begin{aligned}
\Lambda &= \Lambda_0 + \mu\Lambda_1 + \mu^2\Lambda_2 + \cdots, \\
\Lambda' &= \Lambda_0' + \mu\Lambda_1' + \mu^2\Lambda_2' + \cdots, \\
V_i &= V_i^0 + \mu V_i^1 + \mu^2 V_i^2 + \cdots, \\
\lambda_2 &= w_1 + \mu y_1^1 + \mu^2 y_1^2 + \cdots, \\
\lambda_2' &= w_2 + \mu y_2^1 + \mu^2 y_2^2 + \cdots, \\
v_1 &= w_3 + \mu y_3^1 + \mu^2 y_3^2 + \cdots, \\
v_2 &= w_4 + \mu y_4^1 + \mu^2 y_4^2 + \cdots,
\end{aligned}
\tag{11.18}
$$

arranged in powers of μ and formally satisfying the equations of the three-body problem.

Here, Λ_0, Λ_0', and V_i^0 are constants. We have

$$w_i = n_i t + \overline{w}_i .$$

The quantities Λ_k, Λ_k', Λ_i^k, and y_i^k are periodic functions of w which, in addition, depend on the constants Λ_0, Λ_0', Λ_i^0.

On the other hand, the quantities n_i (which also depend on the constants Λ_0, Λ_0', V_i^0) can be expanded in ascending powers of μ, such that we obtain

$$n_i = n_i^0 + \mu n_i^1 + \mu^2 n_i^2 + \cdots .$$

The point to which we would like to draw specific attention is the fact that we have

$$n_3^0 = n_4^0 = 0,$$

$$n_1^0 \gtrless 0, \quad n_2^0 \gtrless 0 .$$

The coefficients of the above series could have been calculated in a more rapid manner and without passing through the entire sequence of change of variables if I had not first set out to establish simply and rigorously the very feasibility of the expansion.

However, there is more to this: The original variables Λ, Λ', $\lambda - \omega_1$, $\lambda' - \omega_2$, ξ, ξ', η, η' can be expanded in series of the same form, i.e., in series whose terms are periodic functions of ω_1, ω_2, ω_3, and ω_4. To convince oneself of this, it is sufficient to replace, in the expressions of the original variables as a function of the new variables, these new variables by their expressions (11.18). It would then be of advantage to calculate directly the coefficients of the expansions of the original variables without going over the new variables which have served for proving the feasibility of this expansion.

We will not discuss further the procedures that permit a direct calculation of these coefficients. The statement made in no. 127 is sufficient for understanding the basic principle; in addition, we will return to this point in Chap. 14.

Let us here only give a means for avoiding the last change of variables, namely, that by which one passes from ρ_i and from ω_i to V_i and to v_i. In cases in which this change cannot be avoided, this will represent the most intractable portion of the calculation.

For this it is sufficient to make a suitable grouping of the terms, which is possible if the eccentricities are small.

In F_1, two types of terms can be distinguished:

(a) Those of degree 0, 1, 2, or 3 with respect to the eccentricities and inclinations;

(b) those of degree 4 at least with respect to the eccentricities and inclinations.

The terms of the second kind are much smaller than those of the first. Let F_1' be the ensemble of terms of the first kind, and let $\epsilon F_1''$ be the ensemble of the terms of the second kind. It can be assumed that ϵ is a very small constant and that F_1'' is finite, so that we can write

$$F = F_0 + \mu F_1' + \mu\epsilon F_1'' + \mu^2 F_2 + \cdots .$$

Nothing will now prevent our combining the terms $\mu\epsilon F_1''$ with the terms $\mu^2 F_2$ since $\mu\epsilon$ is much smaller than μ; or else to attempt an expansion in powers of μ and of ϵ.

Then we retain the variables

$$\Lambda, \quad \Lambda', \quad \lambda_1, \quad \lambda_1', \quad \rho_i, \quad \omega_i .$$

The mean value of F_1' will reduce to

$$H + 2A_1\rho_1 + 2A_2\rho_2 + 2A_3\rho_3 + 2A_4\rho_4$$

(cf. no. 131) and, consequently, is independent of the variables of the second series. However, the last change of variables had no other reason but to render R independent of the variables of the second series. Consequently, this has now become unnecessary.

General Case of the Three-Body Problem

141. Let us now pass to the case of the three-body problem in space. The number of variables ω_i and ρ_i will then be equal to four and Eq. (11.6) will be written as

$$R\left(\frac{dT}{d\omega_1}, \frac{dT}{d\omega_2}, \frac{dT}{d\omega_3}, \frac{dT}{d\omega_4}, \omega_1, \omega_2, \omega_3, \omega_4\right) = C . \qquad (11.19)$$

This equation can no longer be integrated by the procedure used in no. 138. Actually, one does not even know the means of integrating it exactly, but a simple formal integration method can be developed which will be sufficient from the viewpoint used here.

The quantities

$$\frac{dT}{d\omega_i} = \rho_i$$

are of the order of the squares of the eccentricities. Thus if we put

$$T = \mu' T'' ,$$

where μ' is a constant of the order of the square of the eccentricities, then the derivatives $dT''/d\omega_i$ will be finite.

The function R can be expanded in ascending powers of $dT/d\omega_i$, yielding

$$R = R_0 + R_2 + R_4 + \cdots ,$$

where R_k represents the ensemble of terms of the degree $k/2$ with respect to ρ_i (with $R_0, R_2, R_4, \ldots$ not differing from the quantities denoted by the same symbols in no. 131). If $R_k(T)$ is used for denoting what becomes of R_k after replacing ρ_i by $dT/d\omega_i$, then Eq. (11.19) can be written as

$$R_0(T) + R_2(T) + R_4(T) + \cdots = C$$

or else

$$R_0(T'') + \mu' R_2(T'') + \mu'^2 R_4(T'') + \cdots = C .$$

Here, R_0 depends only on Λ and on Λ' and, since for the moment we consider these quantities to be constants, R_0 will also be a constant.

If we then put

$$C = R_0 + \mu' C' ,$$

Eq. (11.19) will become

$$R_2(T'') + \mu' R_4(T'') + \mu'^2 R_6(T'') + \cdots = C' . \qquad (11.20)$$

This leads us to integrating a partial differential equation whose first term depends on the derivatives $dT''/d\omega_i$ and is otherwise periodic with respect to the independent variables ω_i. The left-hand side depends on a parameter μ' and, as soon as this parameter vanishes, reduces to

$$R_2(T'') = \sum_{i=1}^{i=4} 2A_i \frac{dT''}{d\omega_i} .$$

For $\mu' = 0$, the first term thus no longer depends on ω_i but only on the derivatives $dT''/d\omega_i$.

This leads to conditions under which the analysis of no. 125 is applicable, so that we can conclude that there exists a series

$$T_0'' + \mu' T_1'' + \mu'^2 T_2'' + \cdots ,$$

expanded in powers of μ', and which, when substituted for T', will formally satisfy Eq. (11.20); this series is such that the derivatives

$$\frac{dT_k''}{d\omega_i}$$

are periodic with respect to ω_i.

We will then put

$$T_0'' = \Omega_1'\omega_1 + \Omega_2'\omega_2 + \Omega_3'\omega_3 + \Omega_4'\omega_4 ,$$

where $\Omega_1', \Omega_2', \Omega_3', \Omega_4'$ are our four integration constants. The constant C' must satisfy the equation

$$C' = \sum_{i=1}^{i=4} 2 A_i \Omega_i' .$$

It is easy to prove that T_k'' is an integral polynomial of the degree $k+1$ with respect to the four constants Ω_i'.

From this follows that

$$T = \mu' T''$$

is presented in the form of a series expanded in ascending integral powers of the four quantities

$$\mu'\Omega_1', \quad \mu'\Omega_2', \quad \mu'\Omega_3', \quad \mu'\Omega_4' ,$$

which, for abbreviation, will be denoted by

$$\Omega_1, \quad \Omega_2, \quad \Omega_3, \quad \Omega_4 .$$

This series, expanded in powers of the four constants Ω_i which are of the order of the squares of the eccentricities, formally satisfies Eq. (11.19).

As in no. 138, let us put

$$U = \Lambda \lambda_1 + \Lambda' \lambda_1' + T ,$$
$$T = V_1\omega_1 + V_2\omega_2 + V_3\omega_3 + V_4\omega_4 + T' ,$$

where T' is periodic with respect to ω, and V_i represent constants that can be expanded in ascending powers of Ω_i.

The quantities Ω_i are inversely expandable in powers of V_i. It is also possible to expand T in powers of V_i, and the resultant series will still formally satisfy Eq. (11.19).

We will now make a change of variables, similar to that performed in no. 138 [Eqs. (11.13), (11.14), and (11.15)].

For this, we put

$$\rho_i = \frac{dT}{d\omega_i}, \quad v_i = \frac{dT}{dV_i}, \quad \lambda_2 = \lambda_1 + \frac{dT}{d\Lambda}, \quad \lambda_2' = \lambda_1' + \frac{dT}{dV'}.$$

On replacing the old variables

$$\begin{matrix} \Lambda, & \Lambda', & \rho_i, \\ \lambda_1, & \lambda_1', & \omega_i \end{matrix}$$

by the new variables

$$\begin{matrix} \Lambda, & \Lambda', & V_i, \\ \lambda_2, & \lambda_2', & v_i \end{matrix}$$

nothing will change in the canonical form of the equations.

As in no. 138, we can demonstrate the following:

(i) The quantities V_i are of the order of the square of the eccentricities.

(ii) The quantities $\rho_i, \lambda_1 - \lambda_2, \lambda_1' - \lambda_2', \omega_i - v_i$ are periodic functions of v_i.

(iii) The function F can be expanded with respect to ascending powers of μ, of V_i, and of $\sqrt{\rho_i}$ since the terms ρ_i are themselves expandable in ascending powers of V_i.

(iv) The function F is a periodic function of v_i, λ_2, and λ_2'.

(v) The mean value of F, considered as a periodic function of the two variables λ_2 and λ_2', is equal to R and depends only on Λ, Λ', and on V_i.

This brings us to conditions under which the analysis of no. 135 is applicable.

Thus let

$$F(\Lambda, \Lambda', V_i, \lambda_2, \lambda_2', v_i)$$

be the form of the function F after change of variables. One could find a series expandable in powers of μ that formally satisfies the equation

$$F\left(\frac{dS}{d\lambda_2}, \frac{dS}{d\lambda_2'}, \frac{dS}{dv_i}, \lambda_2, \lambda_2', v_i\right) = C_0 + \mu C_1 + \mu^2 C_2 + \cdots$$

and depends on six constants which will be denoted by Λ_0, Λ_0', V_1^0, V_2^0, V_3^0, V_4^0.

This will lead to exactly the same conclusions as those drawn in nos. 138 and 140.

CHAPTER 12
Application to Orbits

Discussion of the Difficulties Involved

142. Cases are in existence in which the application of the methods discussed in the preceding chapter may lead to certain difficulties: These are mainly the cases in which the eccentricities are very small. Below, we will discuss the manner in which these difficulties can be ascertained.

We believe that a simple example, much simpler than the three-body problem, would be most suitable for understanding these difficulties.

Consequently, let

$$F = \Lambda + \mu\sqrt{\Omega}\cos(\omega + \lambda) + \mu A \Omega .$$

Here, μ is a very small parameter, A is a constant, and Λ, Ω, λ, and ω are two pairs of conjugate variables.

Let us next consider the canonical equations

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{dF}{d\Lambda} = 1, & \frac{d\omega}{dt} &= \frac{dF}{d\Omega}, \\ \frac{d\Lambda}{dt} &= -\frac{dF}{d\lambda}, & \frac{d\Omega}{dt} &= -\frac{dF}{d\omega}. \end{aligned} \tag{12.1}$$

These equations are easy to integrate completely, as will be shown below. However, we first will demonstrate their analogy with the equations of the three-body problem.

In no. 137 we demonstrated that, after several changes of variables, the equations of this problem can be brought to a canonical form, with the conjugate variables being

$$\begin{matrix} \Lambda, & \Lambda', & \rho_i, \\ \lambda_1, & \lambda_1' , & \omega_i . \end{matrix}$$

In addition, F can be expanded in powers of

$$\sqrt{\rho_i}\cos\omega_i, \quad \sqrt{\rho_i}\sin\omega_i , \quad \text{and} \quad \mu ,$$

which are periodic in λ_1 and λ_1'. Finally, F_0 depends only on Λ and Λ'. The function F, defined at the beginning of this number, is completely analogous to this. The variable Λ plays the role of Λ and Λ'; the variable Ω plays that of ρ_i; λ plays that of λ_1 and λ_1'; ω that of ω_i. It is clear that F can be expanded in powers of

$$\sqrt{\Omega}\cos\omega, \quad \sqrt{\Omega}\sin\omega,$$

and that, for $\mu=0$, it reduces to Λ.

The analogy is thus obvious. Let us assume that the method of the preceding chapters is to be applied to this equation, i.e., that an attempt is to be made to integrate the partial differential equation

$$\frac{dS}{d\lambda}+\mu\cos(\omega+\lambda)\sqrt{\frac{dS}{d\omega}}+\mu A\frac{dS}{d\omega}=C, \tag{12.2}$$

where C is a constant of integration. It is now a question to find a solution for Eq. (12.2) which can be expanded in powers of μ such that $dS/d\lambda$ and $dS/d\omega$ will be periodic in λ and in ω.

For this, let us put

$$\omega+\lambda=\varphi.$$

Equation (12.2), with the new variables λ and φ, becomes

$$\frac{dS}{d\lambda}+\mu\cos\varphi\sqrt{\frac{dS}{d\varphi}}+(1+\mu A)\frac{dS}{d\varphi}=C.$$

Let Λ be a constant of integration and let us put

$$1+\mu A=B, \quad C-\Lambda=VB.$$

Our equation will be satisfied by setting

$$\frac{dS}{d\lambda}=\Lambda, \quad \frac{dS}{d\varphi}=\frac{2\mu^2\cos^2\varphi+4B^2V\pm 2\mu\cos\varphi\sqrt{\mu^2\cos^2\varphi+4B^2V}}{4B^2}.$$

The function S, defined in this manner, readily satisfies all data of the problem, under condition that the radical

$$\sqrt{\mu^2\cos^2\varphi+4B^2V}$$

can be expanded in powers of μ. This expansion is entirely possible provided that

$$\mu^2<4B^2|V|.$$

The series will be highly convergent if μ^2 is very small not only in an absolute manner but also with respect to V.

If we wish to continue the comparison with the three-body problem, we will see that V represents a quantity analogous to that denoted by Ω_i in the preceding chapter. Let us consider this quantity as being of the order of the square of the eccentricities.

If both μ and V are very small, one could be tempted to expand S in powers of μ and V. Such an expansion is impossible since the radical

$$\sqrt{4B^2+\frac{\mu^2\cos^2\varphi}{V}}$$

can be expanded only in powers of μ (since B depends on μ) and in powers of

$$\frac{\mu^2}{V}.$$

Consequently, if V is sufficiently small to be comparable to μ^2, the method of the preceding chapter stops being applicable.

143. It is obvious that a similar difficulty exists for the three-body problem.

Let us resume the problem in the formulation given in the preceding chapter. Our conjugate variables are

$$\begin{array}{ccc} \Lambda, & \Lambda', & V_i, \\ \lambda_2, & \lambda'_2, & v_i. \end{array}$$

The function S, which formally satisfies our partial differential equation

$$F = \text{const.}$$

and which has been defined in the preceding chapter, depends on the constants Λ_0, Λ'_0, and V_i^0. In general, these latter constants V_i^0, in the applications of very small quantities, will be of the order of the square of the eccentricities. Then, we can put

$$V_i^0 = \epsilon^2 W_i^0,$$

where ϵ is a constant of the order of the eccentricities while W_i^0 are finite constants. Considering, for the moment, the quantities W_i^0 as given, S will depend on three arbitrary constants

$$\Lambda_0, \quad \Lambda'_0, \quad \text{and} \quad \epsilon.$$

It is now a question whether S can be expanded in powers of μ and ϵ.

If this were the case, the solution discussed in the preceding chapter would always be satisfactory no matter how small ϵ might be, i.e., no matter how small the eccentricities were.

However, this is not so, as will be demonstrated below and as the example in the preceding number would let us predict. The quantity S can be expanded only in powers of μ/ϵ and of ϵ. It follows that the method is no longer applicable if μ/ϵ is not very small. Consequently, the method is inapplicable, even if the masses are very small, as soon as the eccentricities are of the same order as the masses.

Let us return to our equation

$$F\left(\frac{dS}{d\lambda_2}, \frac{dS}{d\lambda'_2}, \frac{dS}{dv_i}, \lambda_2, \lambda'_2, v_i\right) = C,$$

which, for short, we will write as

$$F(S) = C\,. \tag{12.3}$$

It has been demonstrated in no. 139 that this equation admits of a particular solution which will be denoted by Σ. This yields

$$F(\Sigma) = C'\,,$$

where C' is a constant.

Putting now

$$S = \Sigma + \epsilon^2 s\,,$$

we obtain

$$F(\Sigma + \epsilon^2 s) = C\,. \tag{12.4}$$

The first term of Eq. (12.4) can be expanded in powers of ϵ; I say, in powers of ϵ, not in powers of ϵ^2: Indeed, F contains terms of odd degree with respect to $\sqrt{\rho_i}$. But, the terms ρ_i, which are related to

$$V_i = \frac{dS}{dv_i} = \epsilon^2 \frac{ds}{dv_i}\,,$$

by the equations

$$\rho_i = \frac{dT}{d\omega_i}\,, \quad V_i = \frac{dT}{dV_i}$$

given in nos. 138 and 141, can be expanded in powers of V_i, and thus also in powers of ϵ^2. Consequently, the terms $\sqrt{\rho_i}$ and thus also F can be expanded in powers of ϵ. We note, in addition, that s will be finite if ϵ is of the order of the eccentricities.

In fact, if ϵ vanishes, S will reduce to Σ. However, this particular solution Σ, as shown above, corresponds to the case in which the terms V_i^0 are zero. In the applications, the terms V_i^0 are not zero but represent very small quantities of the order of the square of the eccentricities. Consequently, the difference $S - \Sigma$ will be of the order of the square of the eccentricities, i.e., of the order of ϵ^2.

For abbreviation, let us set

$$F(\Sigma + \epsilon^2 s) - F(\Sigma) = \epsilon^2 F^*(s)\,,$$

which, by deducting Eq. (12.3) from Eq. (12.4), will yield

$$F^*(s) = K\,, \tag{12.5}$$

where K is a new constant equal to $(C - C')/\epsilon^2$.

The quantity F^* can be expanded in ascending powers of μ, such that

$$F^* = F_0^* + \mu F_1^* + \mu^2 F_2^* + \cdots\,.$$

In addition, F^* is periodic in λ_2 and λ'_2; we will call R^* the mean value of F_1^*.

Since Σ can be expanded in powers of μ, so that

$$\Sigma = \Sigma_0 + \mu\Sigma_1 + \mu^2\Sigma_2 + \cdots ,$$

then $F_0(\Sigma + \epsilon^2 s)$ can also be expanded such that

$$F_0(\Sigma + \epsilon^2 s) = \Phi_0(\epsilon^2 s) + \mu\Phi_1(\epsilon^2 s) + \cdots .$$

It is then obvious that

$$\Phi_0(\epsilon^2 s) = F_0(\Sigma_0 + \epsilon^2 s) ,$$

$$\Phi_1(\epsilon^2 s) = \frac{d\Phi_0}{\epsilon^2 d\dfrac{ds}{d\lambda_2}} \frac{d\Sigma_1}{d\lambda_2} + \frac{d\Phi_0}{\epsilon^2 d\dfrac{ds}{d\lambda'_2}} \frac{d\Sigma_1}{d\lambda'_2} .$$

We thus will have

$$\epsilon^2 F_0^* = F_0(\Sigma_0 + \epsilon^2 s) - F_0(\Sigma_0) ,$$

$$\epsilon^2 F_1^* = F_1(\Sigma_0 + \epsilon^2 s) - F_1(\Sigma_0) + \Phi_1(\epsilon^2 s) - \Phi_1(0) .$$

Here, F_0 depends only on $dS/d\lambda_2$ and on $dS/d\lambda'_2$. On substituting $\Sigma + \epsilon^2 s$ for S, the function F_0 can be expanded in powers of

$$\epsilon^2 \frac{ds}{d\lambda^2}, \quad \epsilon^2 \frac{ds}{d\lambda'_2},$$

from which it follows that

$$\Phi_0(\epsilon^2 s) - \Phi_0(0), \quad \Phi_1(\epsilon^2 s) - \Phi_1(0)$$

are divisible by ϵ^2 and do not depend on ds/dv_i. From this, it follows first that F_0^* can be expanded in positive and ascending powers of ϵ^2.

On the other hand, since the expansion of F_1 contains terms of the first degree in $\sqrt{\rho_i}$, the quantity $F_1(\Sigma_0 + \epsilon^2 s)$ is expandable not in powers of ϵ^2 but in powers of ϵ. The expansion of the difference

$$F_1(\Sigma_0 + \epsilon^2 s) - F_1(\Sigma_0)$$

will start with a term in ϵ.

This leads to the following consequence: F_1^* can be expanded in ascending powers of ϵ but the expansion will start with a term in $1/\epsilon$.

Let us note now that F_1^* is a function periodic in λ_2 and λ'_2 and let us attempt to determine its mean value R^*.

The mean value of $F_1(S)$, by definition, is $R(S)$: On replacing there S by $\Sigma_0 + \epsilon^2 s$, this mean value will not change and will be written as $R(\Sigma_0 + \epsilon^2 s)$. This is due to the fact that

$$\frac{d\Sigma_0}{d\lambda_2}, \quad \frac{d\Sigma_0}{d\lambda'_2}, \quad \frac{d\Sigma_0}{dv_i}$$

reduce, respectively, to

$$\Lambda_0, \quad \Lambda_0', \quad 0,$$

and do not depend on λ_2 and on λ_2'. If, conversely, these derivatives would depend periodically on λ_2 and λ_2', then the mean value could be modified by the substitution.

On the other hand, the mean value

$$[\Phi_1(\epsilon^2 s) - \Phi_1(0)] = \epsilon^2 H$$

depends neither on v_i nor on ds/dv_i since $\Phi_1(\epsilon^2 s)$ itself did not depend thereon. In addition, this value can be expanded in positive and ascending powers of ϵ^2.

Similarly, $R(\Sigma_0 + \epsilon^2 s)$ can be expanded in positive and ascending powers of ϵ^2 since the original expansion of R in powers of $\sqrt{\rho_i}$ (unlike what happens for the expansion of F_1) contains no terms of odd degree and, specifically, no terms of first degree. This leads to

$$\epsilon^2 R^* = R(\Sigma_0 + \epsilon^2 s) - R(\Sigma_0) + \epsilon^2 H,$$

such that R^* can be expanded in positive and ascending powers of ϵ^2.

Thus if we expand s in ascending powers of μ, such that

$$s = s_0 + \mu s_1 + \mu^2 s_2 + \cdots$$

we will have, for determining s_p, recurrence formulas furnished by the methods discussed in the preceding chapters.

Since s_p $(p > 0)$ is a periodic function of λ_2 and λ_2', we can write

$$s_p = s_p' + s_p'',$$

where s_p' is a periodic function of mean value zero, while s_p'' is independent of λ_2 and λ_2'.

We then can write

$$\begin{aligned} \mathsf{S}\frac{dR^*}{dV_i^0}\frac{ds_{p-1}''}{dv_i} &= \psi_p, \\ \mathsf{S}\frac{dF_0^*}{d\Lambda_0}\frac{ds_p'}{d\lambda_2} &= \theta_p. \end{aligned} \tag{12.6}$$

Here, ψ_p must depend on

$$\begin{matrix} s_0, & s_1', & s_2', & \ldots, & s_{p-1}', \\ & s_1'', & s_2'', & \ldots, & s_{p-2}'', \end{matrix}$$

and θ_p on

$$\begin{matrix} s_0, & s_1', & s_2', & \ldots, & s_{p-1}', \\ & s_1'', & s_2'', & \ldots, & s_{p-1}''. \end{matrix}$$

The symbol S represents a summation performed either over the various pairs of conjugate variables V_i and v_i, or over the two pairs of conjugate variables Λ and λ_2, Λ' and λ_2'.

The two sides of relations (12.6) can be expanded in ascending powers of ϵ. However, the first terms contain only positive powers whereas the second terms contain negative powers. Before replacing, in ψ_p and θ_p, the derivatives of s_q' and of s_q'' $(q < p)$ calculated previously by recurrence, the expansions of these two functions already contained terms in $1/\epsilon$ since the expansion of F^* contained such terms, as demonstrated above. This means that the expansion of s_p in ascending powers of ϵ must start with a negative power of ϵ. If, in ψ_p and θ_p, the derivatives of s_q' and s_q'' are replaced by their series in powers of ϵ, calculated previously, then ψ_p and θ_p can be expanded in ascending powers of ϵ; however, now the series, instead of starting with a term in $1/\epsilon$, will start with a term in $1/\epsilon^n$ where n is a positive integer.

The exponent of $1/\epsilon$, in the first term of the expansion of s_p, will thus increase with p.

This means that, if the eccentricities are very small, it might happen that extremely large terms will appear in s_p. This constitutes a difficulty which, as demonstrated above, is due simply to the presence of terms in $1/\epsilon$ in F^*; and these terms in $1/\epsilon$ are simply due to the fact that F contains terms of the first degree with respect to $\sqrt{\rho_i}$ or with respect to ξ, η, ξ', and η'.

Let us see now whether this difficulty, whose nature can be understood in light of the example in the preceding number, might not be entirely artificial and whether a certain detour might permit overcoming it.

Solution of the Difficulty

144. To understand how the above difficulty can be overcome, let us return to the specific example in no. 142.

Putting

$$\sqrt{2\Omega}\cos\omega = x, \quad \sqrt{2\Omega}\sin\omega = y,$$

our canonical equations will become

$$\frac{d\Lambda}{dt} = \frac{\mu}{\sqrt{2}}(y\cos\lambda + x\sin\lambda), \qquad \frac{d\lambda}{dt} = 1,$$
$$\frac{dx}{dt} = \frac{\mu}{\sqrt{2}}\sin\lambda - \mu A y, \qquad \frac{dy}{dt} = \frac{\mu}{\sqrt{2}}\cos\lambda + \mu A x. \tag{12.7}$$

The system of equations obviously is easy to integrate, since the two latter equations are linear and, if we note that $dt = d\lambda$, directly yield

$$x = \alpha \cos \lambda + \beta \cos(\mu A\lambda) - \gamma \sin(\mu A\lambda) ,$$
$$y = -\alpha \sin \lambda + \beta \sin(\mu A\lambda) + \gamma \cos(\mu A\lambda) , \tag{12.8}$$

where

$$\alpha = -\frac{\mu}{\sqrt{2}(1 + \mu A)}$$

and where β and γ are two arbitrary constants.

After this, it is only necessary to effect a quadrature in order to obtain Λ, which is easy to perform. In fact, we have

$$\Lambda = \delta + \beta\alpha \cos(1 + \mu A)\lambda + \gamma\alpha \sin(1 + \mu A)\lambda ,$$

where δ is a new constant of integration.

A remarkable particular solution corresponds to the case in which β and γ are zero. This yields

$$x = \alpha \cos \lambda, \quad y = -\alpha \sin \lambda , \tag{12.9}$$

whence

$$\Lambda = \delta .$$

If one wishes to continue the comparison with the three-body problem, it could be said that this particular solution (12.9) is the analog of the periodic solution of the first kind, defined in Chap. 3.

Equations (12.8) lead to

$$(x - \alpha \cos \lambda)^2 + (y + \alpha \sin \lambda)^2 = \beta^2 - \gamma^2 .$$

If x and y, for the moment, are considered coordinates of a point in a plane, we will have the equation of a circle with the point

$$x = \alpha \cos \lambda, \quad y = -\alpha \sin \lambda ,$$

as center, which would correspond to the periodic solution (12.9). This point is close to the origin, since μ and thus α are small. Nevertheless, it does differ from the origin; if β and γ are also small, the radius of the circle will be small and the origin may become highly eccentric with respect to this circle. In fact, it may even move outside of the circle.

If we go to polar coordinates

$$\sqrt{2\Omega} \quad \text{and} \quad \omega ,$$

the equation of the circle will become

$$2\Omega - 2\alpha\sqrt{2\Omega} \cos(\omega + \lambda) = \beta^2 + \gamma^2 - \alpha^2 .$$

Let us compare this equation with that readily derived from Eq. (12.2):

$$\mu \cos\varphi \sqrt{\frac{dS}{d\varphi}} + (1+\mu A)\frac{dS}{d\varphi} = C - \Lambda = V(1+\mu A)$$

from which, in no. 142, we derived the value of $dS/d\varphi$. Let us recall that

$$\varphi = \omega + \lambda, \quad \frac{dS}{d\varphi} = \frac{dS}{d\omega} = \Omega .$$

We will see that the two equations are identical, provided that

$$2V = \beta^2 + \gamma^2 - \alpha^2 ,$$

from which it follows that the constant $(\beta^2 + \gamma^2 - \alpha^2)/2$ is nothing else but the term denoted above by V and considered as being of the order of the square of the eccentricities. The radius of the circle, which is $\sqrt{\beta^2 + \gamma^2}$, thus is of the order of the eccentricities; if this were of the order of α, i.e., of μ, then the origin would move outside of the circle.

Thus it can be stated that the difficulty encountered in no. 142 is due to the fact that we had used polar coordinates there and that we had poorly selected the origin. This origin must be placed at the center of the circle, i.e., at the point that corresponds to the periodic solution.

We are thus led to change the origin, by putting

$$x' = x - \alpha\cos\lambda, \quad y' = y + \alpha\sin\lambda .$$

To retain the canonical form of the equations, it is then necessary to adopt a new variable Λ' such that

$$\Lambda' = \Lambda - \alpha(x'\cos\lambda - y'\sin\lambda) .$$

Then, our conjugate variables will become

$$\begin{matrix} \Lambda', & x', \\ \lambda, & y' . \end{matrix}$$

The function F which, by hypothesis, was equal to

$$\Lambda + \mu\sqrt{\Omega}\cos(\omega + \lambda) + \mu A\Omega$$

will then become, as a function of the new variables,

$$\Lambda' + \frac{\mu A}{2}(x'^2 + y'^2) + \frac{\mu A\alpha^2}{2} + \frac{\alpha\mu}{\sqrt{2}} .$$

The two last terms are constants and play no role at all since they can be made to go into the constant C.

Our differential equations then become

$$\frac{d\lambda}{dt} = 1, \qquad \frac{d\Lambda'}{dt} = 0,$$

$$\frac{dx'}{dt} = -\mu A y', \quad \frac{dy'}{dt} = \mu A x',$$

while the corresponding partial differential equation will read

$$\frac{dS}{d\lambda} + \frac{\mu A}{2}\left[\left(\frac{dS}{dy'}\right)^2 + y'^2\right] = \text{const} .$$

Returning now to polar coordinates, by putting

$$\sqrt{2\Omega'} \cos \omega' = x', \quad \sqrt{2\Omega'} \sin \omega' = y' ,$$

we obtain

$$F = \Lambda' + \mu A \Omega' + \text{const} ,$$

so that the partial differential equation reduces to

$$\frac{dS}{d\lambda} + \mu A \frac{dS}{d\omega'} = \text{const} .$$

Because of the simplicity of the example used here, the integration of the equation transformed in this manner is immediate. However, the important point here was to show that the terms, that would be analogous to the term in $\sqrt{dS/d\omega}$ in Eq. (12.2) have vanished. This had been exactly the term that caused all the difficulty.

145. Let us now attempt to apply the same method to the three-body problem, first for the two-dimensional case.

We used first the following variables

$$\begin{matrix} \Lambda, & \Lambda', & \xi, & \xi' , \\ \lambda, & \lambda', & \eta, & \eta' , \end{matrix} \tag{12.10}$$

and then

$$\begin{matrix} \Lambda, & \Lambda', & \sigma_i, \\ \lambda_1, & \lambda'_1, & \tau_i , \end{matrix} \tag{12.11}$$

and next

$$\begin{matrix} \Lambda, & \Lambda', & \rho_i, \\ \lambda_1, & \lambda'_1, & \omega_i, \end{matrix} \tag{12.12}$$

and then

$$\begin{matrix} \Lambda, & \Lambda', & V_i . \\ \lambda_2, & \lambda'_2, & v_i . \end{matrix} \tag{12.13}$$

Let us continue our comparison and, for the moment, consider only the two last pairs of conjugate variables, disregarding the two first pairs, i.e., Λ and Λ' and their conjugates.

We can then state that the variables (12.10) and (12.11) are analogous to rectangular coordinates while the variables (12.12) and (12.13) are analogous to polar coordinates.

The difficulty mentioned in no. 143, as has been shown, is due to the presence of terms of degree $\frac{1}{2}$ with respect to V_i, which are themselves due to terms of the first degree with respect to $\sqrt{\rho_i}$ and terms of the first degree in ξ, ξ', η, η'.

If the function F does not contain such terms, this difficulty will not arise.

However, since the difficulty is entirely similar to that mentioned in no. 142 and has been overcome in no. 144, it seems that one could possibly succeed with the same means, i.e., with a transformation analogous to a change in origin. In that case, the variables ξ, ξ', η, η' must be replaced by others which vanish for the periodic solutions of the first kind investigated in no. 40, since these solutions are analogous to the periodic solution (12.12) of the preceding number.

Let us then study the periodic solutions of no. 40. It has been shown that, for these periodic solutions of the first kind,

$$\begin{aligned} &\Lambda, \quad \Lambda', \quad \xi \cos\lambda - \eta \sin\lambda, \qquad \xi \sin\lambda + \eta\cos\lambda, \\ &\qquad\qquad \xi' \cos\lambda' - \eta' \sin\lambda', \quad \xi' \sin\lambda' + \eta' \cos\lambda' \end{aligned} \tag{12.14}$$

represent periodic functions of time and that this also is the case for $\sin(\lambda - \lambda')$, $\cos(\lambda - \lambda')$.

It is also possible to consider the variables (12.14) as periodic functions of $\lambda - \lambda'$ and of two arbitrary constants, which I will call by Λ_1 and Λ_1'.

Thus let

$$\Lambda = A, \quad \Lambda' = A', \quad \xi = B, \quad \xi' = B', \quad \eta = C, \quad \eta' = C'$$

be the equations of these periodic solutions. Here, A, A', B, B', C, and C', will be functions of $\lambda, \lambda', \Lambda_1$, and Λ_1', which are periodic with respect to λ and λ'. Below, we give the form of these functions. The quantities A and A' depend only on $\lambda - \lambda'$, so that we have

$$B = T\cos\lambda + U\sin\lambda, \quad B' = T'\cos\lambda' + U'\sin\lambda',$$

$$C = -T\sin\lambda + U\cos\lambda, \quad C' = -T'\sin\lambda' + U'\cos\lambda',$$

where T, U, T', and U' depend only on $\lambda - \lambda'$.

From this, the following identity is readily derived:

$$\frac{dB}{d\lambda}\frac{dC}{d\lambda'} - \frac{dB}{d\lambda'}\frac{dC}{d\lambda} = -T\frac{dT}{d(\lambda - \lambda')} - U\frac{dU}{d(\lambda - \lambda')}, \tag{12.15}$$

and, by symmetry,

$$\frac{dB'}{d\lambda}\frac{dC'}{d\lambda'} - \frac{dB'}{d\lambda'}\frac{dC'}{d\lambda} = -T'\frac{dT'}{d(\lambda - \lambda')} - U'\frac{dU'}{d(\lambda - \lambda')}.$$

After this, let us form an auxiliary function

$$S = S_0 - \xi_1 C - \xi'_1 C' + \eta B + \eta' B' + \xi_1 \eta + \xi'_1 \eta',$$

where S_0 is a function of λ, λ', Λ_1, Λ'_1 which will be determined below. Then, S is a function of

$$\begin{matrix} \lambda, & \lambda', & \eta, & \eta', \\ \Lambda_1, & \Lambda'_1, & \xi_1, & \xi'_1. \end{matrix}$$

If we next put

$$\begin{aligned} &\frac{dS}{d\Lambda_1} = \lambda_1, \quad \frac{dS}{d\Lambda'_1} = \lambda'_1, \quad \frac{dS}{d\xi_1} = \eta_1, \quad \frac{dS}{d\xi'_1} = \eta'_1, \\ &\frac{dS}{d\lambda} = \Lambda, \quad \frac{dS}{d\lambda'} = \Lambda', \quad \frac{dS}{d\eta} = \xi, \quad \frac{dS}{d\eta'} = \xi', \end{aligned} \tag{12.16}$$

and if, for the new variables, we use

$$\begin{matrix} \Lambda_1, & \Lambda'_1, & \xi_1, & \xi'_1, \\ \lambda_1, & \lambda'_1, & \eta_1, & \eta'_1, \end{matrix}$$

instead of

$$\begin{matrix} \Lambda, & \Lambda', & \xi, & \xi', \\ \lambda, & \lambda', & \eta, & \eta', \end{matrix}$$

then the canonical form of the equations will not be altered. This then yields

$$\Lambda = \frac{dS_0}{d\lambda} - \xi_1 \frac{dC}{d\lambda} - \xi'_1 \frac{dC'}{d\lambda} + \eta \frac{dB}{d\lambda} + \eta' \frac{dB'}{d\lambda},$$

$$\Lambda' = \frac{dS_0}{d\lambda'} - \xi_1 \frac{dC}{d\lambda'} - \xi'_1 \frac{dC'}{d\lambda'} + \eta \frac{dB}{d\lambda'} + \eta' \frac{dB'}{d\lambda'};$$

$$\eta_1 = \eta - C, \quad \eta'_1 = \eta' - C',$$

$$\xi = \xi_1 + B, \quad \xi' = \xi'_1 + B'.$$

Setting

$$\xi_1 = \xi'_1 = \eta_1 = \eta'_1 = 0,$$

the terms ξ, ξ', η, and η' will reduce, respectively, to B, B', C, and C'. We wish to have Λ and Λ' also reduce to A and A'. This involves the conditions

$$\begin{aligned} \frac{dS_0}{d\lambda} &= A - C\frac{dB}{d\lambda} - C'\frac{dB'}{d\lambda}, \\ \frac{dS_0}{d\lambda'} &= A' - C\frac{dB}{d\lambda'} - C'\frac{dB'}{d\lambda'}. \end{aligned} \tag{12.17}$$

These two equations are compatible and determine S_0 provided that the resultant values for the derivatives of S_0 satisfy the integrability condition

$$\frac{d}{d\lambda'}\left(\frac{dS_0}{d\lambda}\right) - \frac{d}{d\lambda}\left(\frac{dS_0}{d\lambda'}\right) = 0 .$$

However, this equation is written as follows:

$$\frac{dA}{d\lambda'} - \frac{dA'}{d\lambda} - \frac{dC}{d\lambda'}\frac{dB}{d\lambda} + \frac{dC}{d\lambda}\frac{dB}{d\lambda'}$$
$$- \frac{dC'}{d\lambda'}\frac{dB'}{d\lambda} + \frac{dC'}{d\lambda}\frac{dB'}{d\lambda'} = 0 .$$

Making allowance for Eqs. (12.15) and noting that A, T, and U depend only on $\lambda - \lambda'$, we obtain

$$-\frac{dA}{d\lambda'} - \frac{dA'}{d\lambda} + T\frac{dT}{d\lambda} + U\frac{dU}{d\lambda} + T'\frac{dT'}{d\lambda} + U'\frac{dU'}{d\lambda} = 0 ,$$

which means that, as a periodic solution, we must have

$$A + A' - \frac{T^2 + U^2 + T'^2 + U'^2}{2} = \text{const.} ,$$

i.e.,

$$\Lambda + \Lambda' - \frac{\xi^2 + \xi'^2 + \eta^2 + \eta'^2}{2} = \text{const} .$$

However, this condition is nothing else but the equation of areas and thus is satisfied.

The function S_0, defined by Eqs. (12.17), thus exists. Its derivatives $dS_0/d\lambda$ and $dS'_0/d\lambda'$ are periodic in λ and λ'. The mean values of these two periodic functions depend solely on the two constants Λ_1 and Λ'_1. Since, until now, we have made no assumptions with respect to the selection of these two constants, they can be chosen in such a manner that the mean values will exactly be Λ_1 and Λ'_1.

This will furnish

$$S_0 = \Lambda_1\lambda + \Lambda'_1\lambda' + \text{function periodic in } \lambda \text{ and } \lambda' .$$

The function S can be expanded in ascending powers of μ and, for $\mu = 0$, reduces to

$$\Lambda_1\lambda + \Lambda'_1\lambda' + \xi_1\eta + \xi'_1\eta' .$$

To perform the transformation, let us attempt to express the old variables as a function of the new variables, making use of Eqs. (12.16). We primarily have

$$\begin{aligned} \eta &= \eta_1 + C, & \eta' &= \eta'_1 + C', \\ \xi &= \xi_1 + B, & \xi' &= \xi'_1 + B', \end{aligned} \qquad (12.18)$$

after which we obtain the two first equations of system (12.16):

$$\lambda_1 = \frac{dS}{d\Lambda_1}, \quad \lambda_1' = \frac{dS}{d\Lambda_1'}.$$

In these two equations, we will substitute η and η' by their values [Eq. (12.18)] so that they can be written in the form of

$$\lambda_1 = \lambda + \mu\psi, \quad \lambda_1' = \lambda' + \mu\psi',$$

where ψ and ψ' are functions of μ, λ, λ', ξ_1, ξ_1', η_1, η_1', Λ_1, Λ_1' of the following form:

(i) They can be expanded in powers of μ.
(ii) They are periodic in λ and λ'.
(iii) They are linear in ξ_1, ξ_1', η_1, η_1'.

By applying the principles of Chap. 2, of which we have made such frequent use, we can then derive from these equations

$$\lambda = \lambda_1 + \mu\psi_1, \quad \lambda' = \lambda_1' + \mu\psi_1',$$

where ψ_1 and ψ_1' are functions of λ_1, Λ_1, ξ_1, η_1 and of the same primed symbols which are

(a) expandable in powers of μ, ξ_1, η_1, ξ_1', η_1';
(b) periodic in λ_1 and λ_1'.

Let us substitute, in Eqs. (12.18), these values of λ and λ'. This will furnish the terms ξ and η as a function of the new variables. We note that ξ and η, expressed in this manner, can be expanded in powers of μ, ξ_1, and η_1, and are periodic in λ_1 and λ_1'. In addition, for $\mu = 0$, the terms ξ and η reduce to ξ_1 and η_1.

If, in the two equations

$$\Lambda = \frac{dS}{d\lambda}, \quad \Lambda' = \frac{dS}{d\lambda},$$

the terms λ, ξ, and η are replaced by their expressions as a function of the new variables, then we will have Λ and Λ' expressed as functions of Λ_1, λ_1, ξ_1, and η_1 which are periodic in λ_1 and λ_1' and can be expanded in powers of μ, ξ_1, and η_1, reducing to Λ_1 and Λ_1' for $\mu = 0$.

What will now happen to F when the new variables are adopted? It is obvious that F will be expandable in powers of μ, ξ_1, and η_1, and will be periodic in λ_1 and λ_1'.

Let

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$$

be the expansion of F in powers of μ, when the old variables are used; in addition, let, similarly,

$$F = F_0' + \mu F_1' + \mu^2 F_2' + \cdots$$

be the expansion of F when the new variables are used.

First of all, it is obvious that, to obtain F'_0, it is sufficient to replace Λ and Λ' in F_0 by Λ_1 and Λ'_1.

Next, let us calculate F'_1.

Let F''_1 be the result obtained on replacing, in F_1, each old variable by the corresponding new variable, i.e., Λ by Λ_1, λ by λ_1, ξ by ξ_1, etc.

Let

$$\Lambda = \Lambda_1 + \mu\Lambda_2 + \cdots ,$$
$$\Lambda' = \Lambda'_1 + \mu\Lambda'_2 + \cdots$$

be the expansions of Λ and Λ' in powers of μ. It is clear that

$$F'_1 = F''_1 + \frac{dF'_0}{d\Lambda_1}\Lambda_2 + \frac{dF'_0}{d\Lambda'_1}\Lambda'_2 .$$

Next, let us calculate Λ_2. It is easy to find

$$\Lambda = A - \xi_1\frac{dC}{d\lambda} - \xi'_1\frac{dC'}{d\lambda} + \eta_1\frac{dB}{d\lambda} + \eta'_1\frac{dB'}{d\lambda} .$$

Thus to obtain Λ_2, it is necessary, in the expression

$$\frac{A - \Lambda_1}{\mu} - \xi_1\frac{dC}{\mu d\lambda} - \xi'_1\frac{dC'}{\mu d\lambda} + \eta_1\frac{dB}{\mu d\lambda} + \eta'_1\frac{dB'}{\mu d\lambda} ,$$

to set $\mu = 0$ and, consequently, $\lambda = \lambda_1, \lambda' = \lambda'_1$. Consequently, Λ_2 (and this is the same for Λ'_2) is a periodic function of λ_1 and a linear function of ξ_1 and η_1, with its mean value (with respect to λ_1 and λ'_1) depending neither on ξ_1 nor on η_1.

Thus F'_1 will be periodic in λ_1 and λ'_1. Let R' be its mean value and let R'' be that of F''_1. The quantity R'' will be obtained by replacing, in R, each old variable by the corresponding new variable; then R' will differ from R'' only by a quantity independent of ξ_1 and of η_1.

In Chap. 10, the importance of the equations

$$\frac{d\xi}{dt} = \frac{dR}{d\eta}, \quad \frac{d\eta}{dt} = - \frac{dR}{d\xi}$$

for studying the secular variations of the elements has been demonstrated. After the change of variables performed above, these will be replaced by the following:

$$\frac{d\xi_1}{dt} = \frac{dR''}{d\eta_1}, \quad \frac{d\eta_1}{dt} = - \frac{dR''}{d\xi_1} .$$

However, according to the above statements, the two systems of equations are identical and the second differs from the first only by the fact that the symbols have subscripts.

Up to now, it seems that the transformation made here has produced no changes in the form of our equations; I can at last discuss the advantages of this procedure.

Let us first look at the fate of the equations of our periodic solutions of the first kind, using the new variables. Because of the choice of our auxiliary function S, these equations can be written

$$\xi_1 = \eta_1 = \xi_1' = \eta_1' = 0, \quad \Lambda_1 = \text{const.}; \quad \Lambda_1' = \text{const.}$$

Finally, λ_1 and λ_1' will be given functions of time, of two new constants Λ_1 and Λ_1', and of two new arbitrary constants.

This might be of some interest despite the fact that we do not require this for our particular purpose, namely, to define the manner in which λ_1 and λ_1' depend on these two constants to be denoted here by α and β. We will have

$$\lambda_1 = \alpha + \varphi(t+\beta, \Lambda, \Lambda_1'), \quad \lambda_1' = \alpha + \varphi'(t+\beta, \Lambda_1, \Lambda_1'),$$

where φ and φ' are two functions of $t+\beta$, Λ_1, Λ_1' which, when $t+\beta$ increases by a certain constant γ depending on Λ_1 and Λ_1', will themselves increase by a certain constant δ (the same for both φ and φ') which also depends on Λ_1 and Λ_1'. The first of these two constants γ is the period of the periodic solution considered while the second δ is the angle through which the three-body system rotates during one period.

Of all this, we will retain only one point:

If ξ_1, η_1, ξ_1', and η_1' are zero at the origin of time, the solution will be periodic of the first kind and these four variables ξ_1, η_1, ξ_1', and η_1' will always remain zero.

Now, we have the following differential equations:

$$\frac{d\xi_1}{dt} = \frac{dF}{d\eta_1}, \quad \frac{d\xi_1'}{dt} = \frac{dF}{d\eta_1'},$$

$$\frac{d\eta_1}{dt} = -\frac{dF}{d\xi_1}, \quad \frac{d\eta_1'}{dt} = -\frac{dF}{d\xi_1'}.$$

Consequently, it is necessary that the four derivatives

$$\frac{dF}{d\xi_1}, \quad \frac{dF}{d\xi_1'}, \quad \frac{dF}{d\eta_1}, \quad \frac{dF}{d\eta_1'}$$

vanish together as soon as the four variables

$$\xi_1, \quad \xi_1', \quad \eta_1, \quad \eta_1'$$

also vanish together, i.e., as soon as F contains no terms of first degree with respect to these four variables.

Thus the expression of F as a function of the new variables has the same form as the expression of F as a function of the old variables. The only difference lies in the fact that there are no terms of first degree with respect to ξ_1, η_1, ξ'_1, η'_1 whereas first-degree terms did exist with respect to the corresponding old variables ξ, η, ξ', and η'. However, it was exactly the first-degree terms that created the entire difficulty; this means that the difficulty has disappeared together with these terms.

Exactly the same happens if, instead of the three-body problem in two-dimensional form, one treats the three-body problem in three-dimensional form.

If, in fact, one selects the following as variables

$$\Lambda_1, \quad \Lambda'_1, \quad \xi_1, \quad \xi'_1, \quad p, \quad p',$$
$$\lambda_1, \quad \lambda'_1, \quad \eta_1, \quad \eta'_1, \quad q, \quad q',$$

F will contain no term of the first degree with respect to ξ_1, η_1, p, and q.[R6]

CHAPTER 13
Divergence of the Lindstedt Series

146. In Chap. 9, we have found that the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{13.1}$$

can be formally satisfied by series of the form

$$\begin{aligned} x_i &= x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots , \\ y_i &= w_i + \mu y_i^1 + \mu^2 y_i^2 + \cdots , \end{aligned} \tag{13.2}$$

where x_i^k and y_i^k are periodic functions of the quantities

$$w_i = n_i t + \overline{\omega}_i \quad (i = 1,2, \ldots ,n)$$

and are represented by series of the sines and cosines of multiples of w, in such a manner that we have

$$x_i^k (\text{or } y_i^k) = A_0 + \sum A \cos(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h) \, . \tag{13.3}$$

The mean value A_0 of these periodic functions can be selected arbitrarily.

It is now a matter of determining whether these series are convergent. However, the question is subdivided; one can actually ask
(a) whether the partial series (13.3) are convergent and whether the convergence is absolute and uniform;
(b) assuming that they do not converge absolutely, whether the terms can be arranged in a manner to obtain semiconvergent series;
(c) assuming that series (13.3) are convergent, whether series (13.2) will converge and whether this convergence will be uniform.

Discussion of Series (13.3)

147. Let us recall the manner in which we obtained series (13.3). We arrived at equations of the form

$$\sum n_p^0 \frac{dx_i^k}{dw_p} = \sum B \cos(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h)$$

[Eqs. (9.13) of no. 127] from which we derived

$$x_i^k = \sum \frac{B \sin(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h)}{m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_h^0} + A_0, \quad (13.4)$$

where A_0 is an arbitrary constant.

Does series (13.4) converge absolutely and uniformly? If this were the case, the sum of this series would have to remain finite for all values of time. However, it has been demonstrated previously[8] that the sum of the terms of a similar series cannot remain constantly below half of any one of its coefficients.

Thus, so as to have series (13.4) converge uniformly, it is necessary that the absolute value of the coefficient

$$\frac{B}{m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0}$$

be limited.[R7]

To be specific, let us assume only two degrees of freedom, and let $n_1^0 = n$, $n_2^0 = -1$. Then series (13.4) becomes

$$A_0 + \sum \frac{B_{m_1 m_2} \sin(m_1 w_1 + m_2 w_2 + h_{m_1 m_2})}{m_1 n - m_2}$$

and the absolute value of the coefficients

$$\frac{B_{m_1 m_2}}{m_1 n - m_2} \quad (13.5)$$

must be limited.

First of all, it is obvious that this cannot take place for the rational values of n, unless $B_{m_1 m_2}$ were zero each time that

$$\frac{m_2}{m_1} = n .$$

This brings us to the case in which n is irrational and specifically to a consideration of those of the divisors $m_1 n - m_2$ that correspond to the successive convergents of n.

We state first that, irrespective of the sequence of the numbers $B_{m_1 m_2}$, an irrational number n can be obtained (as close as desired to a given number) which will be such that the absolute value of the coefficients (13.5) will not be limited.

In fact, let

$$\frac{\alpha_1}{\beta_1}, \quad \frac{\alpha_2}{\beta_2}, \quad \ldots, \quad \frac{\alpha_p}{\beta_p}, \ldots$$

be the successive convergents of n.

Let

$$\lambda_1, \quad \lambda_2, \quad \ldots, \quad \lambda_p, \quad \ldots,$$

be an arbitrary sequence of indefinitely increasing positive numbers. We state that the number n can always be chosen in such a manner that

$$\left| \frac{B\beta_p \alpha_p}{n\beta_p - \alpha_p} \right| > \lambda_p \,. \tag{13.6}$$

In fact, according to the definition of the convergents, we have

$$\alpha_{p+1} = \alpha_{p-1} + \alpha_p a_{p+1}, \quad \beta_{p+1} = \beta_{p-1} + \beta_p a_{p+1} \,,$$

where a_{p+1} is a positive integer which can be arbitrarily selected without changing anything in the first p convergents.

On the other hand, we have

$$|n\beta_p - \alpha_p| < \frac{1}{\beta_{p-1} + \beta_p a_{p+1}} \,.$$

Thus we can select the integer a_{p+1} in such a manner that the absolute value of $n\beta_p - \alpha_p$ will be as small as desired and, consequently, in such a manner as to satisfy the inequality (13.6) no matter what the numbers $B\beta_p \alpha_p$ and λ_p might be.

Since the numbers λ_p are subject only to be indefinitely increasing, it is possible to make an arbitrary selection of the first q of these numbers (irrespective of q) and thus also of the first q convergents. Thus the number n may be as close as possible as desired to any given number.

However, it is also frequently possible to find a number n such that the series (13.4) will be convergent. Let us suppose that the series

$$\sum B_{m_1 m_2} \cos(m_1 w_1 + m_2 w_2 + h)$$

converges in such a manner (as it ordinarily happens) that the following expression is obtained for all values of m_1 and of m_2:

$$|B_{m_1 m_2}| < K \alpha^{|m_1|} \beta^{|m_2|} \,, \tag{13.7}$$

where K is any positive number while α and β are two positive numbers smaller than unity.

Let us take $n = \sqrt{p/q}$ with p and q being two relatively prime integers such that pq will not be a perfect square. This yields

$$\left|\frac{1}{m_1 n - m_2}\right| = \left|\frac{m_1 n + m_2}{m_1^2 n^2 - m_2^2}\right| = \left|\frac{(m_1 n + m_2)q}{pm_1^2 - qm_2^2}\right| < q(|m_1|n + |m_2|) \, ,$$

whence

$$\left|\frac{B_{m_1 m_2} \sin(m_1 w_1 + m_2 w_2 + h)}{m_1 n - m_2}\right| < Kq(|m_1|n + |m_2|)\alpha^{|m_1|}\beta^{|m_2|} \, ,$$

which proves that series (13.4) converges.

However, it is obviously possible to select the integers p and q in such a manner that $\sqrt{p/q} = n$ will be as close as desired to any given number.

This leads to the following result which will be formulated by extending it directly to the general case.

Let K be an arbitrary positive number and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers smaller than unity.

We assume that an inequality analogous to Eq. (13.7) exists and that we can write

$$|B| < K\alpha_1^{|m_1|}\alpha_2^{|m_2|} \cdots \alpha_n^{|m_n|}$$

which is what ordinarily happens.

In this case, the numbers

$$n_1^0, \quad n_2^0, \quad \ldots, \quad n_n^0 \, ,$$

can be selected in such a manner
(a) that they will be as close as desired to n given numbers and that, at the same time, series (13.4) does not converge uniformly;
(b) they can equally be chosen in such a manner that they will be as close as possible to the same n given numbers and that series (13.4) does converge uniformly.

One can easily grasp the importance of this remark. In fact, the observations, no matter what their accuracy might be, can define the mean motions only with a certain approximation. Therefore, one can remaining within the bounds of this approximation, always arrange matters such that series (13.4) will converge.

From another viewpoint, one could ask whether series (13.4) can be made to converge for values of the integration constants x_i^0 comprised within a certain interval (it will be recalled that the terms n_i^0 depend on x_i^0). According to what has been stated above, this will be possible only if the series

$$\sum B \cos(m_1 w_1 + \cdots + m_n w_n + h)$$

contained only a limited number of terms, i.e., if in the function

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots ,$$

each of the functions F_1, F_2, . . . would contain only a finite number of terms in its expansion in sines and cosines of multiples of y.

This will not be so in general, and the function F_1—for example—will be a series with an infinity of terms. However, in practical application, the calculation can be arranged such as to obtain the case in which the functions F_i have only a finite number of terms. In fact, since the series F_1 is convergent, all terms, except for a finite number of them, will be extremely small. Thus it would be of no interest to take these into consideration already in the first approximation.

Thus the accepted procedure will be: In the series F_1, all terms, except for a certain finite number, can be considered of the same order of magnitude as μ; however, there are some terms of the same order as μ^2 and others, even smaller, that will be of the same order as μ^3, etc. In the other series F_2, F_3, terms of these various orders of magnitude will also be encountered.

Thus we can write, in general,

$$F_i = F_{i0} + F_{i1} + F_{i2} + \cdots + F_{ik} + \cdots ,$$

where F_{ik} represent those terms of F_i that can be considered as being of the same order of magnitude as μ^k. These terms are finite in number. This manner of decomposing F_i obviously admits of a high degree of arbitrariness.

Let now μ' be a quantity of the same order of magnitude as μ and let us put

$$F_{ik} = \mu'^k \Phi_{ik} .$$

All terms of Φ_{ik} will be finite, so that we can write

$$F = \sum \mu^i \mu'^k \Phi_{ik} .$$

Because of this artifice, F now depends on two parameters, and Φ_{ik} contains only a finite number of terms. Since the two parameters μ and μ' are of the same order of magnitude, we will set $\mu = \lambda\mu'$, yielding

$$F = \sum \mu'^k \Phi_k ,$$

where Φ_k contains only a finite number of terms.

This artifice, which has been discussed here at some length but whose application can be performed quite rapidly, shows that in practical use one can always assume to have returned to the case in which each of the functions F_i contains only a finite number of terms.

Discussion of Series (13.2)

148. Since the question of the convergence of series (13.13) has thus been settled, it becomes necessary to test whether series (13.2) converge.

This question again must be subdivided.

Series (13.2) actually depend on μ and on the integration constants x_i^0. Consequently, one can ask

(a) whether series (13.2) converge uniformly for all values of μ and x_i^0 comprised within a certain interval;

(b) whether series (13.2) converge uniformly for sufficiently small values of μ when assigning properly chosen values to x_i^0.

The first question must be answered in the negative.

Indeed, let us assume that series (13.2) converge uniformly and let us write them in the form

$$x_i = x_i^0 + \mu\varphi_i(w_k, x_k^0, \mu) ,$$
$$y_i = w_i + \mu\psi_i(w_k, x_k^0, \mu) , \tag{13.8}$$

where φ_i and ψ_i are functions that can be expanded in ascending powers of μ, periodic with respect to w, and further depend in an arbitrary manner on x_i^0.

Let us solve Eqs. (13.8) with respect to x_i^0 and w_i. From these equations, the quantities x_i^0 and w_i can be derived in the form of series arranged in powers of μ whose coefficients depend on x_i and y_i.

It is easy to verify this. Actually, to prove that the theorem of no. 30 is applicable, it is only necessary to note that, for $\mu = 0$, the equations reduce to

$$x_i^0 = x_i, \quad w_i = y_i$$

and that the functional determinant of the left-hand sides is equal to one. Otherwise, it is only necessary to apply the generalized Lagrange formula.

This will yield

$$x_i^0 = x_i + \mu\varphi_i'(y_k, x_k, \mu) , \tag{13.9}$$
$$w_i = y_i + \mu\psi_i'(y_k, x_k, \mu) , \tag{13.10}$$

where φ_i' and ψ_i' are functions that can be expanded in powers of μ, being uniform with respect to x and y and periodic with respect to y.

Equations (13.9) thus define n uniform integrals of our differential equations.

From another angle, we have put

$$w_i = n_i t + \overline{\omega}_i ,$$

and the coefficients n_i defined in this manner depend on μ and on x_i^0. If these quantities are able to vary within certain limits, one could manipulate them in such a manner that the coefficients n_i became mutually commensurable.

In this case, one can find a number T such that the $n_i T$ become multiples of 2π. Consequently, assigning these particular values of μ and x_i^0, Eqs. (13.8) will represent a periodic solution with period T. The existence of n uniform integrals would force us to conclude that $n+1$ of the characteristic exponents relative to this periodic solution are zero.

However, there is more to this.

By hypothesis, series (13.8) must satisfy the differential Eqs. (13.1). We have demonstrated that, on assigning certain particular values to n integration constants x_k^0, series (13.8) will represent a periodic solution of these equations. In order to determine this solution, we will also assign certain particular values to the n integration constants $\overline{\omega}_k$.

Let

$$x_i = f_i(t), \quad y_i = f_i'(t) \tag{13.11}$$

be the resultant periodic solution. Let us put

$$x_i = f_i + \xi_i, \quad y_i = f_i' + \eta_i$$

and form the variational equations of Eqs. (13.1) (see no. 53). Since series (13.8) must satisfy the differential equations no matter what the constants x_k^0 and $\overline{\omega}_k$ might be, we will obtain $2n$ particular solutions linearly independent of our variational equations, by setting

$$\xi_i = \epsilon_{ik} + t\mu \sum \frac{d\varphi_i}{dw_p}\frac{dn_p}{dx_k^0} + \mu\frac{d\varphi_i}{dx_k^0},$$

$$\eta_i = \frac{dn_i}{dx_k^0} t + t\mu \sum \frac{d\psi_i}{dw_p}\frac{dn_p}{dx_k^0} + \mu\frac{d\psi_i}{dx_k^0},$$

$$\xi_i = \mu\frac{d\varphi_i}{dw_k}, \quad \eta_i = \epsilon_{ik} + \mu\frac{d\psi_i}{dw_k};$$

$$\epsilon_{ik} = 1, \quad \text{if } i=k \quad \text{and} \quad 0 \text{ if } i \lessgtr k$$

$$(i,k = 1,2, \ldots, n).$$

In the functions

$$\frac{d\varphi_i}{dw_p}, \quad \frac{d\varphi_i}{dx_k^0}, \quad \frac{d\psi_i}{dw_p}, \quad \frac{d\psi_i}{dx_k^0}, \tag{13.12}$$

the constants x_k^0 and w_k must be replaced by the values corresponding to the periodic solution (13.11); the functions (13.12) will thus become periodic in t.

From this it follows that the $2n$ characteristic exponents are zero. However, it is known that this is not usually the case.

Thus, in general, series (13.2) will not converge uniformly when μ and x_i^0 vary in a certain interval.[R8] Q.E.D.

149. This leaves the second question to be treated. Actually, one could ask whether these series might not converge for small values of μ if certain properly chosen values are attributed to x_i^0.

Here again, we must differentiate two cases.

In general, the quantities n_i depend not only on x_i^0 but also on μ and can be expanded in powers of μ.

We have seen, furthermore, that the mean values of the functions φ_i and ψ_i can be arbitrarily selected. In addition, we have seen that these mean values can be selected in such a manner that we will have

$$n_i = n_i^0 ,$$

$$n_i^1 = n_i^2 = \cdots = n_i^p = \cdots = 0 ,$$

i.e., that the n_i no longer depend on μ.

Consequently, we can distinguish the case in which the quantities n_i depend on μ and the case in which the n_i do not depend on μ.

Let us first assume that the quantities n_i depend on μ and, at the same time, that there are only two degrees of freedom.

Then, let

$$n_1 = n_1^0 + \mu n_1^1 + \mu^2 n_1^2 + \cdots, \quad w_1 = n_1 t + \overline{\omega}_i ,$$

$$n_2 = n_2^0 + \mu n_2^1 + \mu^2 n_2^2 + \cdots, \quad w_2 = n_2 t + \overline{\omega}_2 .$$

On the other hand, x_1, x_2, y_1, and y_2 should be expandable in powers of μ in such a manner that x_1, x_2, $y_1 - w_1$, $y_2 - w_2$ will be periodic in w_1 and w_2.

This should take place for sufficiently small values of μ. However, among the values of μ below a certain limit, it is always possible to find values such that the ratio n_1/n_2 becomes rational, since this ratio is a continuous function of μ.

If the ratio n_1/n_2 is rational, then series (13.2) will represent a periodic solution of Eqs. (13.1), irrespective of the two integration constants $\overline{\omega}_1$ and $\overline{\omega}_2$.[R9]

If series (13.2) were to converge, then a double infinity of periodic solutions of Eqs. (13.1) would correspond to this rational value of n_1/n_2.

However, it has been demonstrated in no. 42 that this cannot take place except in highly peculiar cases.

Therefore, it seems permissible to conclude that series (13.2) do not converge.

Nevertheless, the above reasoning is not sufficient for establishing this point with complete rigorousness.

In fact, what we have demonstrated in no. 42 is that for all values of μ below a certain limit there cannot be a double infinity of periodic solutions, and it will be sufficient here that this double infinity exists for a determined value of μ, differing from zero and generally very small.

Thus we would have an infinity of periodic solutions for $\mu = 0$ and for $\mu = \mu_0$ whereas we would only have a finite number (not considering as distinct the solutions that can be derived from each other by changing t into $t + h$) for values of μ between zero and μ_0.

It is rather unlikely that this would be so, which in itself is already sufficient to render the convergence of series (13.2) highly improbable.

However, there is more: It would be of interest to determine the convergence of the series (13.2) only if such convergence would take place for an infinity of value systems of the constants x_i^0, in such a manner that it would be always possible to single out one of these systems that differs as little as desired from a given arbitrary value system of these same constants. However, if a case of this type would occur, an infinity of values of μ would exist for which the periodic solutions (corresponding to a given rational value of the ratio n_1/n_2) have an infinite number.

It would also be possible to find an infinity of similar values of μ in each interval no matter how small, provided that it be sufficiently close to zero. The characteristic exponents would have to be zero for all these values of μ (see no. 54); since these exponents are continuous functions of μ (see no. 74) they should be identically zero.

It has been shown that this is generally not the case so that one must conclude that the convergence of series (13.2), assuming that it takes place for certain value systems of x_i^0, cannot take place for an infinity of these systems.

This is one more reason to consider a convergence of series (13.2) improbable in all cases, since the values of x_i^0 for which this convergence would take place cannot be distinguished from all others.

Finally, it would be of interest to determine the events that would occur if the mean values of the functions φ_i and ψ_i were selected in such a manner that

$$n_i = n_i^0$$

$$n_i^1 = n_i^2 = \cdots = n_i^p = \cdots = 0 \, .$$

In this case, the quantities n_i no longer depend on μ but only on x_i^0.

Might it not happen that series (13.2) converge when assigning certain properly chosen values to x_i^0 ?

For simplification, let us assume that there are two degrees of freedom. Could then the series, for example, converge when x_1^0 and x_2^0 are selected in such a manner that the ratio n_1/n_2 becomes irrational while its square, on the other hand, becomes rational (or when the ratio n_1/n_2 is subjected to another condition analogous to the one I have more or less guessed here)?

The reasoning in this chapter does not permit the definite statement that this event will never occur. All we can say now is that it is highly improbable.[R10]

Comparison with the Old Methods

150. Let us add one more thought here: In the absence of means of ensuring convergence of the series, what is the best choice for the mean values of x_i^p and y_i^p? I believe that it is advisable to select these mean values in such a manner that x_i^p and y_i^p (starting from x_i^1 and y_i^1) vanish for $t = 0$, in such a manner that x_i^0 represent the initial values of x_i and $\overline{\omega}_i$ the initial values of y_i.

If, next, we consider the resultant series

$$\begin{aligned} x_i &= x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots, \\ y_i &= w_i + \mu y_i^1 + \mu^2 y_i^2 + \cdots, \end{aligned} \tag{13.13}$$

it will be found that the terms x_i^p, w_i, and y_i^p depend on μ. If these quantities are expanded in powers of μ and if the right-hand sides of Eqs. (13.13) are arranged is ascending powers of μ, we will obtain the expansion in powers of μ of that of the particular solution of our differential equations which admits x_i^0 and $\overline{\omega}_i$ as the initial values of x_i and y_i.

It is known that this series converges for sufficiently small values of t.

Let

$$\begin{aligned} x_i &= x_i^0 + \mu\xi_i^1 + \mu^2\xi_i^2 + \cdots, \\ y_i &= n_i^0 t + \overline{\omega}_i + \mu\eta_i^1 + \mu^2\eta_i^2 + \cdots. \end{aligned} \tag{13.14}$$

Here ξ_i^p and η_i^p are not periodic functions of time but no longer depend on μ. In addition, these functions vanish for $t = 0$, as will the quantities x_i^p and y_i^p.

From the manner in which we have derived series (13.14) from series (13.13) it is possible to draw a few consequences with respect to the form of series (13.14).

Thus to obtain ξ_i^1 it is sufficient to set $\mu = 0$ in the expression of x_i^1. Let us recall how x_i^1 depends on μ; x_i^1 is a periodic function of the quantities which we had denoted by

$$w_1, \quad w_2, \quad \ldots, \; w_n$$

and, in addition, we have

$$w_i = n_i t + \overline{\omega}_i \, .$$

Here, $\overline{\omega}_i$ is a constant of integration and n_i depends on μ. Thus if we set $\mu = 0$, n_i reduces to n_i^0, since we have

$$n_i = n_i^0 + \mu n_i^1 + \mu^2 n_i^2 + \cdots .$$

Consequently, w_i reduces to $n_i^0 t + \overline{\omega}_i$ and x_i^1 remains a periodic function of the quantities $n_i^0 t + \overline{\omega}_i$.

Therefore, ξ_i^1 contains no secular terms.

To obtain ξ_i^2, it is sufficient to set $\mu = 0$ in

$$\frac{dx_i^1}{d\mu} + x_i^2 \, .$$

Reasoning as above, it can be demonstrated that, by setting $\mu = 0$ in x_i^2, no secular term is introduced there. On the other hand, we have

$$\frac{dx_i^1}{d\mu} = t \sum \frac{dn_k}{d\mu} \frac{dx_i^1}{dw_k} \, ,$$

or, for $\mu = 0$,

$$\frac{dx_i^1}{d\mu} = t \sum n_k^1 \frac{dx_i^1}{dw_k} \, .$$

This shows that the expression of $dx_i^1/d\mu$ contains secular terms but a distinction still must be made: The expression "mixed secular terms" is to denote terms of the form

$$t^p \sin \alpha t \quad \text{or} \quad t^p \cos \alpha t \, ,$$

while "pure secular terms" will be terms of the form t^p.

We then can write

$$\sum n_k^1 \frac{dx_i^1}{dw_k} = A_0 + \sum A_\alpha \sin \alpha t + \sum B_\alpha \cos \alpha t \, .$$

Actually, the left-hand side is a periodic function of w and, for $\mu = 0$, we have $w_i = n_i^0 t + \overline{\omega}_i$. If A_0 is zero, the expression ξ_i^2 will contain no pure secular terms but might contain mixed secular terms. If A_0 is not zero, the expression ξ_i^2 will contain pure secular terms.

A case exists in which A_0 is definitely zero, which is the case in which none of the quantities n_i^0 is zero and in which no linear relation with integral coefficients exists between the n_i^0 terms (case of no. 125). In fact, we then have

$$A_0 = \sum n_k^1 \left[\frac{dx_i^1}{dw_k} \right] \quad \text{and} \quad \left[\frac{dx_i^1}{dw_k} \right] = 0 \,,$$

denoting by $[U]$ the mean value of a mean function of a periodic function U of $w_1, w_2, \ldots, w_n$.

Below, we give still another case in which A_0 is zero.

We assume that

$$n_3^0 = n_4^0 = \cdots = n_n^0 = 0 \,,$$

and that, on the other hand, the ratio of n_1^0 to n_2^0 is incommensurable. Let us put

$$x_i^1 = \sum C \sin(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + h) \,,$$

where $m_1, m_2, \ldots, m_n$ are integers while C and h are any constants.

This then is the form of the expansion of x_i^1, since this function is periodic with respect to the w.

From this it follows that

$$\sum n_k^1 \frac{dx_i^1}{dw_k} = \sum CS \cos(m_1 w_1 + m_2 w_2^1 + \cdots + m_n w_n + h) \,,$$

where

$$S = \sum n_k^1 m_k \,.$$

For $\mu = 0$, we obtain

$$\sum n_k^1 \frac{dx_i^1}{dw_k} = \sum CS \cos(\alpha t + \beta) \,,$$

where

$$\alpha = m_1 n_1^0 + m_2 n_2^0, \quad \beta = m_1 \overline{\omega}_1 + m_2 \overline{\omega}_2 + \cdots + m_n \overline{\omega}_n + h \,.$$

According to the hypotheses made above, α can be zero only if $m_1 = m_2 = 0$. Hence

$$A_0 = \sum CS \cos \beta \,,$$

where the summation extends over all terms with $m_1 = m_2 = 0$.

Now, let

$$F_1 = \sum D \cos(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n + k) \,,$$

where D and k are functions of $x_1, x_2, \ldots, x_n$. This necessarily must be the form of the functions F_1 which is periodic with respect to y.

Let D_0 and k_0 be what becomes of D and k when replacing there x_i by x_i^0. Then let

$$F_1^0 = \sum D_0 \cos(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + k_0),$$

which will be the form of F_1 after replacing there x_i by x_i^0 and y_i by w_i. The function x_i^1 will be defined by the equation

$$\sum n_k^0 \frac{dx_i^1}{dw_k} = \frac{dF_1^0}{dw_i},$$

whence

$$x_i^1 = \sum \frac{D_0 m_i}{m_1 n_1^0 + m_2 n_2^0} \cos(m_1 w_1 + m_2 w_2 + \cdots + m_n w_n + k_0),$$

from which it follows that

$$C = \frac{D_0 m_i}{m_1 n_1^0 + m_2 n_2^0}, \quad h = k_0 .$$

If $i = 1$ or 2, C will vanish for $m_1 = m_2 = 0$ and A_0 becomes zero. Thus ξ_1^2 and ξ_2^2 will contain mixed secular terms but no pure secular terms.

Contrariwise, the expressions

$$\xi_3^2, \quad \xi_4^2, \quad \ldots, \quad \xi_n^2$$

may contain pure secular terms.

Let us apply this to the three-body problem.

Let us return to the series in no. 140.

The quantities n_i^0 are zero, with the exception of n_1^0 and n_2^0.

Let us expand the quantities Λ and V_i in ascending powers of μ. This yields

$$\Lambda = \Lambda_0 + \mu\Lambda_1^1 + \mu^2\Lambda_1^2 + \cdots,$$
$$\Lambda' = \Lambda_0' + \mu\Lambda_2^1 + \mu^2\Lambda_2^2 + \cdots,$$
$$V_i = V_i^0 + \mu V_i'^1 + \mu^2 V_i'^2 + \cdots,$$

where Λ_i and $V_i'^k$ are functions of t, independent of μ and vanishing with t.

According to the above considerations, the quantities Λ_i^1 contain no secular terms. This represents the Lagrange theorem on the invariance of the major axes, neglecting the squares of the masses.

The quantities Λ_i^2 will contain mixed secular terms but no pure secular term; this is Poisson's theorem on the invariance of the major axes, when one neglects the cubes of the masses.

The quantities $V_i'^1$ contain no secular terms, but the quantities $V_i'^2$ will contain secular terms, both of the pure and mixed type.

Let us return to the case in which the quantities n_i^0 all differ from zero and are not interconnected by any linear relation with integral coefficients. One then has

$$\xi_i^3 = x_i^3 + \frac{dx_i^2}{d\mu} + \frac{1}{2}\frac{d^2 x_i^1}{d\mu^2} \quad \text{for } \mu = 0 .$$

One can see that as above, x_i^3 furnishes no secular term and that $dx_i^2/d\mu$ furnishes no pure secular term. On the other hand, we have

$$\frac{d^2 x_i^1}{d\mu^2} = \sum \frac{dx_i^1}{dw_k}\frac{d^2 w_k}{d\mu^2} + \sum \frac{d^2 x_i^1}{dw_k dw_h}\frac{dw_k}{d\mu}\frac{dw_h}{d\mu} .$$

The right-hand side can be written as follows:

$$2t \sum n^2 \frac{dx_i^1}{dw_k} + t^2 \sum \frac{d^2 x_i^1}{dw_k dw_h} n_k^1 n_h^1 .$$

Thus we again have mixed secular terms but no pure secular terms, since the mean value of the derivatives dx_i^1/dw_k, $d^2x_i^1/dw_k dw_h$ is always zero.

Obviously, the same reasoning will apply to the following terms of the series, i.e., to the ξ_i^k.

Thus in the particular case of the three-body problem, defined in no. 9, the major axis remains invariant in the sense of Poisson, no matter how far the approximation is continued.

Similarly, with any law of attraction other than Newton's, the expansions of quantities corresponding to the major axes will contain no purely secular terms no matter how far the approximation is carried. Thus these quantities are invariant in Poisson's sense.

The Lindstedt method is thus connected with the famous theorems by Lagrange and Poisson.

The idea of the possibility of this connection is due to Tisserand.

These considerations lead to a final remark:

It could seem that the series developed in the preceding chapters yield no conclusions, since they are all divergent.

Let us consider the expansion of arcsin u and let us write

$$\arcsin u = u + A_1 u^3 + A_2 u^5 + \cdots ,$$

from which we can derive

$$\mu t = \sin \mu t + A_1 \sin^3 \mu t + A_2 \sin^5 \mu t + \cdots .$$

Since the powers $\sin^3 \mu t$, $\sin^5 \mu t$ are easy to expand in sines of multiples of μt, it would seem that the expansion—at least formally—of the function μt in a trigonometric series could be derived from this.

Obviously, it will be the same for $\mu^2 t^2$, for $\mu t \sin \alpha t, \ldots$, and for all terms that might be encountered in series (13.14).

Consequently, stating that the functions represented by these series (13.14) can be expanded in purely trigonometric series, whenever a purely formal expansion is involved, seems to affirm nothing and to give no information with respect to the form of these series (13.14).

However, one would be mistaken in this. If an attempt were made to use the rough artifice which we applied to the function μt (I would not dare to say that this has never been done before) to reduce the expansions (13.14) to a purely trigonometric form, an infinity of different arguments would be introduced. What we have learned from the theorems of the preceding chapters is that the formal expansions are possible with a limited number of arguments. This could not have been predicted and allows numerous conclusions as to the coefficients of series (13.14) or as to the coefficients of other analogous series encountered in the three-body problem.[R11]

CHAPTER 14

Direct Calculation of the Series

Generalities

151. It might be of some interest to return to the results obtained in the three preceding chapters and to define their significance. Above all, we intend to demonstrate a method for directly calculating the coefficients of the expansions which we have learned to form in an indirect manner and of which we have thus proved the existence. Once this existence has been established, the calculation of these coefficients can be done in a more rapid manner without being restricted by the numerous changes of variables which had been necessary before.

Let us start by considering the particular case of the equations in no. 134.

In no. 134 we showed, using procedures analogous to those in no. 125 but somewhat modified, that one can formally satisfy our canonical equations by setting

$$x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots,$$
$$y_i = y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + \cdots,$$

where x_i^k and y_i^k are periodic functions of quantities that will be denoted by w, except for y_i^0 which must reduce to w_i. The terms x_i^0 are arbitrary constants on which the other functions x_i^k and y_i^k depend, and we have

$$w_i = n_i t + \overline{\omega}_i,$$

where $\overline{\omega}_i$ is a constant of integration while n_i is a constant depending on μ and on x_i^0 which can be expanded in powers of μ.

Using the procedures in no. 126, it is possible to assign an infinity of forms to the series, in such a manner that the mean values of the periodic functions x_i^k and y_i^k will be any desired arbitrary functions of x_i^0.

Let us note that, after as well as before the transformation of these series by the methods used in no. 126, the expression $\Sigma x_i dy_i$ (considered as a function of w_i while the terms x_i^0 are considered constants) must be an exact differential.

Let, again,

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots.$$

Let us assume that $2n$ pairwise conjugate variables exist and that the variables of the first series are of two kinds. Those of the first kind will be denoted by x_i and those of the second kind, by x_i'.

The variables of the second series conjugate to the x_i will be denoted by y_i, and those that are conjugates of x_i' will be designated y_i', such that our canonical equations will read

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{dF}{dy_i}, & \frac{dx_i'}{dt} &= \frac{dF}{dy_i'}, \\ \frac{dy_i}{dt} &= -\frac{dF}{dx_i}, & \frac{dy_i'}{dt} &= -\frac{dF}{dx_i'}. \end{aligned} \tag{14.1}$$

It is assumed that F_0 depends on x_i but not on y_i, y_i', or x_i', that F is periodic with respect to y_i and with respect to y_i', and that, if R denotes the mean portion of F_1 (considering, for the moment, F_1 as a periodic function only of y_i but not of y_i'), then R will not depend on y_i' but only on x_i and x_i'. In sum, these are the same hypotheses as those in no. 134.

We have shown that Eqs. (14.1) can be formally satisfied by series of the following form:

$$\begin{aligned} x_i &= x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \cdots, \\ x_i' &= x_i'^0 + \mu x_i'^1 + \mu^2 x_i'^2 + \cdots, \\ y_i &= w_i + \mu y_i^1 + \mu^2 y_i^2 + \cdots, \\ y_i' &= w_i' + \mu y_i'^1 + \mu^2 y_i'^2 + \cdots, \end{aligned} \tag{14.2}$$

where $x_i^k, x_i'^k, y_i^k, y_i'^k$ are periodic functions of w_i and of w_i' and, in addition, depend on the constants x_i^0 and $x_i'^0$ whose mean values can be arbitrarily selected functions of these constants; this can be demonstrated by using a method of reasoning as in no. 126. In addition, we have

$$w_i = n_i t + \overline{\omega}_i, \quad w_i' = n_i' t + \overline{\omega}_i',$$

where $\overline{\omega}_i$ and $\overline{\omega}_i'$ are constants of integration, while n_i and n_i' can be expanded in powers of μ such that

$$n_i = \sum \mu^k n_i^k, \quad n_i' = \sum \mu^k n_i'^k$$

with

$$n_i^0 \gtrless 0, \quad n_i'^0 = 0.$$

Since the possibility of such an expansion has been established in no. 134, we will now directly calculate the coefficients.

For this, we will assume that, in Eqs. (14.1), series (14.2) have been substituted and that, consequently, our variables no longer are considered as being expressed directly as a function of time but rather as being dependent on time over the intermediary of w and w_i'. Then equations (14.1) will become

$$\sum_k n_k \frac{dx_i}{dw_k} + \sum_k n_k' \frac{dx_i}{dw_k'} = \frac{dF}{dy_i}. \tag{14.3}$$

After substituting the expansions (14.2), we will also obtain

$$\frac{dF}{dy_i} = \sum \mu^k X_i^k, \quad -\frac{dF}{dx_i} = \sum \mu^k Y_i^k,$$

which are equations analogous to Eqs. (9.10) and (9.11) in no. 127; similarly,

$$\frac{dF}{dy_i'} = \sum \mu^k X_i'^k, \quad -\frac{dF}{dx_i'} = \sum \mu^k Y_i'^k.$$

The quantities X_i^k, Y_i^k, $X_i'^k$, $Y_i'^k$ will be functions of w_i, x_i^k, y_i^k, x_i^0, and of the same primed quantities. These functions will be periodic with respect to w and w'.

As in no. 127, let us see on what variables all these quantities depend. Since

$$\frac{dF_0}{dy_i} = \frac{dF_0}{dx_i'} = \frac{dF_0}{dy_i'} = 0,$$

it is obvious that X_i^k, $X_i'^k$, $Y_i'^k$ depend solely on

$$\begin{array}{llll} x_i^0, & x_i^1, & \ldots, & x_i^{k-1}, \\ y_i^0, & y_i^1, & \ldots, & y_i^{k-1} \end{array}$$

and on the same primed quantities, whereas Y_i^k will depend, in addition, on x_i^k but not on $x_i'^k$, y_i^k, and $y_i'^k$.

Let us next consider the expression

$$\sum_k n_k \frac{dx_i}{dw_k}.$$

Let us there substitute the term x_i by its expansion (14.2) and the terms n_k by their expansion in powers of μ. This expression can then be expanded in powers of μ; in order to use notations analogous to those given in no. 127, we will write its expansion in the following form:

$$\begin{aligned} \sum_k n_k \frac{dx_i}{dw_k} &= \sum{}' \mu^p n_k^0 \frac{dx_i^p}{dw_k} + \sum \mu^p n_k^p \frac{dx_i^0}{dw_k} - \sum \mu^p Z_i^p, \\ \sum_k n_k \frac{dy_i}{dw_k} &= \sum{}' \mu^p n_k^0 \frac{dy_i^p}{dw_k} + \sum \mu^p n_k^p \frac{dy_i^0}{dw_k} - \sum \mu^p T_i^p. \end{aligned} \tag{14.4}$$

We will agree that the sign Σ expresses a summation extending over all values of k and over all values of p from zero to infinity, and that the sign Σ' expresses a summation extending over all values of k and over all values of p from one to infinity.

It should be recalled here that $y_i^0 = w_i, y_i'^0 = w_i'$; in addition, two other equations of the same form can be added to Eqs. (14.4), where the symbols x_i, y_i, x_i^p, y_i^p, x_i^0, y_i^0, Z_i^p, and T_i^p are replaced by the same primed symbols.

In the same manner, we write

$$\sum_k n_k' \frac{dx_i}{dw_k'} = \sum{}' \mu^p n_k'^1 \frac{dx_i^{p-1}}{dw_k'} + \sum \mu^p n_k'^p \frac{dx_i^0}{dw_k'} - \sum \mu^p U_i^p. \tag{14.5}$$

We will agree that the summation Σ extends over all values of p, from one to infinity, while the summation Σ' extends over all values of p from two to infinity; to Eq. (14.5) we will add three more equations of the same form in which the symbols

$$x_i,\quad x_i^{p-1},\quad x_i^0,\quad U_i^p$$

will be, respectively, replaced by

$$y_i,\quad y_i^{p-1},\quad y_i^0,\quad V_i^p,$$

or by

$$x_i',\quad x_i'^{p-1},\quad x_i'^0,\quad U_i'^p,$$

or else by

$$y_i',\quad y_i'^{p-1},\quad y_i'^0,\quad V_i'^p.$$

This will yield a series of equations analogous to Eqs. (9.15) of no. 127, which we will write, noting that x_i^0 and $x_i'^0$ are constants and that y_i^0 and $y_i'^0$ reduce to w_i and w_i',

$$\begin{aligned}
&\sum_k n_k^0 \frac{dx_i^p}{dw_k} + \sum_k n_k'^1 \frac{dx_i^{p-1}}{dw_k'} = X_i^p + Z_i^p + U_i^p,\\
&\sum_k n_k^0 \frac{dy_i^p}{dw_k} + \sum_k n_k'^1 \frac{dy_i^{p-1}}{dw_k'} = Y_i^p + T_i^p + V_i^p - n_i^p,\\
&\sum_k n_k^0 \frac{dx_i'^p}{dw_k} + \sum_k n_k'^1 \frac{dx_i'^{p-1}}{dw_k'} = X_i'^p + Z_i'^p + U_i'^p,\\
&\sum_k n_k^0 \frac{dy_i'^p}{dw_k} + \sum_k n_k'^1 \frac{dy_i'^{p-1}}{dw_k'} = Y_i'^p + T_i'^p + V_i'^p - n_i'^p.
\end{aligned} \tag{14.6}$$

For $p = 0$, the left-hand side of each of the equations of system (14.6) must be eliminated, as must be the second term on the left-hand side for

$p = 1$. To demonstrate this, it is sufficient to recall the conventional meaning attributed to the signs Σ and Σ' in Eqs. (14.4) and (14.5).

Now, let U be any periodic function of w_i and w_i'. Let us agree to represent, by $[U]$, the mean value of U which, for the moment, is considered as a periodic function of w_i only. From this definition, it follows that

$$\left[\frac{dU}{dw_i}\right] = 0, \quad \left[\frac{dU}{dw_i'}\right] = \frac{d[U]}{dw_i'}.$$

Then we will represent, by $[[U]]$, the mean value of U considered as a periodic function of both w_i and w_i'. This is a constant independent of w and w', whereas $[U]$ is independent of w but is still a periodic function of w'. Next, let us take the mean values of the two sides of Eqs. (14.6), so that

$$\begin{aligned}
\sum n_k'^1 \frac{d[x_i^{p-1}]}{dw_k'} &= [X_i^p + Z_i^p + U_i^p], \\
\sum n_k'^1 \frac{d[y_i^{p-1}]}{dw_k'} &= [Y_i^p + T_i^p + V_i^p] - n_i^p, \\
\sum n_k'^1 \frac{d[x_i'^{p-1}]}{dw_k'} &= [X_i'^p + Z_i'^p + U_i'^p], \\
\sum n_k'^1 \frac{d[y_i'^{p-1}]}{dw_k'} &= [Y_i'^p + T_i'^p + U_i'^p] - n_i'^p.
\end{aligned} \tag{14.7}$$

The left-hand sides must be eliminated for $p = 1$.

Let us see how Eqs. (14.6) and (14.7) will permit calculation of the coefficients of the expansion (14.2).

In Eqs. (14.6), let us first put $p = 0$. This yields (since

$$Z_i^0, \quad U_i^0, \quad \ldots, \quad \text{are zero}$$

and since the first members must be omitted as stipulated above)

$$0 = 0, \quad Y_i^0 = n_i^0, \quad Y_i'^0 = n_i'^0.$$

These equations furnish the values of n_i^0, which are already known, and they indicate that the quantities $n_i'^0$ are zero since the quantities $Y_i'^0$ are zero.

Let us now consider Eqs. (14.7) by setting $p = 1$; this yields (noting that Z_i^1, U_i^1, . . . are zero)

$$\begin{aligned}
&[X_i^1] = [X_i'^1] = 0, \\
&[Y_i^1] = n_i^1, \quad [Y_i'^1] = n_i'^1.
\end{aligned} \tag{14.8}$$

To interpret these equations, it is useful to define the form of the quantities $[X_i^1]$, ... To obtain X_i^1, $X_i'^1$, and $Y_i'^1$, it is necessary to consider the derivatives

$$\frac{dF_1}{dy_i}, \quad \frac{dF_1}{dy'_i}, \quad -\frac{dF_1}{dx'_i}$$

and to replace there the terms x_i, x'_i, y_i, y'_i by $x_i^0, x_i'^0, w_i, w'_i$.

Let F_1^* be the result of this substitution in F_1, so that

$$X_i^1 = \frac{dF_1^*}{dw_i}, \quad X_i'^1 = \frac{dF_1^*}{dw'_i}, \quad Y_i'^1 = -\frac{dF_1^*}{dx_i'^0},$$

$$[F_1^*] = R^*,$$

where R^* is the result of the same substitution in R.

From this, we obtain

$$[X_i^1] = \frac{dR^*}{dw_i} = 0, \quad [X_i'^1] = \frac{dR^*}{dw'_i},$$

$$[Y_i'^1] = -\frac{dR^*}{dx_i'^0}.$$

According to our hypotheses, R does not depend on y and y' and thus R^* does not depend on w and w'.

Consequently, $[X_i^1]$ and $[X_i'^1]$ are zero and $[Y_i'^1]$ depends only on x_i^0 and $x_i'^0$ and therefore is a constant.

Of the four equations (14.8), the two first ones are satisfied identically. The fourth equation can yield $n_i'^1$ since the left-hand side is a constant.

From this (denoting by F_0^* the result of the substitution of x_i^0 for x_i in F_0) it follows that

$$Y_i^1 = -\sum_k \frac{d^2F_0^*}{dx_i^0 dx_k^0} x_k^1 - \frac{dF_1^*}{dx_i^0},$$

whence

$$n_i^1 = -\sum_k \frac{d^2F_0^*}{dx_i^0 dx_k^0} [x_k^1] - \frac{dR^*}{dx_i^0}. \tag{14.9}$$

The quantities n_i^1 must be constants, which is also true of dR^*/dx_i^0. Consequently, the same holds for the quantities $[x_k^1]$.

In fact, to obtain x_k^1 it is necessary to replace, in dS_1/dy_k (no. 134), the terms y and y' by w and w'; or, which comes to the same, if this substitution is made in S_1, we obtain

$$x_k^1 = \frac{dS_1}{dw_k}.$$

However,

$$S_1 = \sum_k \alpha_{1k} y_k + S'_1,$$

where S'_1 is periodic with respect to y and y' while the α_{1k} are constants. This will yield

$$x_k^1 = \alpha_{1k} + \frac{dS'_1}{dw_k},$$

whence

$$[x_k^1] = \alpha_{1k}.$$

Q. E. D.

Let us consider the first of the equations of system (14.7) by setting there $p = 2$. If the terms $[x_i^1]$ are constants, the following will remain:

$$[X_i^2 + Z_i^2 + U_i^2] = 0.$$

However, it follows from the definitions that

$$[Z_i^p] = 0, \quad U_i^2 = 0.$$

Consequently, we have

$$[X_i^2] = 0.$$

This conclusion, which has been obtained on the basis of the feasibility of the expansion demonstrated in the preceding chapters, can also be obtained directly.

In fact, we have

$$\begin{aligned} X_i^2 = \sum \frac{d^2F_1}{dw_i dw_k} y_k^1 + \sum \frac{d^2F_1}{dw_i dw'_k} y_k'^1 \\ + \sum \frac{d^2F_1}{dw_i dx_k^0} x_k^1 + \sum \frac{d^2F_1}{dw_i dx_k'^0} x_k'^1 + \frac{dF_2}{dw_i}. \end{aligned} \tag{14.10}$$

It goes without saying that, in F_1 and F_2, the quantities $y_i, x_i, \ldots$ are assumed as replaced by $w_i, x_i^0, \ldots$.

It is obvious that the mean value of dF_2/dw_i is zero. It remains to prove that the algebraic sum of the mean values of the first four terms on the right-hand side of Eq. (14.10) is also zero.

In fact, let us assume that the expressions

$$\frac{d^2F_1}{dw_i dw_k}, \quad y_k^1, \quad \frac{d^2F_1}{dw_i dw'_k}, \quad y_k'^1, \quad \ldots$$

are expanded in trigonometric series, in the sines and cosines of multiples of w_i. Thus X_i^2 will be expanded in a series of the same form, and it becomes a question of calculating the terms of this series that are independent of w_i.

For this, it is sufficient to calculate the terms independent of w_i in the product

$$\frac{d^2F_1}{dw_i dw_k} y_k^1$$

and in any other analogous products.

However, the constant terms of this product are obtained by considering a term of

$$\frac{d^2F_1}{dw_i dw_k},$$

depending on

$$\cos(m_1w_1 + m_2w_2 + \cdots + m_qw_q)$$

(if it is assumed that the number of w_i is equal to q) or on

$$\sin(m_1w_1 + m_2w_2 + \cdots + m_qw_q),$$

by means of a term of y_k^1 depending on the same cosine or on the same sine.

Let us note first that we can disregard the case in which

$$m_1 = m_2 = \cdots = m_q = 0.$$

In fact, since

$$\frac{d^2F_1}{dw_i dw_k}$$

is a derivative with respect to w_i of a function periodic with respect to w, it cannot contain terms independent of w. This is of some importance; in fact, it follows from this that it no longer is necessary to calculate the quantities

$$[y_k^1], \quad [x_k^1], \quad \ldots .$$

Now, Eqs. (14.6) will indeed yield y_k^1, x_k^1, ... to within an arbitrary function of w, but they will not yield the mean values of these functions. Fortunately, as we have just seen, we do not need them.

Thus let

$$m_1, \quad m_2, \quad \ldots, \quad m_q$$

be any system of positive or negative integers not all of which are zero at the same time. Let us put

$$m_1w_1 + m_2w_2 + \cdots + m_qw_q = h.$$

In the two factors of each of the terms on the right-hand side of Eq. (14.10), we will search for the terms of cos h and in sin h, and we will check whether they give the terms independent of w in X_i^2.

Therefore, let

$$A \cos h + B \sin h$$

be the terms of F_1 depending on h. It is obvious that A and B will be functions of x_i^0, $x_i'^0$, and w_i'.

The corresponding terms will be

$$\text{in } \frac{d^2F_1}{dw_i dw_k}: \quad -m_i m_k (Ac + B_5),$$

$$\text{in } \frac{d^2F_1}{dw_i dw_k'}: \quad m_i\left(-\frac{dA}{dw_k'}s + \frac{dB}{dw_k'}c\right),$$

$$\text{in } \frac{d^2F_1}{dw_i dw_k^0}: \quad m_i\left(-\frac{dA}{dx_k''}s + \frac{dB}{dx_k^0}c\right),$$

$$\text{in } \frac{d^2F_1}{dw_i dx_k'^0}: \quad m_i\left(-\frac{dA}{dx_k'^0}s + \frac{dB}{dx_k'^0}c\right)$$

(where, for abbreviation, the symbols s and c are used instead of $\sin h$ and $\cos h$).

We now have Eqs. (14.6), which become, setting $p = 1$,

$$\sum n_k^0 \frac{dx_i^1}{dw_k} = +\frac{dF_1}{dw_i},$$

$$\sum n_k^0 \frac{dy_i^1}{dw_k} = -\frac{dF_1}{dx_i},$$

together with two other equations where the symbols x_i^1, y_i^1, w_i (but not w_k), and x_i^0 are replaced by the same primed symbols. Thus if we put

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_q n_q^0 = \frac{1}{M},$$

it will be seen that the terms in $\sin h$ and $\cos h$ will be

$$\text{in } y_k^1: \quad -M\left(\frac{dA}{dx_k^0}s - \frac{dB}{dx_k^0}c\right),$$

$$\text{in } y_k'^1: \quad -M\left(\frac{dA}{dx_k'^0}s - \frac{dB}{dx_k'^0}c\right),$$

$$\text{in } x_k^1: \quad +Mm_k(Ac + Bs),$$

$$\text{in } x_k'^1: \quad +M\left(\frac{dA}{dw_k'}s - \frac{dB}{dw_k'}c\right).$$

Substituting into the right-hand side of Eq. (14.10), we will find that all these terms vanish. As predicted, we will thus have

$$[X_i^2] = 0.$$

After this, setting $p = 1$ in Eqs. (14.6), it is easy to calculate

$$y_i^1, \quad x_i^1, \quad y_i'^1, \quad x_i'^1$$

to within an arbitrary function of w'.

In this manner, we know n_i^0, n_i^1, $n_i'^1$, and

$$y_i^1 - [y_i^1], \quad x_i^1 - [x_i^1], \quad y_i'^1 - [y_i'^1], \quad x_i'^1 - [x_i'^1].$$

We also know that $[x_i^1]$ is a constant, i.e., a function of x_i^0 and $x_i'^0$. According to the remark made at the beginning and analogous to that at the end of no. 126, we know that this function can be arbitrarily selected. Thus we can conclude that x_i^1 is completely known.

Next, we have to determine

$$[x_i'^1] \quad \text{and} \quad [y_i'^1].$$

For this, we make use of Eqs. (14.7) by setting there $p = 2$. Noting that

$$[Z_i'^2] = [U_i'^2] = [T_i'^2] = [V_i'^2] = 0,$$

we can show that these equations become

$$\begin{aligned} \sum n_k'^1 \frac{d\,[x_i'^1]}{dw_k'} &= [X_i'^2], \\ \sum n_k'^1 \frac{d\,[y_i'^1]}{dw_k'} &= [Y_i'^2] - n_i'^2. \end{aligned} \tag{14.11}$$

We have given above [Eq. (14.10)] the expression for X_i^2. To derive from this the expression for $X_i'^2$ it is sufficient to change there w_i into w_i'; for deriving the expression for $Y_i'^2$, it is sufficient to change w_i into $x_i'^0$.

Thus in $[X_i'^2]$, for example, we will have terms of the form

$$\left[\frac{d^2F_1}{dw_i'dw_k} y_k^1\right], \quad \left[\frac{d^2F_1}{dw_i'dx_k^0} x_k^1\right], \quad \ldots \tag{14.12}$$

from which one easily finds

$$\left[\frac{d^2F_1}{dw_i'dw_k} y_k^1\right] = \left[\frac{d^2F_1}{dw_i'dw_k} (y_k^1 - [y_k^1])\right] + \frac{d^2R^*}{dw_i'dw_k} [y_k^1];$$

$$\left[\frac{d^2F_1}{dw_i'dx_k^0} x_k^1\right] = \left[\frac{d^2F_1}{dw_i'dx_k^0} (x_k^1 - [x_k^1])\right] + \frac{d^2R^*}{dw_i'dx_k^0} [x_k^1]; \quad \ldots .$$

However, by hypothesis, R depends only on x_i and x_i' so that the derivatives of R^* with respect to w_i' are zero. From this, we can draw the following conclusion:

The terms (14.12) that enter the second member of the first equation of the system (14.11) depend only on

$$y_k^1 - [y_k^1], \quad x_k^1 - [x_k^1], \quad \ldots$$

which are known but do not depend on $[y_k^1]$, $[x_k^1]$, ... which are unknown. Consequently, $[X_i'^2]$ is a known function of w' and therefore, it is possible to derive from this the value of $[x_i'^1]$, under one condition, namely, that

$$[[X_i'^2]] = 0.$$

This condition must be satisfied identically, since we know in advance that the expansion is feasible.

For the same reason, $[Y_i'^2]$ is a known function. In fact, we now know $[x_k^1]$, $[x_k'^1]$ but do not yet know $[y_k^1]$ nor $[y_k'^1]$. However, the terms of $[Y_i'^2]$ that depend on $[y_k^1]$ and on $[y_k'^1]$ are written in the form

$$-\sum \frac{d^2R^*}{dx_i'^0 dw_k}[y_k^1] - \sum \frac{d^2R^*}{dx_i'^0 dw_k'}[y_k'^1],$$

and, since R^* does not depend on w or on w', they are zero so that the second equation of (14.11), added to

$$[[Y_i'^2]] = n_i'^2,$$

will yield $n_i'^2$ and $[y_i'^1]$.

Having thus determined $[x_i'^1]$ and $[y_i'^1]$ by means of Eqs. (14.7:3,2) and (14.7:4,2), i.e., by the third and fourth equations of system (14.7) where p is set equal to 2, we can now proceed to determine $[x_i^2]$.

The simplest way is to make use of the fact that the expression

$$\sum x_i dy_i + \sum x_i' dy_i'$$

must be an exact differential.

If, in this expression, we replace $x_i, y_i, \ldots$ by their expansions (14.2), then the coefficient of each of the powers of μ will become an exact differential. The following differentials

$$\sum' x_i^0 dw_i,$$

$$\sum' (x_i^1 dw_i + x_i^0 dy_i^1),$$

$$\sum' (x_i^2 dw_i + x_i^1 dy_i^1 + x_i^0 dy_i^2),$$

$$\sum' (x_i^3 dw_i + x_i^2 dy_i^1 + x_i^1 dy_i^2 + x_i^0 dy_i^3),$$

$$\vdots$$

thus must be also exact. The sign Σ' is to mean that the summation must be extended over all indices i and, in addition, over all primed symbols.

If, for example, there are q symbols y_i without prime and λ symbols y_i' with prime, we will have

$$\sum{}' x_i^0 dw_i = x_1^0 dw_1 + x_2^0 dw_2 + \cdots + x_q^0 dw_q$$

$$+ x_1'^0 dw_1' + x_2'^0 dw_2' + \cdots + w_\lambda'^0 dw_\lambda .$$

Since, on the other hand, the quantities x_i^0 and $x_i'^0$ are constants, then

$$\sum{}' x_i^0 dy_i^k$$

will always by an exact differential in such a manner that we can write

$$\begin{aligned} &\sum{}' x_i^1 dw_i = d\varphi_1, \\ &\sum{}' \left(x_i^2 dw_i + \sum{}' x_i^1 dy_i^1 \right) = d\varphi_2, \\ &\sum{}' \, (x_i^3 dw_i + x_i^2 dy_i^1 + x_i^1 + x_i^1 dy_i^2) = d\varphi_3, \\ &\qquad \vdots \end{aligned} \tag{14.13}$$

In addition, $\varphi_1, \varphi_2, \varphi_3, \ldots$ must be functions of w and w' whose derivatives are periodic.

Let us see how the equation

$$\sum{}' x_i^2 dw_i + \sum{}' x_i^1 dy_i^1 = d\varphi_2$$

will permit us to determine $[x_i^2]$. This equation yields

$$x_k^2 + \sum{}' x_i^1 \frac{dy_i^1}{dw_k} = \frac{d\varphi_2}{dw_k} .$$

However, since the derivatives of φ_2 must be periodic, we will have

$$\left[\frac{d\varphi_2}{dw_k} \right] = \text{const.},$$

which furnishes

$$[x_k^2] + \sum \left[x_i^1 \frac{dy_i^1}{dw_k} \right] + \sum \left[x_i'^1 \frac{dy_i'^1}{dw_k} \right] = \text{const.} \tag{14.14}$$

In Eq. (14.14) everything is known except $[x_k^2]$. In fact, we do know x_i^1, $x_i'^1$, $y_i'^1$, and we also know dy_i^1/dw_k since $y_i^1 - [y_i^1]$ is known. With respect to the constant on the right-hand side, one of our previous remarks shows that it can be arbitrarily chosen.

Thus it becomes possible to calculate $[x_k^2]$.

Let us now calculate $[y_i^1]$ by means of Eq. (14.7:2,2). This equation is written as

$$\sum n_1'^k \frac{d\,[y_i^1]}{dw_k'} = [Y_i^2] - n_i^2, \tag{14.15}$$

from which we obtain by equating the mean values taken with respect to w',

$$[[Y_i^2]] = n_i^2. \tag{14.16}$$

However, Y_i^2 depends only on $x_i^1, y_i^1, x_i'^1, y_i'^1$, and x_i^2. The quantities x_i^1, $x_i'^1$, and $y_i'^1$ are completely known. Conversely, we only know $y_i^1 - [y_i^1]$ and $[x_i^2]$. Let us see how Y_i^2 depends on x_i^2 and y_i^1. We obtain

$$Y_i^2 = -\sum \frac{d^2F_0}{dx_i^0 dx_k^0} x_k^2 - \sum \frac{d^2F_1}{dx_i^0 dw_k} y_k^1 + A,$$

where A is completely known.

From this we derive

$$\begin{aligned}[Y_i^2] = &-\sum \frac{d^2F_0}{dx_i^0 dx_k^0}\,[x_k^2] \\ &-\sum \left[\frac{d^2F_1}{dx_i^0 dw_k}\,(y_k^1 - [y_k^1])\right] \\ &-\sum \frac{d^2R^*}{dx_i^0 dw_k}\,[y_k^1] + [A].\end{aligned}$$

Since R^* does not depend on w_k and since the quantities $d^2R^*/dx_i^0dw_k$ are zero, $[Y_i^2]$ is completely known and Eqs. (14.16) and (14.15) will yield n_i^2 and $[y_i^1]$.

Subsequently and successively, we will determine $x_i^2 - [x_i^2]$ from Eqs. (14.6:1,2); $x_i'^2 - [x_i'^2]$ from Eq. (14.6:3,2); $y_i'^2 - [y_i'^2]$ from Eq. (14.6:4,2); $y_i^2 - [y_i^2]$ from Eq. (14.6:2,2); $[x_i'^2]$ from Eq. (14.7:3,3); $[y_i'^2]$ and $n_i'^3$ from Eqs. (14.7:4,3); $[x_i^3]$ from Eq. (14.14:3) [namely by an equation derived from the third equation of system (14.13) as Eq. (14.14) had been derived from the second equation of system (14.13) above]; $[y_i^2]$ and n_i^3 from Eq. (14.7:2,3); $x_i^3 - [x_i^3]$, $x_i'^3 - [x_i'^3]$, $y_i'^3 - [y_i'^3]$, $y_i^3 - [y_i^3]$, $[x_i'^3]$, $[y_i'^3]$ and $n_i'^4$, $[x_i^4]$, $[y_i^3]$ and n_i^4, etc., etc.

If care is taken to proceed with the calculation in this sequence, one would never be stopped, since each equation contains only one unknown that must be determined.

In addition, let us recall that the mean values

$$[[x_i^k]], \quad [[y_i^k]], \quad [[x_i'^k]], \quad [[y_i'^k]]$$

can be arbitrarily chosen as functions of x_i^0 and $x_i'^0$.

To make the integration possible, certain conditions must be satisfied. However, we know they are satisfied (which, no doubt, is also easy to demonstrate directly) since we know in advance that the expansion is feasible.

Application to the Three-Body Problem

152. In Chap. 11, we demonstrated the manner in which the principles of no. 134 are applicable to the three-body problem. Obviously, this is the same for the results of the preceding number, which are derived directly from these principles. In Chap. 11 we adopted successively the variables

$$\begin{matrix} \Lambda, & \Lambda', & \sigma_i, \\ \lambda_1, & \lambda_1', & \tau_i, \end{matrix} \tag{14.17}$$

$$\begin{matrix} \Lambda, & \Lambda', & \rho_i, \\ \lambda_1, & \lambda_1', & \omega_i, \end{matrix} \tag{14.18}$$

$$\begin{matrix} \Lambda, & \Lambda', & V_i, \\ \lambda_2, & \lambda_2', & V_i. \end{matrix} \tag{14.19}$$

Using the system (14.19), the equations of motion assume the same form as those in no. 134 and no. 151.

However, the change of variables that permits passing from the system (14.18) to the system (14.19), is rather laborious, and in most cases the eccentricities are so small that this change can be avoided by the artifice described at the end of no. 140. Let us recall here the basic principle. In the function F, the terms of μF_1 that depend on powers of the eccentricities and inclinations higher than the third are very small. Thus if we put

$$\mu F_1 = \mu F_1' + \mu^2 F_2',$$

where $\mu F_1'$ represents the ensemble of terms of at most the third degree and $\mu^2 F_2'$ represents terms of at least the fourth degree, then $\mu^2 F_2'$ will be very small and F_2' will be finite. In that case, we can write

$$F = F_0 + \mu F_1' + \mu^2 (F_2' + F_2) + \mu^3 F_3 + \cdots$$

and F will still be expanded in powers of μ. However, considering F_1' as a periodic function of λ_1 and λ_1', its mean value will not depend on ω_i, such that with variables (14.18) the conditions in no. 134 will be satisfied.

It is true that the significance of the parameter μ thus differs slightly from that ordinarily attributed to it; however, this is of little importance since the purpose of this parameter is merely to demonstrate the order of magnitude of the various terms.

Once these conventions are made, the results of the preceding number become directly applicable to the problems in question here. However, to avoid the difficulties discussed in Chap. 12, we will replace variables (14.18) by variables (14.17); this will cause some modifications in these results, which must be discussed in some detail. To obtain greater symmetry in the notations, we will substitute, in the remainder of this chapter, λ and λ' for λ_1 and λ_1'; F_1 for F_1'; and F_2 for $F_2' + F_2$. This cannot lead to confusion.

It is known that variables (14.18), from the formal viewpoint, can be expanded in ascending powers of μ in the following manner:

$$\Lambda = \sum \mu^k \Lambda_k, \quad \Lambda' = \sum \mu^k \Lambda_k',$$

$$\lambda = \sum \mu^k \lambda_k, \quad \lambda' = \sum \mu^k \lambda_k', \tag{14.20}$$

$$\rho_i = \sum \mu^k \rho_i^k, \quad \omega_i = \sum \mu^k \omega_i^k .$$

In this, $\Lambda_k, \Lambda_k', \lambda_k, \lambda_k', \rho_i^k, \omega_i^k$ are periodic functions of w and w' except for the case of $k = 0$; Λ_0, Λ_0', and ρ_i^0 are constants, while λ_0 and λ_0' reduce to w_1 and w_2 and ω_i^0 to w_i'.

Adopting variables (14.17) we will have at the same time

$$\sigma_i = \sum \mu^k \sigma_i^k, \quad \tau_i = \sum \mu^k \tau_i^k, \tag{14.21}$$

where σ_i^k and τ_i^k will be periodic functions of w and w' so that, on abbreviating the constant $\sqrt{2\rho_i^0}$ as x_i^0, we obtain

$$\sigma_i^0 = \sqrt{2\rho_i^0} \cos \omega_i^0 = x_i^0 \cos \omega_i',$$
$$\tau_i^0 = \sqrt{2\rho_i^0} \sin \omega_i^0 = x_i^0 \sin \omega_i'.$$ [R12]

Let us add that, since

$$\Lambda\, d\lambda + \Lambda' d\lambda' + \sum \rho_i d\omega_i$$

must be an exact differential, this must also be so for

$$\Lambda\, d\lambda + \Lambda' d\lambda' + \sum \sigma_i d\tau_i,$$

since

$$\sum \sigma_i d\tau_i = \sum \rho_i d\omega_i + \tfrac{1}{2} \sum d(\sigma_i \tau_i).$$

If the same meaning as in the preceding number is given to w_i, w_i', n_i, n_i', . . . , then our equations will be written as

$$\sum n_k \frac{d\Lambda}{dw_k} + \sum n_k' \frac{d\Lambda}{dw_k'} = \frac{dF}{d\lambda}, \tag{14.22}$$

which is an equation analogous to Eq. (14.19) of the preceding number, just as expansions (14.21) are analogous to expansions (14.18) of the preceding number.

Naturally, Eq. (14.22) must be supplemented by other equations in which the symbols Λ and λ are replaced, respectively, by Λ' and λ'; λ and $-\Lambda$ by λ' and $-\Lambda'$; σ_i and τ_i by τ_i and $-\sigma_i$. We should add that the number of parameters w is 2 in the three-body problem and $n-1$ in the n-body problem; the number of parameters w' is 4 in the three-body problem and $2n-2$ in the n-body problem in the three-dimensional case while it is only $n-1$ in the n-body problem in the two-dimensional case.

Let us substitute expansions (14.21) and those of n_i and n_i' into Eqs. (14.22), so that the two sides of these equations can be expanded in powers of μ, and we can write

$$\frac{dF}{d\lambda} = \sum \mu^p L_p, \qquad \frac{dF}{d\lambda'} = \sum \mu^p L_p',$$

$$\frac{dF}{d\Lambda} = \sum \mu^p l_p, \qquad -\frac{dF}{d\Lambda'} = \sum \mu^p l_p',$$

$$\frac{dF}{d\tau_i} = \sum \mu^p S_i^p, \qquad -\frac{dF}{d\sigma_i} = \sum \mu^p \Theta_i^p,$$

which are equations analogous to Eqs. (9.10) and (9.11) of no. 127 and to other equations encountered in the preceding number.

Let us continue with the calculation as in the preceding number and let us put

$$\sum_k n_k \frac{d\Lambda}{dw_k} = \sum' \mu^p n_k^0 \frac{d\Lambda_p}{dw_k} + \sum \mu^p n_k^p \frac{d\Lambda_0}{dw_k} - \sum \mu^p Z_p,$$

which are equations in which the signs Σ and Σ' have the same meaning as in Eq. (14.21) of the preceding number and to which other equations must be added in which the symbols

$$\Lambda, \quad \Lambda_p, \quad \Lambda_0, \quad Z_p$$

are replaced by the same primed symbols, or by

$$\lambda, \quad \lambda_p, \quad \lambda_0, \quad T_p,$$

or by the same primed symbols, or by

$$\sigma_i, \quad \sigma_i^p, \quad \sigma_i^0, \quad Z_i^p,$$

or, finally, by

$$\tau_i, \quad \tau_i^p, \quad \tau_i^0, \quad T_i^p.$$

Clearly one must try not to confuse the symbols Z_p and Z_i^p, T_p and T_i^p.

Similarly, we will put

$$\sum_k n_k' \frac{d\Lambda}{dw_k'} = \sum{}' \mu^p n_k'^1 \frac{d\Lambda_{p-1}}{dw_k'} + \sum \mu^p n_k'^p \frac{d\Lambda_0}{dw_k'} - \sum \mu^p U_p,$$

which is an equation in which the signs Σ and Σ' have the same meaning as in Eq. (14.22) of the preceding number, and to which other equations of the same form should be added, in which the symbols

$$\Lambda, \quad \Lambda_{p-1}, \quad \Lambda_0, \quad U_p$$

are replaced by the same primed symbols or by

$$\lambda, \quad \lambda_{p-1}, \quad \lambda_0, \quad V_p,$$

or by the same primed symbols, or by

$$\sigma_i, \quad \sigma_i^{p-1}, \quad \sigma_i^0, \quad U_i^p$$

or, finally, by

$$\tau_i, \quad \tau_i^{p-1}, \quad \tau_i^0, \quad V_i^p.$$

Now, we can write a series of equations analogous to Eqs. (14.6).

If, to shorten the notation, we put for any function u,

$$\sum_k n_k^0 \frac{du}{dw_k} = \Delta u,$$

$$\sum_k n_k'^1 \frac{du}{dw_k'} = \Delta' u,$$

then these equations will be written in the form

$$\begin{aligned}
\Delta\Lambda_p + \Delta'\Lambda_{p-1} &= L_p + Z_p + U_p,\\
\Delta\lambda_p + \Delta'\lambda_{p-1} &= l_p + T_p + V_p - n_1^p,\\
\Delta\sigma_i^p + \Delta'\sigma_i^{p-1} &= S_i^p + Z_i^p + U_i^p + n_i'^p x_i'^0 \sin w_i',\\
\Delta\tau_i^p + \Delta'\tau_i^{p-1} &= \Theta_i^p + T_i^p + V_i^p - n_i'^p x_i'^0 \cos w_i'.
\end{aligned} \tag{14.23}$$

To the two first equations of system (14.23), it is convenient to add two other equations which differ from the former since all symbols there are primed except for n_1^p which is replaced by n_2^p. Just as in the preceding number, the left-hand side must be eliminated for $p=0$ and the second term on the left-hand side, for $p=1$.

On equating the mean values of the two sides with respect to w, equations analogous to Eqs. (14.7) will be obtained. These are written as

$$\begin{aligned}\Delta'[\Lambda_{p-1}] &= [L_p+Z_p+U_p],\\ \Delta'[\lambda_{p-1}] &= [l_p+T_p+V_p]-n_1^p,\\ \Delta'[\sigma_i^{p-1}] &= [S_i^p+Z_i^p+U_i^p]+n_i'^p x_i'^0 \sin w_i',\\ \Delta'[\tau_i^{p-1}] &= [\Theta_i^p+T_i^p+V_i^p]-n_i'^p x_i'^0 \cos w_i'.\end{aligned} \tag{14.24}$$

For $p=0$ and 1, the left-hand side must be eliminated.

Let us then add equations analogous to Eqs. (14.13) and (14.14).

Indeed, we have seen that

$$\Lambda\, d\lambda + \Lambda' d\lambda' + \sum \sigma_i d\tau_i$$

must be the exact differential of a function all of whose derivatives are periodic; the same will then also be true for the following expressions:

$$\sum \Lambda_0\, d\lambda_0 + \sum \sigma_i^0 d\tau_i^0,$$

$$\sum (\Lambda_1 d\lambda_0 + \Lambda_0\, d\lambda_1) + \sum (\sigma_i^1 d\tau_i^0 + \sigma_i^0 d\tau_i^1),$$

$$\sum (\Lambda_2 d\lambda_0 + \Lambda_1 d\lambda_1 + \Lambda_0 d\lambda_2) + \sum (\sigma_i^2 d\tau_i^0 + \sigma_i^1 d\tau_i^1 + \sigma_i^0 d\tau_i^2), \quad \ldots .$$

In each of these expressions, the first sign Σ extends over the two planets, in such a manner—for example—that

$$\sum \Lambda_0\, d\lambda_0 = \Lambda_0\, d\lambda_0 + \Lambda_0' d\lambda_0'.$$

If, for the moment, we consider w' as constants and w as the only variables, these expressions will *a fortiori* remain exact differentials while $d\tau_i^0$ and $d\sigma_i^0$ will be zero, so that

$$\sigma_i^p d\tau_i^0 \quad \text{and} \quad \sigma_i^0 d\tau_i^p = d(\sigma_i^0 \tau_i^p)$$

will be exact differentials. Since this holds also for $\Lambda_0\, d\lambda_p$ and since $\lambda_0 = w_1$, $\lambda_0' = w_2$, the expressions

$$\Lambda_1 dw_1 + \Lambda_1' dw_2,$$

$$\Lambda_2\, dw_1 + \Lambda_2' dw_2 + \sum \Lambda_1 d\lambda_1 + \sum \sigma_i^1 d\tau_i^1,$$

$$\Lambda_3\, dw_1 + \Lambda_3' dw_2 + \sum (\Lambda_2\, d\lambda_1 + \Lambda_1 d\lambda_2) + \sum (\sigma_i^2 d\tau_i^1 + \sigma_i^1 d\tau_i^2), \quad \ldots$$

will also be exact differentials of functions whose derivatives are periodic and, consequently, whose derivatives with respect to w_1 and w_2 have a mean value independent of w'.

Reasoning as in the preceding number, where we derived Eqs. (14.14) from Eqs. (14.13), we will find

$$
\begin{aligned}
&[\Lambda_1] = \text{const.},\\
&[\Lambda_2] + \sum\left[\Lambda_1 \frac{d\lambda_1}{dw_1}\right] + \sum\left[\sigma_i^1 \frac{d\tau_i^1}{dw_1}\right] = \text{const.},\\
&[\Lambda_3] + \sum\left[\Lambda_2 \frac{d\lambda_1}{dw_1} + \Lambda_1 \frac{d\lambda_2}{dw_1}\right] + \sum\left[\sigma_i^2 \frac{d\tau_i^1}{dw_1} + \sigma_i^1 \frac{d\tau_i^2}{dw_1}\right] = \text{const.},\\
&\qquad\vdots
\end{aligned}
\tag{14.25}
$$

Let us first consider Eqs. (14.23) by setting there $p = 0$. It is easy to demonstrate that these equations are satisfied identically, provided that (as we are assuming here) Λ_0 and Λ_0' are constants, that λ_0 and λ_0' reduce to w_1 and w_2, that σ_i^0 and τ_i^0 reduce to $x_i'^0 \cos w_i'$ and $x_i'^0 \sin w_i'$, that $n_i'^1$ is zero, and that n_i^0 has a suitable value.

Let us now pass to Eqs. (14.24) by setting there $p = 1$ so that, as in Eqs. (14.8), we obtain

$$
\begin{aligned}
&[L_1] = 0, \quad [l_1] = n_1^1,\\
&[S_i^1] = -n_i'^1\tau_i^0, \quad [\Theta_i^1] = n_i'^1\sigma_i^0.
\end{aligned}
\tag{14.26}
$$

As in the preceding number, it is obvious that $[L_1]$, $[S_i^1]$, and $[\Theta_i^1]$ are derivatives of R with respect to λ, τ_i, and $-\sigma_i$. Of course, it is necessary to replace, in R, the quantities $\Lambda, \lambda, \sigma_i$, and τ_i by $\Lambda_0, \lambda_0, \sigma_i^0$, and τ_i^0. Now, in Chap. 10 we found the expression for R, which is

$$\sum A_i(\sigma_i^2 + \tau_i^2) + B,$$

where B and A_i are functions of Λ and Λ'.

This demonstrates that Eqs. (14.26) except for the second one, are satisfied identically, provided that

$$n_i'^1 = -2A_i^0$$

(where A_i^0 and B_0 are what A_i and B become on replacing Λ by Λ_0), since

$$[L_1] = 0, \quad [S_i^1] = 2A_i^0\tau_i^0, \quad [\Theta_i^1] = -2A_i^0\sigma_i^0.$$

On the other hand,

$$[l_1] = -\frac{d^2F_0}{d\Lambda_0^2}[\Lambda_1] - \frac{d^2F_0}{d\Lambda_0\, d\Lambda_0'}[\Lambda_1'] - \sum \frac{dA_i^0}{d\Lambda_0}(x_i'^0)^2 - \frac{dB_0}{d\Lambda_0},$$

just as $[\Lambda_i]$ and $[\Lambda_i']$ must be constants as shown above, with $[l_1]$ also being a constant which makes it possible to equate it to n_1^1.

To continue the calculation, using the same sequence as in the preceding number, it will now be necessary to consider Eqs. (14.6:1,1), (14.6:3,1), (14.6:4,1).

The left-hand sides will reduce to

$$\Delta\Lambda_1, \quad \Delta\sigma_i^1, \quad \Delta\tau_i^1,$$

while the right-hand sides will be known and periodic functions of w and of w' whose mean value, with respect to w, will be zero since Eqs. (14.24) ($p = 1$) are satisfied.

Thus the integration can be performed as in the preceding number and as in no. 127, yielding

$$\Lambda_1 - [\Lambda_1], \quad \sigma_i^1 - [\sigma_i^1], \quad \tau_i^1 - [\tau_i^1].$$

Since we know that $[\Lambda_1]$ reduces to a constant and that this constant can be chosen arbitrarily, we can regard Λ_1 as completely known.

Let us consider Eq. (14.6:2,1) whose left-hand side reduces to $\Delta\lambda_i$. Since the right-hand side contained no unknown quantity other than Λ_1, this side becomes a known function of w and w' so that the above-applied procedure will furnish

$$\lambda_1 - [\lambda_1].$$

Next, $[\sigma_i^1]$ and $[\tau_i^1]$ must be determined on the basis of Eqs. (14.7:3,2) and (14.7:4,2). The right-hand side of these equations is not completely known. They actually depend not on $[\lambda_1]$ but on $[\Lambda_1]$, on $[\sigma_k^1]$, and on $[\tau_k^1]$. The terms that depend on these quantities can be written in the following form:

(i) In Eq. (14.7:3,2), for example,

$$\sum \frac{d^2R^*}{d\tau_i^0 d\Lambda_0}[\Lambda_1] + \sum \frac{d^2R^*}{d\tau_i^0 d\sigma_k^0}[\sigma_k^1] + \sum \frac{d^2R^*}{d\tau_i^0 d\tau_k^0}[\tau_k^1].$$

The first term is known since $[\Lambda_1]$ is known. According to the form of the function R^* given above, all second derivatives are zero except $d^2R^*/d\tau_i^{0\,2}$. The two last terms thus reduce to

$$2A_i^0[\tau_i^1] = -n_i'^1[\tau_i^1].$$

In addition, the right-hand side of Eq. (14.7:3,2) contains a term in $n_i'^2 x_i'^0 \sin w_i'$ and the right-hand side of Eq. (14.7:4,2) contains a term in $-n_i'^2 x_i'^0 \cos w_i'$, both of which contain the unknown quantity $n_i'^2$.

Thus the right-hand side of Eq. (14.7:3,2) is equal to $-n_i'^1[\tau_i^1]$ plus a known function of w' (and of $n_i'^2$). Similarly, the right-hand side of Eq.

(14.7:4,2) reduces to a known function of w' (and of $n_i'^2$), in such a manner that our equations become

$$\begin{aligned} \Delta'[\sigma_i^1] + n_i'^1[\tau_i^1] &= \varphi_1 + n_i'^2 x_i'^0 \sin w_i', \\ \Delta'[\tau_i^1] - n_i'^1[\sigma_i^1] &= \varphi_2 - n_i'^2 x_i'^0 \cos w_i', \end{aligned} \tag{14.27}$$

where φ_1 and φ_2 are known periodic functions of w'. For abbreviation, let

$$h = m_1' w_1' + m_2' w_2' + \cdots + m_q' w_q',$$

where m' are any integers and, similarly,

$$N = m_1' n_1'^1 + m_2' n_2'^1 + \cdots + m_q' n_q'^1.$$

Let

$$A_1 \cos h + B_1 \sin h,$$
$$A_2 \cos h + B_2 \sin h$$

be the terms in h in the known functions φ_1 and φ_2. In addition, let

$$C_1 \cos h + D_1 \sin h,$$
$$C_2 \cos h + D_2 \sin h$$

be the terms in h in the unknown functions $[\sigma_i^1]$ and $[\tau_i^1]$. It is now a question of calculating the coefficients C and D as a function of the coefficients A and B.

Equations (14.27) will yield, on identifying,

$$\begin{aligned} ND_1 + n_i'^1 C_2 &= A_1, \\ -NC_1 + n_i'^1 D_2 &= B_1, \\ ND_2 - n_i'^1 C_1 &= A_2, \\ -NC_2 - n_i'^1 D_1 &= B_2. \end{aligned} \tag{14.28}$$

These equations (14.28) make it possible to know the unknown coefficients C and D at least so long as the determinant is not zero. However, this determinant is equal to

$$[N^2 - (n_i'^1)^2]^2.$$

Thus the determinant can vanish only if

$$N = \pm n_i'^1,$$

i.e. (since no linear relation with integral coefficients exists between the quantities $n_k'^1$), if

$$h = \pm w_i'$$

or (since we need not consider separately the terms in h and in $-h$) if

$$h = w_i'.$$

Let us then identify all terms in w_i' by equating the two sides of Eq.

(14.27). For abbreviation, we will write h instead of w_i' (and N instead of $n_i'^1$), since it is assumed here that $h = w_i'$, and we continue to denote by A_1, $B_1, \ldots$, the coefficients of cos h and sin h in the functions φ_1, etc. Only, here Eqs. (14.28) no longer have the same form since the terms in $n_i'^2$ that enter the right-hand side of Eqs. (14.27) must be taken into consideration. Consequently, we obtain

$$\begin{aligned} N(D_1 + C_2) &= A_1, \\ N(-C_1 + D_2) &= B_1 + n_i'^2 x_i'^0, \\ N(D_2 - C_1) &= A_2 - n_i'^2 x_i'^0, \\ -N(C_2 + D_1) &= B_2. \end{aligned} \tag{14.29}$$

To have these equations become compatible, it is obviously necessary that

$$A_1 + B_2 = 0$$

and

$$A_2 - B_1 = 2n_i'^2 x_i'^0. \tag{14.30}$$

The first condition must be satisfied identically since we know that the expansion is feasible. The second condition will yield the value of $n_i'^2$.

Since these conditions are satisfied, Eqs. (14.29) are no longer distinct. They will yield C_1 and D_1 if C_2 and D_2 are known. We state that C_2 and D_2 can be selected arbitrarily. This will be proved by a reasoning analogous to that in no. 126. In fact, the form of the series is not changed by adding, to Λ_0, Λ_0', $x_i'^0$, w, and w', arbitrary functions of μ, A_0, and $x_i'^0$ divisible by μ. The number of these arbitrary functions is the same as that of the variables, i.e., 12 for the three-body problem in space. Thus these can be used for satisfying 12 conditions. One way is to utilize them such that the mean values of Λ, Λ', λ, λ' as well as the coefficients of cos w_i' and sin w_i' in the four functions τ_i, become arbitrary functions of μ and of the constants Λ_0 and $x_i'^0$. These functions must be expandable in powers of μ; when one considers separately the various terms of this series, one can see that the coefficients of cos w_i' and sin w_i' can be arbitrarily chosen in the various functions τ_i^p and, specifically, C_2 and D_2.

Consequently, Eqs. (14.27) will permit determining $[\sigma_i^1]$ and $[\tau_i^1]$.

Let us next determine $[\Lambda_2]$. For this, we will make use of the second equation of system (14.25) where everything is known except $[\Lambda_2]$.

This is the same for $[\Lambda_2']$.

Let us now calculate $[\lambda_1]$ by means of Eq. (14.7:2,2). This equation—compare with Eq. (14.7:2,2) of the preceding number and with our reasoning in using this equation for determining $[y_i^1]$—can be written as follows:

$$\Delta'[\lambda_1] = A - n_1^2,$$

where A is a wholly known periodic function of w'. This equation can be integrated if one equates n_1^2 to the mean value of the periodic function A, so that the mean value of the right-hand side becomes zero. The terms $[\lambda'_1]$ and n_2^2 are determined in the same manner. This leaves to determine, by the same procedures,

$\Lambda_2 - [\Lambda_2]$ by (14.6:1,2),

$\sigma_i^2 - [\sigma_i^2]$ by (14.6:3,2),

$\tau_i^2 - [\tau_i^2]$ by (14.6:4,2),

$\lambda_2 - [\lambda_2]$ by (14.6:2,2).

$[\sigma_i^2]$, $[\tau_i^2]$ and $n_i'^3$ by (14.7:3,3) and (14.7:4,3),

$[\Lambda_3]$ by the third equation of (14.25)

$[\lambda_2]$ and n_1^3 by (14.7:2,3),

and so on.

Diverse Properties

153. The six quantities Λ_p, λ_p, σ_i^p, τ_i^p, n_i^p, $n_i'^p$, defined in the preceding number are functions of Λ_0, w, $x_i'^0$, and w_i'. However, since we have

$$\sigma_i^0 = x_i'^0 \cos w_i', \quad \tau_i^0 = x_i'^0 \sin w_i',$$

we can also consider these as functions of Λ_0, w, σ_i^0, and τ_i^0. We intend to demonstrate that these functions can be expanded in powers of σ_i^0 and of τ_i^0.

This proposition is subject to another, obviously equivalent, statement. Let us return to the variables Λ_0, w, $x_i'^0$, w_i'; our functions $\Lambda_p, \ldots$ will be periodic with respect to w and to w' and, consequently, will be expandable in trigonometric series. Let

$$A \cos h \quad \text{or} \quad A \sin h$$

be a term of one of these series. We assume that

$$h = \sum m_k w_k + \sum m_k' w_k',$$

where m_k and m_k' are positive or negative integers. The coefficients A are functions of Λ_0 and of $x_i'^0$.

Thus our proposition can be formulated as follows:

The quantity A can be expanded in powers of $x_k'^0$. The series is divisible by

$$(x_i'^0)^{|m_i'|}$$

and all its terms contain $x_i'^0$ to an even power if m_i' is even or to an odd power if m_i' is odd.

To demonstrate this proposition, I will use a recursive argument. In the preceding number, we successively determined the functions $\Lambda_p, \ldots$ by a series of equations for which we will keep the same numbering as in the preceding number.

It is now a question of demonstrating that the values of the functions determined by these equations can be expanded in powers of σ_i^0 and τ_i^0.

We will note first that, since F can be expanded in powers of σ_i and τ_i, the functions denoted by L_p, l_p, S_i^p, Θ_i^p, can be expanded in powers of

$$\begin{array}{llllll} & \Lambda_1, & \Lambda_2, & \ldots, & \Lambda_p, & \ldots, \\ & \lambda_1, & \lambda_2, & \ldots, & \lambda_p, & \ldots, \\ \sigma_i^0, & \sigma_i^1, & \sigma_i^2, & \ldots, & \sigma_i^p, & \ldots, \\ \tau_i^0, & \tau_i^1, & \tau_i^2, & \ldots, & \tau_i^p, & \ldots. \end{array} \quad \text{(and the same letters primed)} \tag{14.31}$$

Recalling the significance of the quantities Z_p, etc., this means that the right-hand side of Eqs. (14.23) can be expanded in powers of quantities (14.31), of their derivatives with respect to w and w', and finally of n_i^p and $n_i'^p$.

We mean to prove that all these quantities, just as the right-hand side of Eqs. (14.23) and (14.24), can be expanded in powers of σ_i^0 and of τ_i^0. For this, we will review the sequence of operations by which, in the preceding number, we derived these quantities from one another, showing that none of them can change this property.

These operations are as follows:

(i) Replace on the right-hand side of Eqs. (14.23), the quantities (14.31), their derivatives, and the quantities n_i^p, and $n_i'^p$ by their previously calculated values. Since the right-hand sides of Eqs. (14.23) can be expanded in powers of the substituted quantities and since these substituted quantities are themselves expandable in powers of σ_i^0 and τ_i^0 (since we are reasoning by recurrence and assume that the already calculated quantities possess the indicated property), it is clear that the result of the substitution can also be expanded in powers of σ_i^0 and τ_i^0.

(ii) Take the mean value of a known periodic function either with respect to w alone or with respect to w and w'.

This is what happens when the right-hand side of Eqs. (14.24) are deducted from those of Eqs. (14.23) or else if the mean value of the right-hand side of Eq. (14.7:2,2) is canceled by equating n_1^2 to the mean value of A (see above, toward the end of the preceding number).

Since this operation consists in eliminating terms in the trigonometric expansion of the considered function, it is obvious that this cannot change the enunciated proposition.

(iii) Differentiate one of the quantities (14.31) with respect to w or w'.

Let, as above

$$A \cos h \quad \text{or} \quad A \sin h$$

be a term of the expansion of the quantity differentiated here.

The derivative of this term with respect to w_i will be

$$-Am_i \sin h \quad \text{or} \quad Am_i \cos h.$$

Its derivative with respect to w_i' will be

$$-Am_i' \sin h \quad \text{or} \quad Am_i' \cos h.$$

It is obvious that, if A satisfies the stipulated condition, this will also be the same for

$$\pm Am_i \quad \text{and for} \quad \pm Am_i'.$$

(iv) Integrate Eqs. (14.23), (14.24), and (14.25).

Some of these equations will directly yield the unknown; these include Eqs. (14.25) and those that furnish n_i^p, which must be chosen so as to cancel the mean value of the right-hand side of Eq. (14.7:2, p). However, other equations require an integration; these include, for example, Eqs. (14.23) which have the form

$$n_1^0 \frac{dx}{dw_1} + n_2^0 \frac{dx}{dw_2} = y, \tag{14.32}$$

where x is the unknown function and y is a known periodic function. Let then

$$A \cos h \quad \text{or} \quad A \sin h$$

be a term of y. The corresponding term of x will be written as

$$\frac{A}{n_1^0 m_1 + n_2^0 m_2} \sin h \quad \text{or} \quad \frac{A}{n_1^0 m_1 + n_2^0 m_2} \cos h.$$

It is clear that, if A satisfies the postulated condition, this will also be so for

$$\pm \frac{A}{n_1^0 m_1 + n_2^0 m_2}.$$

The same reasoning is applicable to Eq. (14.7:2, p) which, after one has selected n_i^p in such a manner as to cancel the mean value of the right-hand side, will take the form

$$\sum n_k'^1 \frac{dx}{dw_i'} = y, \tag{14.33}$$

where y is known and x is unknown, and is thus of the same form as Eq. (14.32). It should be noted, in addition, that the quantities $n_i'^1$, just as n_i^0, depend on Λ_0 but not on $x_i'^0$. It should be added that Eq. (14.33) determines the unknown x only to within a constant which can be arbitrarily selected as a function of Λ_0 and of $x_i'^0$. Naturally, to have the theorem hold, it is necessary to select this arbitrary function in such a manner that it can be expanded in integral powers of $(x_i'^0)^2$.

Similarly, Eqs. (14.25) determine $[\Lambda_p]$ only to within a constant which can be arbitrarily chosen. It is necessary to make this choice in such a manner that

$$[[\Lambda_p]]$$

becomes expandable in powers of $(x_i'^0)^2$.

(v) The integration of Eqs. (14.7:3, p) and (14.7:4, p) is treated in approximately the same manner.

For example, let us consider Eqs. (14.27) and let us make use of the equations which had been previously applied in studying these equations.

Let us first consider the case in which h is not equal to $\pm w_i'$ and in which the determinant of the linear equations (14.28) is not zero. It is then obvious that, if the coefficients A_1, B_1, A_2, B_2 satisfy the stated condition this must also be the case for the coefficients C_1, D_1, C_2, D_2 derived from Eqs. (14.28).

Let us now pass to the case in which $h = w_i'$ and in which Eqs. (14.28) must be replaced by Eqs. (14.29).

We first have the equation

$$n_i'^2 = \frac{A_2 - B_1}{2x_i'^0}.$$

We assume that A_2 and B_1, which are the coefficients of the expansion of a previously calculated function, will satisfy the postulated condition, i.e., that they are expandable in powers of $x_k'^0$, that they are divisible by $x_i'^0$, and that the quotient contains none but even powers of $x_k'^0$. It follows from this that $n_i'^2$ also will contain only even powers of $x_k'^0$ and thus satisfies our proposition.

Returning to Eqs. (14.29), one can see that C_1 and D_1 satisfy the stated condition provided that C_2 and D_2 satisfy it. However, we have seen that C_2 and D_2 can be chosen arbitrarily; we can always make this choice so as to satisfy the condition and, naturally, the theorem will hold only under this particular condition.

Since none of our operations is able to change the mentioned property, this will hold in all its generality.

154. Let us now note the equations of motion will not change when, keeping Λ and ρ_i unchanged, the quantities λ and ω_i are increased by the same amount.

Let us return to series (14.21), retaining the numbering of no. 152. Since the mean values of the quantities Λ_p, λ_p, Λ'_p, λ'_p, ρ_i^p, ω_i^p can be arbitrarily chosen with respect to w and w', we will choose all these mean values in an arbitrary manner.

Then series (14.21) are the only ones that formally satisfy the equations of motion and that, in addition, satisfy the double condition that all these mean values be determined and that

$$\sum \Lambda \, d\lambda + \sum \rho_i d\omega_i \qquad (14.34)$$

be an exact differential.

In fact, the calculation of no. 152 determines, without ambiguity, the coefficients of the series subjected to these diverse conditions.

Let us now add the same constant α to λ, λ', and the ω_i. We will again satisfy the equations of motion, according to the remark made at the beginning of this number, and our series (14.21) will not have changed except that λ_0, λ'_0, and ω_i^0 have become $w_1 + \alpha$, $w_2 + \alpha$, and $w'_i + \alpha$.

Let us then change w_i and w'_i into $w_i - \alpha$ and $w'_i - \alpha$. The series will retain the same form, i.e., Λ^p, λ'^p, ρ_i^p, and ω_i^p $(p > 0)$ will still be periodic functions of w and w' whose mean value will remain the same. These formally satisfy the equations of motion since we only have taken away a constant α from the constants $\overline{\omega}_i$ and $\overline{\omega}'_i$ which are arbitrary.

Finally, expression (14.34) will remain an exact differential.

Consequently, these series cannot differ from series (14.21), which are the only ones that satisfy all these conditions.

This means that the quantities Λ_p, λ_p, ω_i^p $(p > 0)$ do not change when simultaneously reducing w and w' by the same amount.

This also means that, if

$$A \cos h \quad \text{or} \quad A \sin h$$

is a term of the expansion of Λ_p, λ_p, ρ_i^p, or ω_i^p and if

$$h = \sum m_i w_i + \sum m'_i w'_i,$$

then the algebraic sum of the integers m_i and m'_i must be zero.

From this, one can readily conclude that, if

$$\text{A} \cos h \quad \text{or} \quad A \sin h$$

is a term of σ_i^p or τ_i^p, then this same algebraic sum must be equal to ± 1; we should add that this sum is zero in the expansion of

$$\sigma_i^p \cos w_k + \tau_i^p \sin w_k,$$

$$\tau_i^p \cos w_k - \sigma_i^p \sin w_k$$

(and also in that of the same expressions where w_k is replaced by w'_k).

Symmetry considerations and an analogous reasoning will then lead to other properties.

Thus, since everything is symmetric with respect to the xz plane the equations of motion will not change when one changes the signs of λ, λ', and τ_i without changing Λ, Λ', and σ_i.

Let us now assume that—in expansions (14.21)—the mean values of λ_p and of τ_i^p, which can be arbitrarily selected, are zero. Next, let us change

$$\lambda, \quad \lambda', \quad \tau_i$$

into

$$-\lambda, \quad -\lambda', \quad -\tau_i$$

and, at the same time, w_i and w'_i into

$$-w_i \quad \text{and} \quad -w'_i.$$

Series (14.21) will retain the same form and will not cease satisfying the equations of motion. The mean value of Λ_p and σ_i^p will not change since the quantities λ_p and τ_i^p remain zero. Finally, expression (14.34) will remain an exact differential.

For this to be the case, series (14.21) must not change. Consequently, Λ_p and σ_i^p do not change, while λ_p and σ_i^p change sign when w and w' change sign.

This means that the expansion of Λ and of σ_i contains only cosines whereas the expansion of λ and of τ_i contains only sines.

At the same time, everything is symmetric with respect to the xy plane so that other conclusions can be drawn from this.

Let us assume that we have to do with the three-body problem in three-dimensional space and consider

$$\Lambda, \quad \Lambda', \quad \sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4,$$

$$\lambda, \quad \lambda', \quad \tau_1, \quad \tau_2, \quad \tau_3, \quad \tau_4.$$

The third and fourth pairs of variables define the eccentricities and the perihelions. The first two pairs of variables define the inclinations and the nodes.

Because of the symmetry mentioned above, the equations will not change when σ_3, σ_4, τ_3, and τ_4 change sign, the other variables remaining unaltered.

By entirely the same reasoning as above, it can then be demonstrated that series (14.21) do not change when we simultaneously change

$$\sigma_3, \quad \sigma_4, \quad \tau_3, \quad \tau_4, \quad w_3', \quad w_4'$$

into

$$-\sigma_3, \quad -\sigma_4, \quad -\tau_3, \quad -\tau_4, \quad +w_3'+\pi, \quad w_4'+\pi.$$

From this, it can be concluded that, in series (14.21) arranged in cosines and sines of

$$h = \sum m_i w_i + \sum m_i' w_i',$$

the sum $m_3' + m_4'$ must be even in the expansion of

$$\begin{matrix} \Lambda & \Lambda' & \sigma_1 & \sigma_2 \\ \lambda & \lambda' & \tau_1 & \tau_2 \end{matrix}$$

and odd in the expansion of

$$\begin{matrix} \sigma_3 & \sigma_4, \\ \tau_3, & \tau_4. \end{matrix}$$

155. For simplifying the discussion and the calculations, we used an artifice in no. 152, which had already been described at the end of no. 140 and which we recalled at the beginning of no. 152. It consists in regarding all terms containing the masses only to the first degree as being of the second order.

This artifice is legitimate because of the extreme smallness of these terms but it does have some drawbacks. In fact, the significance of the parameter μ is somewhat modified by this. Setting $\mu = 0$, one obtains a particular case of the three-body problem, namely, the case in which the perturbing masses are zero and the motion is Keplerian. Assigning a certain determined very small value to μ, one comes to another particular case of the three-body problem, namely, that corresponding to the real masses of the bodies under consideration. However, assigning an intermediate value of μ, the equations will be those of a dynamics problem which has no relation at all with the three-body problem.

This will no longer be the same if the original meaning had been retained for the symbol μ, as defined in no. 11. Irrespective of the value attributed to μ, the equations will be those of a particular case of the three-body problem corresponding to certain mass values.

Therefore, it will be much more satisfactory to restore the original meaning to the symbol μ and to attempt to expand our variables not only in powers of μ but also in powers of the constants we called $x_i'^0$ and which are of the order of the eccentricities.

The equations of motion still have the same form; only the mean value of F_1, again denoted by R, has a more complex expression. More simply, we have as in no. 152,

$$R = B + \sum A_i(\sigma_i^2 + \tau_i^2). \tag{14.35}$$

However, R can be expanded in ascending powers of σ_i and τ_i, and the right-hand side of Eq. (14.35) represents only the first terms of the series, namely, those of zero degree and those of second degree (all terms, as is known, being of an even degree).

Let us then expand our variables (14.17) in powers of μ and $x_i'^0$. Let us retain series (14.21) and let, on the other hand,

$$\begin{aligned} \Lambda_p &= \Lambda_{p0} + \Lambda_{p1} + \Lambda_{p2} + \cdots, \\ \lambda_p &= \lambda_{p0} + \lambda_{p1} + \lambda_{p2} + \cdots, \\ \sigma_i^p &= \sigma_i^{p0} + \sigma_i^{p1} + \sigma_i^{p2} + \cdots, \\ \tau_i^p &= \tau_i^{p0} + \tau_i^{p1} + \tau_i^{p2} + \cdots, \end{aligned} \tag{14.36}$$

where

$$\Lambda_{pq}, \quad \lambda_{pq}, \quad \sigma_i^{pq}, \quad \tau_i^{pq}$$

represent the collection of terms of degree q with respect to $x_i'^0$.

We still assume

$$\Lambda_0 = \text{const.}, \quad \lambda_0 = \omega_1,$$

and, consequently,

$$\Lambda_{0q} = \lambda_{0q} = 0 \quad \text{for} \quad q > 0;$$

but we no longer suppose that

$$\sigma_i^0 = x_i'^0 \cos w_i', \quad \tau_i^0 = x_i'^0 \sin w_i'.$$

We next assume that

$$\sigma_i^{00} = \tau_i^{00} = 0;$$

$$\sigma_i^{01} = x_i'^0 \cos w_i', \quad \tau_i^{01} = x_i'^0 \sin w_i'.$$

However, σ_i^{0q}, τ_i^{0q} will not be zero.

After making these hypotheses, let us return to the calculation of no. 152.

We first considered Eqs. (14.23) by setting there $p = 0$. These equations will be satisfied provided that, with $n_i'^0$ being zero, σ_i^0 and τ_i^0 do not depend on w_i but only on w_i', which we will assume here.

Next, let us turn to Eqs. (14.24) by setting there $p=1$ [see Eqs. (14.26) of no. 152]; however, it should be noted that the form of Eqs. (14.23) and (14.24) is somewhat modified.

In fact, let us consider, in Eqs. (14.6:3, p), (14.6:4, p), (14.7:3, p), (14.7:4, p), the last term on the right-hand side. This term must be written as

$$\begin{aligned}
&\text{for } (14.6:3, p)\ \ldots\ldots && -\sum_k n_k^p \frac{d\sigma_i^0}{dw_k} - \sum n_k'^p \frac{d\sigma_i^0}{dw_k'},\\
&\text{for } (14.6:4, p)\ \ldots\ldots && -\sum_k n_k^p \frac{d\tau_i^0}{dw_k} - \sum n_k'^p \frac{d\tau_i^0}{dw_k'},\\
&\text{for } (14.7:3, p)\ \ldots\ldots && - \sum n_k'^p \frac{d\sigma_i^0}{dw_k'},\\
&\text{for } (14.7:4, p)\ \ldots\ldots && - \sum n_k'^p \frac{d\tau_i^0}{dw_k'}.
\end{aligned} \tag{14.37}$$

In no. 152, the quantities σ_i^0 and τ_i^0 reduce to

$$x_i'^0 \cos w_i' \quad \text{and} \quad x_i'^0 \sin w_i',$$

after which these four terms reduce to

$$\pm\, n_i'^p x_i'^0 \begin{matrix}\sin\\ \cos\end{matrix}\, w_i'.$$

However, this is no longer the case here, and expressions (14.37) must be retained for these terms.

In that case, Eqs. (14.24) for $p=1$ are written in the form

$$\begin{gathered}
[L_1] = \frac{dR}{d\lambda_0} = 0, \quad [l_1] = n_1^1,\\
\frac{dR}{d\tau_i^0} = \sum n_k'^1 \frac{d\sigma_i^0}{dw_k'}, \qquad -\frac{dR}{d\sigma_i^0} = \sum n_k'^1 \frac{d\tau_i^0}{dw_k'}.
\end{gathered} \tag{14.38}$$

Naturally, it must be assumed that, in R, the terms Λ, λ, σ_i, and τ_i were replaced by Λ_0, λ_0, σ_i^0, and τ_i^0.

These equations, in a different form, are the same as those discussed in Chap. 10. The first of these is satisfied identically. Let us therefore examine the two last equations that must determine σ_i^0 and τ_i^0.

Let us expand $n_i'^1$ in powers of $x_i'^0$ and let

$$n_i'^1 = n_i'^{1.0} + n_i'^{1.2} + n_i'^{1.3} + \cdots, \tag{14.39}$$

where $n_i'^{1.q}$ is the set of terms of the degree q with respect to $x_i'^0$.

In the two last equations of system (14.38), let us substitute expansions

(14.36) and (14.39) for the quantities σ_i^0, τ_i^0, $n_k'^1$ and let us equate all terms of the same degree in both members. For abbreviation, let us also put

$$\Delta'' u = \sum n_k'^{1.0} \frac{du}{dw_k'}.$$

Equating the terms of the first degree with respect to $x_i'^0$, we obtain

$$\Delta'' \sigma_i^{0.1} = 2A_i^0 \tau_i^{0.1}; \quad \Delta'' \tau_i^{0.1} = -2A_i^0 \sigma_i^{0.1}.$$

These equations are satisfied provided that

$$n_i'^{1.0} = -2A_i^0.$$

Let us now assume that we had already determined

$$\begin{array}{cccc} \sigma_i^{0.1}, & \sigma_i^{0.2}, & \ldots, & \sigma_i^{0.q-1}, \\ \tau_i^{0.1}, & \tau_i^{0.2}, & \ldots, & \tau_i^{0.q-1}, \\ n_i'^{1.0}, & n_i'^{1.1}, & \ldots, & n_i'^{1.q-2}, \end{array}$$

and that we wish to determine

$$\sigma_i^{0.q}, \quad \tau_i^{0.q}, \quad n_i'^{1.q-1}.$$

Let us equate, on both sides of the two last equations of system (14.38), all terms of degree q. These terms will be:

In the third equation:

Left-hand side . . . $2A_i \tau_i^{0.q}$ + known quantities.

Right-hand side . . . $\Delta'' \sigma_i^{0.q} + n_i'^{1.q-1} \dfrac{d\sigma_i^{0.1}}{dw_i'}$ + known quantities.

In the fourth equation:

Left-hand side . . . $-2A_i \sigma_i^{0.q}$ + known quantities.

Right-hand side . . . $\Delta'' \tau_i^{0.q} + n_i'^{1.q-1} \dfrac{d\tau_i^{0.1}}{dw_i'}$ + known quantities.

Thus we can write

$$\begin{aligned} \Delta'' \sigma_i^{0.q} + n_i'^{1.0} \tau_i^{0.q} &= \varphi_1 + n_i'^{1.q-1} x_i'^0 \sin w_i', \\ \Delta'' \tau_i^{0.q} + n_i'^{1.0} \sigma_i^{0.q} &= \varphi_2 + n_i'^{1.q-1} x_i'^0 \cos w_i', \end{aligned} \tag{14.40}$$

where φ_1 and φ_2 are known periodic functions of w'.

The analogy of these equations with Eqs. (14.27) is evident. It is possible to pass from one to the other by changing $[\sigma_i^1]$, $[\tau_i^1]$, $n_k'^1$, $n_i'^2$ into $\sigma_i^{0.q}$, $\tau_i^{0.q}$, $n_k'^{1.0}$, $n_i'^{1.q-1}$.

Consequently, Eqs. (14.40) will be treated like Eqs. (14.27). The condition for success of the method [namely, that in the equations analogous to Eq. (14.29), $A_1 + B_2$ be zero] must be satisfied identically since we have demonstrated the feasibility of expansion in advance.

When the two last equations of system (14.38) have been satisfied, R will be a constant (since these two equations admit $R = \text{const.}$ as integral, analogous to the *vis viva* integral); since this must hold, irrespective of the constants Λ_0 and Λ_0', the derivative $-dR/d\Lambda_0$ must also be a constant depending solely on Λ_0 and $x_i'^0$.

However, we have

$$[l_1] = -\frac{d^2F_0}{d\Lambda_0^2}[\Lambda_1] - \frac{d^2F_0}{d\Lambda_0\, d\Lambda_0'}[\Lambda_1'] - \frac{dR}{d\Lambda_0}.$$

The derivatives of F_0 are constants. The first equation of system (14.25) demonstrates that this is true also for $[\Lambda_1]$ and $[\Lambda_1']$. Thus $[l_1]$ will also be a constant that can be equated to n_1^1, which thus satisfies the second equation of system (14.38).

In no. 152 we then determined successively $\Lambda_1 - [\Lambda_1]$ (and, consequently, also Λ_1 since $[\Lambda_1]$ is a constant that can be arbitrarily selected) as well as $\sigma_i^1 - [\sigma_i^1]$, $\tau_i^1 - [\tau_i^1]$, $\lambda_1 - [\lambda_1]$ by Eqs. (14.6:1,1), (14.6:3,1), (14.6:4,1), and (14.6:2,1). We changed nothing in this portion of the calculation.

Let us next determine

$$[\sigma_i^1] \quad \text{and} \quad [\tau_i^1],$$

and, for this, let us consider Eqs. (14.7:3,2) and (14.7:4,2). These equations assume the form

$$\Delta'[\sigma_i^1] = \varphi_1 + \sum\frac{d^2R}{d\tau_i^0 d\sigma_k^0}[\sigma_k^1] + \sum\frac{d^2R}{d\tau_i^0 d\tau_k^0}[\tau_k^1] - \sum n_k'^2\frac{d\sigma_i^0}{dw_k'},$$

$$\Delta'[\tau_i^1] = \varphi_2 - \sum\frac{d^2R}{d\sigma_i^0 d\sigma_k^0}[\sigma_k^1] - \sum\frac{d^2R}{d\sigma_i^0 d\tau_k^0}[\tau_k^1] - \sum n_k'^2\frac{d\tau_i^0}{dw_k'}, \tag{14.41}$$

where φ_1 and φ_2 are known.

These equations are analogous to Eqs. (14.27); however, since R, σ_i^0, τ_i^0 have less simple expressions, it now no longer happens—as was the case in no. 152—that for the first of these equations, for example, the last three terms on the right-hand side reduce, respectively, to

$$0, \quad 2A_i^0[\tau_i^1], \quad n_i^2\tau_i^0,$$

which results in a considerable simplification.

Consequently, in Eq. (14.41) let us replace the quantities σ_i^1 and τ_i^1 by their expansions (14.36) and the quantity $n_k'^1$ by its expansion (14.39) and, finally, the quantity $n_k'^2$ by its expansion

$$n_k'^2 = n_k'^{2.0} + n_k'^{2.1} + n_k'^{2.2} + \cdots$$

analogous to Eq. (14.39). In addition, let φ_1^q and φ_2^q be the collection of the terms of φ_1 and of φ_2 that are of degree q with respect to $x_i'^0$.

We will then equate all terms of the same degree on both sides of Eq. (14.41).

Equating first the terms of zero degree, we obtain simply

$$\begin{aligned} \Delta''[\sigma_i^{1.0}] &= \varphi_1^0 + 2A_i^0[\tau_i^{1.0}], \\ \Delta''[\tau_i^{1.0}] &= \varphi_2^0 - 2A_i^0[\sigma_i^{1.0}]. \end{aligned} \tag{14.42}$$

Here φ_1^0 and φ_2^0 will be constants depending solely on Λ_0 and Λ_0'. In fact, by virtue of the reasoning in no. 153, which remains valid without modification, the quantities φ_1 and φ_2 can be expanded in powers of $x_i'^0 \cos w_i'$ and of $x_i'^0 \sin w_i'$. All terms of zero degree with respect to $x_i'^0$ will thus be independent of $x_i'^0$ as well as of w_i'.

It results from this that $[\tau_i^{1.0}]$ and $[\sigma_i^{1.0}]$ also are constants and that the left-hand side of Eqs. (14.42) are zero. Equations (14.42) thus permit us to determine $[\sigma_i^{1.0}]$ and $[\tau_i^{1.0}]$.

Let us now assume that we had determined

$$\begin{array}{ccccc} [\sigma_i^{1.0}], & [\sigma_i^{1.1}], & [\sigma_i^{1.2}], & \ldots, & [\sigma_i^{1.q-1}], \\ [\tau_i^{1.0}], & [\tau_i^{1.1}], & [\tau_i^{1.2}], & \ldots, & [\tau_i^{1.q-1}], \\ & n_i'^{2.0}, & n_i'^{2.1}, & \ldots, & n_i'^{2.q-2}, \end{array}$$

and that we wish to determine

$$[\sigma_i^{1.q}], \quad [\tau_i^{1.q}], \quad n_i'^{2.q-1}. \tag{14.43}$$

For this, let us equate all terms of degree q on both sides of Eqs. (14.41).

In defining the terms dependent on the unknown quantities (14.43), we obtain

$$\begin{aligned} \Delta''[\sigma_i^{1.q}] + n_i'^{1.0}[\tau_i^{1.q}] &= \psi_1 + n_i'^{2.q-1} x_i'^0 \sin w_i', \\ \Delta''[\tau_i^{1.q}] + n_i'^{1.0}[\sigma_i^{1.q}] &= \psi_2 + n_i'^{2.q-1} x_i'^0 \cos w_i', \end{aligned} \tag{14.44}$$

where ψ_1 and ψ_2 are known functions.

These equations are analogous to Eqs. (14.27). In fact, we pass from one to the other by changing

$$[\sigma_i^1], \quad [\tau_i^1], \quad n_k'^1, \quad n_i'^2$$

into

$$[\sigma_i^{1.q}], \quad [\tau_i^{1.q}], \quad n_k'^{1.0}, \quad n_i'^{2.q-1}.$$

Thus Eqs. (14.44) can be treated like Eqs. (14.27).

After this, one would determine

$$[\Lambda_2], \quad n_1^2, \quad [\lambda_1], \quad \Lambda_2, \quad \sigma_i^2 - [\sigma_i^2], \quad \tau_i^2 - [\tau_i^2], \quad \lambda_2 - [\lambda_2],$$

as in no. 152.

For determining

$$[\sigma_i^2], \quad [\tau_i^2], \quad \text{and} \quad n_i'^3,$$

one would make use of Eqs. (14.7:3,3) and (14.7:4,3). These equations would have the same form as Eqs. (14.41) and would be treated in the same manner.

Noteworthy Particular Cases

156. Series (14.21) and (14.36), as shown in no. 153, have their right-hand sides expanded in powers of $x_i'^0 \cos w_i'$ and of $x_i'^0 \sin w_i'$.

If all arbitrary constants $x_i'^0$ are simultaneously canceled, then our variables will no longer depend on w_i' but only on w_1 and w_2. Their expansions will proceed along trigonometric lines of

$$m_1 w_1 + m_2 w_2,$$

where m_1 and m_2 are integers.

According to our statements in no. 154, the sum $m_1 + m_2$ in the expansion of Λ and of λ must be zero so that these variables will depend solely on $w_1 - w_2$. For the same reason, this will be the same for

$$\begin{aligned} &\sigma_i \cos w_1 + \tau_i \sin w_1, \\ &\tau_i \cos w_1 - \sigma_i \sin w_1. \end{aligned} \tag{14.45}$$

Obviously, this means that these particular hypotheses ($x_i'^0 = 0$) correspond to the case of a periodic solution, and it is easy to prove that the resultant solutions do not differ from those which, in Chap. 3, were designated as periodic solutions of the first kind.

From this, it can be concluded that series (14.21) which ordinarily are not convergent in the geometric sense of the word will become so as soon as the constants $x_i'^0$ vanish.

Since the constants $x_i'^0$ generally are small, it is obvious that the real solution will oscillate about the periodic solution without deviating much from it.

Let us now consider, in the expansions of Λ, λ, and expressions (14.45), all terms of the first degree with respect to $x_i'^0$. Taking the results of nos. 153 and 154 into consideration, it will be found that these have the form

$$\sum_k x_k'^0 \cos(w_k' - w_1)\varphi_k + \sum_k x_k' \sin(w_k' - w_1)\psi_k, \qquad (14.46)$$

where φ_n and ψ_k are periodic functions that can be expanded in multiple sines and cosines of $w_1 - w_2$.

The interpretation of this result is evident. In Chap. 4, we discussed the variational equations relative to a given periodic solution. Let us now consider our equations of motion and the periodic solution of the first kind, obtained on canceling all $x_i'^0$. Expressions (14.46) will then be nothing else but the most general solution of the corresponding variational equations.

From this, it can be concluded that the characteristic exponents, relative to this solution of the first kind, will be

$$\pm\sqrt{-1}\,(n_k' - n_1).$$

It is of importance to note that, in this expression, the constants $x_i'^0$ (on which n_k' and n_1 depend) must be equated to zero.

One could propose to derive from series (14.21) and (14.36) the periodic solutions of the second and third kind, exactly as it had been done for those of the first kind. However, this is somewhat more difficult.

To understand better what has to be done, we will first use a simpler example. Let us return to the series of no. 127 and let us deduce from these the periodic solutions of no. 42. In the series given in no. 127, we have seen that the mean values of the periodic functions x_i^p and y_i^p can be arbitrarily chosen and that, specifically, this selection can be made such that n_i^p will be zero each time that $p > 0$. This condition can also be realized by properly choosing the mean values of x_i^p, while the mean values of y_i^p remain arbitrary.

Thus, let us suppose that we had chosen the mean values in the above-described manner and that, consequently,

$$n_i = n_i^0.$$

Let us suppose, in addition, that the quantities x_i^0 had been chosen such that the quantities n_i^0 would have certain mutually commensurable given values. It then happens, if the calculation of no. 127 is to be performed, that certain coefficients become infinite unless the constants $\overline{\omega}_i$ are properly chosen and the mean values of y_i^p remain arbitrary.

If the choiçe is made in this manner, the series of no. 127 will be valid; these are convergent and do not differ from those of no. 44. Let us now return to the three-body problem.

Let us select our constants Λ_0, Λ_0', and $x_i'^0$ as well as the mean values of the various terms of series (14.21) and (14.36), considered as periodic functions of w and w'. In other words, let us select these quantities in such a manner that

(a) the quantities n_1^0 and n_2^0 will have given values that are mutually commensurable (we note that, if the notations of no. 155 are used, $n_i^{0,p}$ will be zero for $p > 0$);

(b) the quantities n_i^p and $n_i'^p$ will be zero for $p > 1$;

(c) we will have

$$n_1^1 = n_2^1 = n_i'^1 \quad (i = 1,2,3,4).$$

The selection can be made such that these conditions will be realized and that even half of the mean values remain arbitrary.

It then happens, if the calculation of nos. 152 or 155 is to be performed, that certain coefficients become infinite unless the constants $\overline{\omega}_i$ and $\overline{\omega}_i'$ are properly chosen so that the mean values remain arbitrary.

If this is done, then series (14.21) and (14.36) will exist. They converge and do not differ from those that represent solutions of the second and third kind.

Let us now suppose, without canceling $x_1'^0$ and $x_2'^0$, that $x_3'^0$ and $x_4'^0$ vanish. This will yield a series of particular solutions of the three-body problem, which depend only on four arguments

$$w_1, \quad w_2, \quad w_1', \quad w_2'.$$

These are the solutions corresponding to the case of the three-body problem in the plane. Here, the number of arguments is reduced to four, as is the number of degrees of freedom.

However, it should be noted that the quantities Λ and λ and the expressions (14.45) depend only on the differences

$$w_2 - w_1, \quad w_1' - w_1, \quad w_2' - w_1,$$

as demonstrated in no. 154.

Thus, if Λ, λ, and expressions (14.45) are used as variables, the number of arguments will be reduced to three. This corresponds to the case of the problem discussed in no. 5 with three degrees of freedom.

Let us now imagine that the mass of the first planet is infinitely small (case of a minor planet perturbed by Jupiter). It will first happen that

$$\sigma_2, \quad \tau_2, \quad \sigma_4, \quad \tau_4$$

are reduced to

$$\xi', \quad \eta', \quad p', \quad q'.$$

These quantities, like Λ', will be constants, and λ' will reduce to w_2.

From this it follows that

$$n_2' = \frac{dw_2'}{dt} = 0,$$

$$n_4' = \frac{dw_4'}{dt} = 0.$$

The number of our arguments, which had been six, is now reduced to four, namely,

$$w_1, \quad w_2, \quad w_1', \quad w_3'.$$

Here it no longer happens that $\Lambda, \lambda', \ldots$ depend only on the differences

$$w_1 - w_2, \quad w_1' - w_2, \quad w_3' - w_2.$$

The reasoning of no. 154 actually proves only one point, namely, that in the general case Λ depends solely on the five differences

$$w_2 - w_1, \quad w_i' - w_1 \quad (i = 1,2,3,4).$$

When two of the w_i' reduce to constants (which happens in the particular case studied here), two of these five arguments will differ only by one constant and it is for this reason that not more than four arguments will remain; however, there is no call for pushing the reduction still further.

In addition, our variables—in virtue of no. 153—remain expandable in powers of

$$x_i'^0 \cos w_i', \quad x_i'^0 \sin w_i'.$$

Let us assume that $x_3'^0$ and $x_4'^0$ are canceled. This corresponds to the case in which the three bodies move in one and the same plane (always assuming that one of the masses is infinitely small). Then, our variables no longer depend on w_3' and only three arguments remain, namely,

$$w_1, \quad w_2, \quad w_1'.$$

Let us then also cancel the constant $x_2'^0$. This corresponds to the case in which the orbit of the second planet is circular, i.e., to the problem of no. 9.

Since our variables can be expanded in powers of $x_i'^0 \cos w_i'$ and $x_i'^0 \sin w_i'$ and since

$$x_2'^0 = x_3'^0 = x_4'^0 = 0,$$

they no longer depend on w_2', w_3', or w_4'. Because of no. 154, they depend only on the differences

$$w_1 - w_2, \quad w_1 - w_i'.$$

However, we have shown above that there are three of the w_i' that no longer must enter their expression. These will now depend only on

$$w_1 - w_2, \quad w_1 - w_1'.$$

The number of arguments is thus reduced to two. We have also seen that the problem of no. 9 admits of exactly 2 degrees of freedom. If, in addition, we set $x_i'^0 = 0$, we fall back on the periodic solutions studied by Hill (see no. 41 and especially the remark made in the last three lines).

If, in the theory of the moon, this satellite is considered as subject only to the influence of earth and sun and if the relative motion of these two latter celestial bodies is considered as being Keplerian, we come back to one of the particular cases investigated above.

However, it is frequently necessary to allow for perturbations that other planets exert on the earth while still neglecting the direct action of these planets on the moon. If this particular viewpoint is adopted, the relative motion of the earth and sun will no longer be a Keplerian motion but will be known, and the moon will remain subject only to the action of these two mobile bodies that move in accordance with a known law.

Thus, let us assume that the coordinates of the sun with respect to the earth can be expressed by series of the same form as those studied in this chapter and depending on n arguments. It is easy to demonstrate, by reasoning more or less as done in this chapter, that the lunar coordinates are expressed by series of the same form, depending on $n + 2$ arguments.

To explain what we really mean by this, let us return to the problem of no. 9, i.e., let us imagine that the earth and the sun describe concentric circumferences. Then the solar coordinates will depend on $n = 1$ argument; the distances from the moon to the earth and to the sun will depend on two arguments (which are those denoted above by $w_1 - w_2, w_1 - w_1'$). However, the lunar considerates with respect to fixed axes will depend on $n + 2 = 3$ arguments.

Similar considerations are applicable to the case of more than three bodies. For example, let us assume that four such bodies are present. Then the number of w_i will be three and that of w_i', six.

Let us assume that one has set the six constants $x_i'^0$ simultaneously equal to zero. An immediate consequence of this hypothesis is that the motion takes place in a plane. In addition, the quantities Λ and λ and expressions (14.45), and thus also the mutual distances of the four bodies will depend on no more than two arguments

$$w_1 - w_2, \quad w_1 - w_3.$$

It does not follow from this (as in the case in which, considering only three bodies, all terms $x_i'^0$ are canceled) that the series become convergent in the mathematical sense of the word; however, it is possible to deduce from this the periodic solutions of no. 50.

The mode of operation is as follows:

Let us choose our constants of integration and the mean values of the various terms of series (14.21) and (14.36) such that (a) the quantities

$$n_1^0 - n_2^0, \quad n_1^0 - n_3^0$$

will have mutually commensurable given values; (b) that

$$n_1^p = n_2^p = n_3^p$$

for $p > 0$. The constants $\overline{\omega}_i$ and $\overline{\omega}'_i$ as well as half of our mean values become arbitrary.

If one wishes to perform the calculation of no. 152, certain coefficients become infinite unless one chooses the terms $\overline{\omega}_i$ properly, while leaving arbitrary $\overline{\omega}'_i$ and the mean values.

If this choice is made in the above manner, the series will exist, will converge, and will represent the periodic solutions of no. 50.

Conclusions

157. Such are the series obtained by computational methods discussed in the preceding chapters. Newcomb was the first to have conceived these and to have discovered their principal properties.

These series are divergent; however, if one terminates the series at the proper place, namely, before having encountered very small divisors, they will represent the coordinates with an excellent approximation.

The series can be used in still another manner.

Let us imagine that the expansion is stopped at a certain term and that then, applying the method of variation of constants, the quantities Λ_0, $x_i'^0$, $\overline{\omega}_i$, and $\overline{\omega}'_i$ are used as new variables. These new variables will vary extremely slowly and the old methods can be advantageously applied to the differential equations that define these variations. For example, it would be possible to expand these new variables in powers of time.

CHAPTER 15

Other Methods of Direct Calculation

Problem of No. 125

158. Let us return to the equations

$$\frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{15.1}$$

and

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}. \tag{15.2}$$

We have in mind to satisfy these equations by means of series arranged in sines and cosines of multiples of n arguments

$$w_1, \quad w_2, \quad \ldots, \quad w_n,$$

a series whose existence we have proved in no. 125.

Let us recall also that we have

$$w_k = n_k t + \varpi_k$$

and, consequently,

$$\begin{aligned} \sum_k n_k \frac{dx_i}{dw_k} &= \frac{dx_i}{dt}, \\ \sum_k n_k \frac{dy_i}{dw_k} &= \frac{dy_i}{dt}. \end{aligned} \tag{15.3}$$

In no. 127 we have used Eqs. (9.10) and (9.11) for determining the series; however, one can also operate differently.

First, we have the kinetic energy integral

$$F = \text{const.} \tag{15.4}$$

On the other hand, the expression

$$\sum x_i dy_i \tag{15.5}$$

must be an exact differential and, since the quantities x_i^0 are constants, this must also be so for

$$\sum (x_i - x_i^0) dy_i = dS,$$

which yields

$$\frac{dS}{dw_k} = \sum (x_i - x_i^0) \frac{dy_i}{dw_k}. \tag{15.6}$$

I now say that Eqs. (15.2) are a consequence of Eqs. (15.1), (15.3), (15.4), and (15.6). In fact, Eqs. (15.6) mean that expression (15.5) is an exact differential and that the integrability conditions for this expression can be written

$$\sum_i \left(\frac{dx_i}{dw_q} \frac{dy_i}{dw_k} - \frac{dx_i}{dw_k} \frac{dy_i}{dw_q} \right) = 0. \tag{15.7}$$

Let us multiply this equation by n_q; then, retaining a constant value for k, let us successively set $q = 1, 2, \ldots, n$.

Next, let us add the resultant n equations. Allowing for Eq. (15.3), this yields

$$\sum_i \left(\frac{dx_i}{dt} \frac{dy_i}{dw_k} - \frac{dx_i}{dw_k} \frac{dy_i}{dt} \right) = 0$$

or, taking Eq. (15.1) into consideration,

$$\sum \frac{dx_i}{dt} \frac{dy_i}{dw_k} + \sum \frac{dF}{dx_i} \frac{dx_i}{dw_k} = 0. \tag{15.8}$$

Let us now differentiate Eq. (15.4) with respect to w_k; we obtain

$$\sum \frac{dF}{dx_i} \frac{dx_i}{dw_k} + \sum \frac{dF}{dy_i} \frac{dy_i}{dw_k} = 0$$

or, taking Eq. (15.8) into account

$$\sum \frac{dx_i}{dt} \frac{dy_i}{dw_k} = \sum \frac{dF}{dy_i} \frac{dy_i}{dw_k} \quad (k = 1, 2, \ldots, n),$$

whence

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}.$$

Thus we can determine our series using Eqs. (15.4) and (15.6) and

$$\sum n_k \frac{dy_i}{dw_k} = -\frac{dF}{dx_i}. \tag{15.1a}$$

In these various equations, let us replace the quantities x_i, y_i, n_k, and S by their expansion in powers of μ:

$$\sum \mu^p x_i^p, \quad \sum \mu^p y_i^p, \quad \sum \mu^p n_k^p, \quad \sum \mu^p S_p.$$

Then, let us equate the coefficients of like powers of μ on both sides.

This will yield a series of equations that permits us to determine by recurrence the coefficients of the series.

Let us assume that we had calculated

$$\begin{array}{lllll} x_i^0 & x_i^1, & x_i^2, & \dots, & x_i^{p-1}, \\ y_i^0, & y_i^1, & y_i^2, & \dots, & y_i^{p-1}, \\ n_k^0, & n_k^1, & n_k^2, & \dots, & n_k^{p-1}, \\ S_0, & S_1, & & \dots, & S_{p-i}, \end{array}$$

and that we wish to determine

$$x_i^p, \quad y_i^p, \quad n_k^p, \quad S_p.$$

In Eq. (15.4), let us equate the coefficients of μ^p so that

$$\sum n_k^0 x_k^p = \Phi + \text{const.} \tag{15.9}$$

Here, as in this entire chapter, I designate by Φ a wholly known and periodic arbitrary function of w. There is no need to mention that the various functions, thus denoted by Φ, are not identical. As to the constant on the right-hand side of Eq. (15.9), it is arbitrary just like the constant on the right-hand side of Eq. (15.4).

Let us now equate the coefficients of μ^p on both sides of Eq. (15.6), so that

$$\frac{dS_p}{dw_k} = x_k^p + \Phi, \tag{15.10}$$

from which, taking Eq. (15.9) into consideration, we obtain

$$\sum n_k \frac{dS_p}{dw_k} = \Phi + \text{const.} \tag{15.11}$$

The function S_p must have all its derivatives periodic with respect to w, i.e., the function must have the form

$$\alpha_{1p} w_1 + \alpha_{2p} w_2 + \cdots + \alpha_{np} w_n + \varphi,$$

where α_{kp} are constants while φ is a periodic function.

Equation (15.11), by means of a calculation very similar to the integration of Eq. (15.6) in no. 125, will yield the value of S_p. It should be mentioned here that the constants α_{kp} can be arbitrarily selected as a function of the constants x_i^0 since the constant on the right-hand side of Eq. (15.11) itself is arbitrary.

After S_p has been determined in this manner, Eqs. (15.10) will yield the quantities x_k^p whose mean value α_{kp} as will be demonstrated below, can be chosen arbitrarily.

Since now the quantities x_i^p are known, let us equate the coefficients μ^p on both sides of Eq. (15.1a). This yields

$$\sum n_k^0 \frac{dy_i^p}{dw_k} = \Phi - n_i^p. \tag{15.12}$$

We start by determining the constant n_i^p in such a manner as to cancel the mean value of the right-hand side of Eq. (15.12). Then, Eq. (15.12) will yield y_i^p by a calculation entirely similar to that in no. 127. Let us note, in passing, that the mean value of y_i^p can be arbitrarily selected as a function of x_i^p.

Another Example

159. Let

$$\begin{array}{llll} \xi_1, & \xi_2, & \ldots, & \xi_n, \\ \eta_1, & \eta_2, & \ldots, & \eta_n \end{array}$$

be our n pairs of conjugate variables.

Let us assume that F can be expanded in ascending powers of ξ_i and of η_i; let us also assume that this expansion contains no terms of either the zero or first degree and that the terms of the second degree are written as follows:

$$\sum A_i(\xi_i)^2 + \sum A_i(\eta_i)^2.$$

We are using parentheses $(\xi_i)^2$ for writing the square of ξ_i so as to avoid confusion with the notation ξ_i^2 which will be used below and in which the numeral 2 is an index rather than an exponent.

Then let

$$\frac{d\xi_i}{dt} = \frac{dF}{d\eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{dF}{d\xi_i} \tag{15.13}$$

be our differential equations.

We now suppose that we wish to expand ξ_i and η_i in powers of certain integration constants α_i, and we write

$$\xi_i = \xi_i^1 + \xi_i^2 + \cdots + \xi_i^p + \cdots ,$$
$$\eta_i = \eta_i^1 + \eta_i^2 + \cdots + \eta_i^p + \cdots .$$

The quantities ξ_i^p and η_i^p will represent the terms of the series that are of order p with respect to α_i. These must be periodic functions with respect to n arguments:

$$w_1, \quad w_2, \quad \ldots, \quad w_n.$$

In addition, we should have

$$\xi_i^1 = \alpha_i \cos w_i, \quad \eta_i^1 = \alpha_i \sin w_i.$$

We have, moreover,

$$n_k = n_k^0 + n_k^1 + \cdots + n_k^p + \cdots ,$$

where n_k is expanded in powers of x_i^0 and n_k^p, representing the collection of terms of the order p with respect to α_i. Our differential equations then become

$$\sum_k n_k \frac{d\xi_i}{dw_k} = \frac{dF}{d\eta_i}, \quad \sum_k n_k \frac{d\eta_i}{dw_k} = -\frac{dF}{d\xi_i}. \tag{15.14}$$

On the other hand,

$$\sum \xi_i \, d\eta_i$$

must be an exact differential which, naturally, will also be the case for

$$dS = \sum \xi_i d\eta_i - \sum d(\xi_i^1 \eta_i).$$

Finally, let us note that S must also be expanded in powers of α_i; we denote the collection of terms of the degree p by S_p.

Let us also put

$$x_i = \xi_i \cos w_i + \eta_i \sin w_i,$$
$$y_i = \xi_i \sin w_i - \eta_i \cos w_i$$

and

$$x_i = \sum x_i^p, \quad y_i = \sum y_i^p,$$

whence

$$x_i^p = \xi_i^p \cos w_i + \eta_i^p \sin w_i,$$
$$y_i^p = \xi_i^p \sin w_i - \eta_i^p \cos w_i,$$
$$x_i^1 = \xi_i^0 \cos w_i + \eta_i^0 \sin w_i = \alpha_i,$$
$$y_i^1 = 0.$$

One immediately finds

$$n_k^0 = -2A_k.$$

Let us then note that Eqs. (15.14) furnish

$$\sum_k n_k \frac{dx_i}{dw_k} = \frac{dF}{d\eta_i}\cos w_i - \frac{dF}{d\xi_i}\sin w_i - n_i y_i, \tag{15.15}$$

$$\sum_k n_k \frac{dy_i}{dw_k} = \frac{dF}{d\eta_i}\sin w_i + \frac{dF}{d\xi_i}\cos w_i + n_i x_i. \tag{15.16}$$

We calculate our series by means of Eq. (15.16), from the equation

$$F = \text{const.} \tag{15.17}$$

and from

$$\frac{dS}{dw_k} = \sum \xi_i \frac{d\eta_i}{dw_k} - \sum \frac{d(\xi_i^1 \eta_i)}{dw_k}. \tag{15.18}$$

Equations (15.15) and thus also Eqs. (15.14) and (15.13) are readily derived from this. Let us thus assume that we had determined

$$\begin{array}{llll} \xi_i^1, & \xi_i^2, & \ldots, & \xi_i^{p-1}; \\ \eta_i^1, & \eta_i^2, & \ldots, & \eta_i^{p-1}; \\ x_i^1, & x_i^2, & \ldots, & x_i^{p-1}; \\ y_i^1, & y_i^2 & \ldots, & y_i^{p-1}; \\ n_k^0, & n_k^1, & \ldots, & n_k^{p-2}; \\ S_1, & S_2, & \ldots, & S_p, \end{array}$$

and that we wish to determine

$$\xi_i^p, \quad \eta_i^p, \quad x_i^p, \quad y_i^p, \quad n_k^{p-1}, \quad S_{p+1}.$$

Let us equate all terms of order p on both sides of Eq. (15.16) and all terms of the order $p+1$ on both sides of Eqs. (15.17) and (15.18).

For abbreviation, we will put, as in the preceding chapter,

$$\Delta u = \sum n_k^0 \frac{du}{dw_k}.$$

From this, it follows that

$$\Delta y_i^p = \Phi + 2A_i(\xi_i^p \cos w_i + n_i^p \sin w_i) + n_i^0 x_i^p + n_i^{p-1} x_i^1, \tag{15.19}$$

$$\sum 2A_i(\xi_i^1 \xi_i^p + \eta_i^1 \eta_i^p) + \Phi + \text{const.} = 0, \tag{15.20}$$

$$\frac{dS_{p+1}}{dw_k} = \sum \xi_i^p \frac{d\eta_i^1}{dw_k} - \sum \eta_i^p \frac{d\xi_i^1}{dw_k} + \Phi.$$

Noting that

$$d\xi_i^1 = -\eta_i^1 dw_i, \quad d\eta_i^1 = \xi_i^1 dw_i,$$

we can write

$$\frac{dS_{p+1}}{dw_k} = \xi_k^1 \xi_k^p + \eta_k^1 \eta_k^p + \Phi. \tag{15.21}$$

Combining Eqs. (15.20) and (15.21) will yield

$$\Delta S_{p+1} = \Phi + \text{const.} \tag{15.22}$$

On the other hand, Eq. (15.21) can be written as

$$\frac{dS_{p+1}}{dw_k} = \alpha_k x_k^p + \Phi, \tag{15.23}$$

while Eq. (15.19) can be written as

$$\Delta y_i^p = \Phi + n_i^{p-1} \alpha_i. \tag{15.24}$$

Then Eq. (15.22) will yield the value of S_{p+1} and Eq. (15.23) will furnish the quantities x_k^p. By writing that the mean value on the right-hand side of Eq. (12) is zero, we will obtain n_i^{p-1} and Eq. (15.24) will then yield y_i^p. Having thus found y_i^p and x_i^p, we will have ξ_i^p and η_i^p.

For determining these quantities it would also have been possible to use the following equations, which are derived from Eq. (15.24) by equating all terms of the order p on both sides and which are analogous to Eqs. (15.21) of no. 152:

$$\Delta \xi_i^p = 2A_i \eta_i^p + n_i^{p-1} \eta_i^1 + \Phi, \tag{15.25}$$

$$\Delta \eta_i^p = -2A_i \xi_i^p - n_i^{p-1} \xi_i^1 + \Phi. \tag{15.26}$$

By an argument similar to that given in no. 153, it would then have been obvious that the quantities ξ_i^p, η_i^p can be expanded in powers of

$$\alpha_i \cos w_i, \quad \alpha_i \sin w_i,$$

and that this also holds for n_i^p (i.e., those quantities that do not depend on w_i can be expanded in even powers of α_i).

This obviously also is true for the periodic terms of S_{p+1}, in view of Eq. (15.22).

It is known that

$$S_{p+1} = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n + S'_{p+1},$$

where β_k are constants and S'_{p+1} is periodic.

For S'_{p+1}, we learn from Eq. (15.22) and from a reasoning analogous to that in no. 153 that the condition has been satisfied. As to the quantities

β_k, these can be arbitrarily chosen. Thus we can assume that β_k can be expanded in even powers of α_i and is divisible by $(\alpha_k)^2$.

There is no need to repeat here the argument given in no. 153.

Let us merely note what happens when treating Eq. (15.23). This equation yields the value of $\alpha_k x_k^p$ which, naturally, must be divisible by α_k; in fact, I say that dS_{p+1}/dw_k and Φ are divisible by α_k.

It should be noted that, if ψ is a function expandable in powers of $\alpha_i \cos w_i$ and $\alpha_i \sin w_i$ and if this function is expanded in a trigonometric series, then the coefficient of the cosine or of the sine of

$$m_1 w_1 + m_2 w_2 + \cdots + m_n w_n,$$

in this expansion will be divisible by

$$\alpha_1^{|m_1|} \cdot \alpha_2^{|m_2|} \cdots \alpha_n^{|m_n|}.$$

Thus the coefficients of the terms depending on w_k are divisible by α_k; consequently, $d\psi/dw_k$ is divisible by α_k.

$$\frac{dS_{p+1}}{dw_k} = \beta_k + \frac{dS'_{p+1}}{dw_k}$$

and β_k has been selected as divisible by α_k, so that dS'_{p+1}/dw_k must also be divisible according to what we have seen above. Consequently, this holds also for dS_{p+1}/dw_k.

On the other hand, Φ is a sum of terms. Each of these terms is the product of factors one of which has the form

$$\frac{d\xi_i^p}{dw_k} \quad \text{or} \quad \frac{d\eta_i^p}{dw_k},$$

and, consequently, is divisible by α_k.

Therefore, Φ is also divisible by α_k. Q.E.D.

160. Let us assume that F depends on a very small parameter μ and has the form

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots.$$

We are still assuming that F can be expanded in powers of ξ_i and of η_i, that the expansion of F_0 starts with terms of the second degree, and that these terms are written as

$$\sum A_i(\xi_i)^2 + \sum A_i(\eta_i)^2.$$

However, we also assume that the expansion of $F_1, F_2, \ldots,$ starts with terms of the first degree.

We now propose to expand

$$\xi_i, \quad \eta_i, \quad x_i, \quad y_i, \quad n_k, \quad S,$$

not only in powers of the constants α_i but also in powers of these constants and in powers of μ.

Let us denote by

$$\xi_i^{p.q}, \quad \eta_i^{p.q}, \quad x_i^{p.q}, \quad y_i^{p.q}, \quad n_i^{p.q}, \quad S_{p.q}$$

the terms of these series that are of the degree p with respect to α_i and of the degree q with respect to μ.

In addition, we will have

$$\xi_i^{0.0} = \eta_i^{0.0} = 0,$$

$$\xi_i^{1.0} = \alpha_i \cos w_i, \quad \eta_i^{1.0} = \alpha_i \sin w_i,$$

$$n_k^{0.0} = -2A_k.$$

We also have

$$dS = \sum \xi_i d\eta_i - \sum d(\xi_i^{1.0}\eta_i),$$

whence

$$S_{0.0} = S_{1.0} = 0,$$

$$S_{2.0} = -\tfrac{1}{2}\sum (\alpha_i)^2 w_i - \tfrac{3}{4}\sum (\alpha_i)^2 \sin 2w_i.$$

Let us then assume that we had calculated

$$\xi_i^{a.b}, \quad \eta_i^{a.b}, \quad x_i^{a.b}, \quad y_i^{a.b}, \quad n_k^{a-1.b}, \quad S_{a+1.b}$$

$$(a \leqslant p, \quad a+b \leqslant p+q),$$

with exception of the combination $a = p$, $b = q$, and that we wish to calculate

$$\xi_i^{p.q}, \quad \eta_i^{p.q}, \quad x_i^{p.q}, \quad y_i^{p.q}, \quad n_i^{p-1.q}, \quad S_{p+1.q}.$$

Let us return to Eqs. (15.13)–(15.18). On both sides of Eq. (15.16), let us equate all terms of order p with respect to α_i and of order q with respect to μ. Similarly, let us equate in Eqs. (15.17) and (15.18) all terms of order $p+1$ with respect to α_i and of order q with respect to μ.

Let

$$\Delta u = \sum n_k^{0.0} \frac{du}{dw_k}.$$

This will return us to Eqs. (15.19)–(15.26), with the one difference that the single indices (superscripts or subscripts) p, $p+1$, or $p-1$ will be replaced by the double indices $p.q, p+1.q$ or $p-1.q$ and that the single indices 1 or 0 will be replaced by the double indices 1.0 or 0.0.

These equations, as in the preceding number, will be used for successively determining $S_{p+1.q}$, $x_k^{p.q}$, $n_i^{p-1.q}$, $y_i^{p.q}$, and thus $\xi_i^{p.q}$ and $\eta_i^{p.q}$.

As in no. 153, it will be seen that $\xi_i^{p.q}$ and $\eta_i^{p.q}$ can be expanded in powers of

$$\alpha_k \cos w_k \quad \text{and} \quad \alpha_k \sin w_k .$$

It results from this that $\xi_i^{0.q}$ and $\eta_i^{0.q}$ are constants.

On the other hand, it should be noted that the remark in no. 126 in virtue of which the mean values of x_i^p and y_i^p can be arbitrarily chosen is applicable here only with certain restrictions.

Let us return to the reasoning in no. 126. Let us consider the expansion of ξ_i and of η_i in powers of μ and of α_i.

Let us change there α_i and w_i into

$$\alpha_i(1+\varphi_i), \quad w_i+\psi_i,$$

where φ_i and ψ_i are two functions that can be expanded in powers of μ and of $(\alpha_k)^2$ and reduce to zero when these quantities vanish. The values of $\xi_i^{0.q}$, $\eta_i^{0.q}$ will not be modified by this change. As a result, the mean values of

$$x_i^{p.q}, \quad y_i^{p.q} \quad (p>0)$$

can be chosen arbitrarily, but this is not the case for

$$x_i^{0.q}, \quad y_i^{0.q}.$$

It is easy to see that these latter mean values must be zero.

Let us now suppose that we return to Eqs. (15.13)–(15.18) and that, in Eqs. (15.13)–(15.16), we consider the terms of zero degree with respect to α_i and in Eqs. (15.17) and (15.18) the terms of zero or first degree with respect to α_i; this will yield equations whose form will differ somewhat from that of Eqs. (15.19)–(15.26) to which we must thus return.

This difference in form is due primarily to the fact that $n_i^{p-1.q}$ is zero at $p=0$ and to the fact that, since $\xi_i^{0.q}$ and $\eta_i^{0.q}$ are constants,

$$\Delta\xi_i^{0.q} = \Delta\eta_i^{0.q} = 0.$$

It is sufficient to consider Eqs. (15.13), (15.14), (15.17), and (15.18) from which Eqs. (15.15) and (15.16) are directly deduced. For abbreviation, let us put

$$\xi_i^0 = \xi_i^{0.1} + \xi_i^{0.2} + \cdots ,$$
$$\xi_i^1 = \xi_i^{1.0} + \xi_i^{1.1} + \xi_i^{1.2} + \cdots .$$

Let us similarly define η_i^0 and η_i^1 and let F^* be the result of the substitution of ξ_i^0 and η_i^0 in F for ξ_i and η_i.

The terms of zero degree in Eqs. (15.13) and (15.14) will yield

$$\frac{dF^*}{d\xi_i^0} = \frac{dF^*}{d\eta_i^0} = 0.$$

These two equations permit determination, by recurrence, of the quantities $\xi_i^{0.q}$ and $\eta_i^{0.q}$.

The terms of zero and first degree of Eq. (15.17) then yield

$$F^* = \text{const.},$$

$$\frac{dF^*}{d\xi_i^0}\xi_i^1 + \frac{dF^*}{d\eta_i^0}\eta_i^1 = \text{const.}$$

The first of these two equations permits us to determine the constant on the right-hand side [which cannot be arbitrarily chosen as had been done for the constant of Eq. (15.20) when assuming $p > 1$].

The second equation is self-satisfied and the constant on the right-hand side must be zero since the two derivatives of F^* are zero.

This leaves Eq. (15.18); the zero-degree terms furnish

$$d(S_{0.0} + S_{0.1} + S_{0.2} + \cdots) = 0,$$

noting that, since η_i^0 are constants, $d\eta_i^0$ will be zero. It is sufficient, for satisfying this equation, to assume that the quantities $S_{0.q}$ are constants.

The first-degree terms then furnish

$$d(S_{1.0} + S_{1.1} + S_{1.2} + \cdots) = \sum \xi_i^0 d\eta_i^1 - \sum d\xi_i^{1.0}\eta_i^0.$$

It is sufficient, for satisfying this expression, to assume that

$$S_{1.q} = \sum (\xi_i^{0.1}\eta_i^{1.q} + \xi_i^{0.2}\eta_i^{1.q-2} + \cdots + \xi_i^{0.q}\eta_i^{1.0}) - \sum (\xi_i^{1.0}\eta_i^{0.q}).$$

The terms of zero and first degree will thus raise no difficulties, unlike what could have been feared.

Problem of No. 134

161. The same method obviously is applicable to the problem of no. 134. Let us resume the notations of no. 151.

Let us return to Eqs. (15.1)–(15.6) of no. 158, by agreeing that the signs Σ extend not only over all x_i (or over all y_i, or over all w_i, etc.) but also over x_i and x_i' (or over y_i and y_i', or over w_i and w_i', etc.).

As in no. 158, we will then see that Eqs. (15.2) are consequences of Eqs. (15.1), (15.3), (15.4), and (15.6). Therefore, we will retain Eqs. (15.4), (15.6), and (15.1a), which are useful for determining our unknowns.

As in no. 158, we will replace, in these various equations, all quantities x_i, y_i, n_i, and S by their expansions in powers of μ and then equate the coefficients of like powers of μ on both sides .

However, the resulting equations are not the only ones which we will use; we will also employ those derived from the former by equating on both sides the mean values taken only with respect to w_k (but not with respect to w'_k).

Let U be an arbitrary periodic function with respect to w and w'. As in no. 151, we will use the symbol $[U]$ for denoting its mean value taken with respect to w alone and by $[[U]]$ its mean value taken with respect to both w and w'.

This will furnish

$$\frac{d[U]}{dw_k}=0,$$

but, in general,

$$\frac{d[U]}{dw'_k}\gtrless 0.$$

So far as S is concerned, this is not a periodic function but merely a function whose derivatives are periodic.

Thus we will only have

$$\left[\frac{dS}{dw_k}\right]=\text{const.}$$

Let us now suppose that we had completely calculated

$$\begin{array}{llll} x_i^0, & x_i^1, & x_i^2, & \ldots, & x_i^{p-2}, \\ y_i^0, & y_i^1, & & \ldots, & y_i^{p-2}, \\ n_k^0, & n_k^1, & & \ldots, & n_k^{p-1}, \\ S_0, & S_1, & & \ldots, & S_{p-2}, \end{array}$$

as well as x_i^{p-1}, y_i^{p-1}, and S_{p-1} to within an arbitrary function of w' and that we now wish to determine x_i^{p-1}, y_i, p_i^{p-1}, and S_{p-1} and then to calculate n_k^p completely and to calculate x_i^p, y_i^p, and S_p to within an arbitrary function of w'.

Equation (15.9) of no. 158, obtained by equating, in Eq. (15.4), all terms in μ^p, will assume a slightly different form since the right-hand side will no longer be wholly known. This formula will be written as

$$\sum n_k^0 x_k^p = \sum \frac{dF_1}{dy_i^0} y_i^{p-1} + \sum \frac{dF_1}{dx_i^0} x_i^{p-1} + \Phi + \text{const.} \quad (15.9a)$$

In the case of $p = 1$, we simply have

$$\sum n_k^0 x_k^1 = F_1 + \text{const.} \quad (15.9b)$$

It goes without saying that in F_1 the term x_i is assumed to be replaced by x_i^0 and the term y_i by $y_i^0 = w_i$.

The right-hand side of Eq. (15.9a) is not entirely known since x_i^{p-1} and y_i^{p-1} are known only to within an arbitrary function of w'.

Let us now take Eq. (15.10); here again, since the right-hand side is not entirely known, the form will be somewhat modified so that we must write

$$\frac{dS_p}{dw_k} = x_k^p + \sum_i x_i^{p-1} \frac{dy_i^1}{dw_k} + \sum_i x_i^1 \frac{dy_i^{p-1}}{dw_k} + \Phi. \quad (15.10a)$$

Noting now that

$$x_i^{p-1} - [x_i^{p-1}] \quad \text{and} \quad y_i^{p-1} - [y_i^{p-1}]$$

are known, Eqs. (15.9a) and (15.10a) can be written as

$$\sum n_k^0 x_k^p = \sum \frac{dF_1}{dy_i^0} [y_i^{p-1}] + \sum \frac{dF_1}{dx_i^0} [x_i^{p-1}] + \Phi + \text{const.}, \quad (15.9c)$$

$$\frac{dS_p}{dw_k} = x_k^p + \sum_i [x_i^{p-1}] \frac{dy_i^1}{dw_k} + \sum_i x_i^1 \frac{d[y_i^{p-1}]}{dw_k} + \Phi. \quad (15.10b)$$

One sees the role played by dy_i^1/dw_k; it is this that leads me to first determine this quantity by a detailed study of the first approximation. For this, we have the above equation (15.9b) as well as the equation

$$\frac{dS_1}{dw_k} = x_k^1, \quad (15.10c)$$

such that Eq. (15.9b) will become

$$\sum n_k^0 \frac{dS_1}{dw_k} = F_1 + \text{const.}$$

We note first that the quantities $n_k'^0$ are zero and that, consequently, we can write

$$\mathsf{S}\, n_k^0 \frac{dS_1}{dw_k} = F_1 + \text{const.}, \quad (15.9d)$$

where S designates a summation extending over only x_i or over only x_i', while Σ is to denote, as above, a summation extending simultaneously over x_i and x_i'. Using the mean values of the two sides, and since

$$\left[\frac{dS_1}{dw_k}\right] = \text{const.},$$

we will obtain

$$[F_1] = \text{const.}$$

However, $[F_1]$ is actually R which is independent of x_i^0 and of $x_i'^0$. Since these are arbitrary constants, the constant on the right-hand side will also be arbitrary and Eq. (15.9d) can be integrated without difficulty.

Let us now, on both sides of Eq. (151a), equate the first-degree terms, so that

$$\sum_k n_k^0 \frac{dy_i^1}{dw_k} + n_i^1 = -\sum_k \frac{d^2F_0}{dx_i^0 dx_k^0} x_k^1 - \frac{dF_1}{dx_i^0}.$$

The right-hand side is wholly known. In fact, the second term there depends only on x_i^0 and on $y_i^0 = w_i$. The first term, in addition, depends on x_k^1 (but not on $x_k'^1$ since F_0, by hypothesis, does not depend on x_i'). However, these quantities are equal to dS_1/dw_k which are known, since S_1 had been determined to within an arbitrary function of w_k'.

In addition, the mean value of this right-hand side, taken with respect to w_i along, is a constant.

In fact, this mean value is equal to

$$-\sum \frac{d^2F_0}{dx_i^0 dx_k^0} [x_k^1] - \frac{d[F_1]}{dx_i^0}.$$

However,

$$[x_k^1] = \left[\frac{dS_1}{dw_k}\right] = \text{const.},$$

$$\frac{d[F_1]}{dx_i^0} = \frac{dR}{dx_i^0} = \text{const.},$$

since R depends only on x_i^0 and $x_i'^0$ which both are constants.

Thus since the mean value of the right-hand side is a constant, we can equate it to n_i^1. The quantity $n_i'^1$ will be calculated in a similar manner except that, in this case, the first term will be absent and the equation reduces simply to

$$n_i'^1 = -\frac{dR}{dx_i'^0}.$$

The equation can then be integrated without difficulty and will yield y_i^1 to within an arbitrary function of w'.

What we stated with respect to y_i^1 holds without change for $y_i'^1$. As to $x_i'^1$, this is equal to dS_1/dw_i' and, consequently, is known to within an arbitrary function of w'.

Let us now return to Eqs. (15.9c) and (15.10b).

Let us take the mean values of both sides with respect to w alone, and let us first perform this operation for Eq. (15.10b) by assuming that the derivative dS_p/dw_k has been taken with respect to one of the quantities w_k rather than with respect to one of the quantities w_k'. We will have

$$\left[\frac{dS_p}{dw_k}\right] = \text{const.}, \qquad \frac{d\,[y_i^{p-1}]}{dw_k} = 0,$$

$$\left[[x_i^{p-1}]\,\frac{dy_i^1}{dw_k}\right] = [x_i^{p-1}]\left[\frac{dy_i^1}{dw_k}\right] = 0,$$

from which it finally follows that

$$[x_k^p] = \Phi + \text{arb. const.} \tag{15.10d}$$

Let us operate in the same manner for Eq. (15.9c), so that (since $n_k'^0 = 0$),

$$\sum n_k^0 x_k^p = \mathsf{S} n_k^0 x_k^p,$$

$$\left[\frac{dF_1}{dy_i^0}\right] = \frac{dR}{dy_i^0} = 0,$$

$$\left[\frac{dF_1}{dx_i^0}\right] = \frac{dR}{dx_i^0} = \text{a given const.}$$

Thus we will have

$$\mathsf{S} n_k^0\,[x_k^p] = \sum \frac{dR}{dx_i^0}\,[x_i^{p-1}] + \Phi + \text{const.}, \tag{15.9e}$$

whence, using Eq. (15.10d),

$$\sum \frac{dR}{dx_i^0}\,[x_i^{p-1}] = \Phi + \text{const.}$$

or

$$\mathsf{S}\frac{dR}{dx_i^0}\,[x_i^{p-1}] + \mathsf{S}\frac{dR}{dx_i'^0}\,[x_i'^{p-1}] = \Phi + \text{const.}$$

However, $[x_i^{p-1}]$ is known to within a constant. In fact, we have an equation analogous to Eq. (15.10d) on changing p into $p-1$:

$$[x_i^{p-1}] = \Phi + \text{const.}$$

Taking the equality

$$n_i'^1 = -\frac{dR}{dx_i'^0}.$$

into consideration, we can thus write

$$\mathsf{S} n_i'^1 [x_i'^{p-1}] = \Phi + \text{const.}$$

Let us now return to Eq. (15.10b) but change there w_k into w_k' and p into $p - 2$. Noting that $[x_i^{p-2}]$ and $[y_i^{p-2}]$ are known, this equation can be written as

$$\frac{dS_{p-1}}{dw_k'} = x_k'^{p-1} + \Phi. \tag{15.10e}$$

Here, S_{p-1} is a sum of terms some of which are periodic with respect to w and to w' while others reduce to a constant multiplied by one of the quantities w or by one of the quantities w'. This results from the above hypothesis that the derivatives of S_{p-1} are periodic.

If, in this sum of terms, we omit all those that depend on w, we will be left with a function of w', which can be designated by $[S_{p-1}]$; since we had assumed that the function S_{p-1} is known to within an arbitrary function of w', we can say that we know $S_{p-1} - [S_{p-1}]$ but not $[S_{p-1}]$.

We then have

$$\frac{d[S_{p-1}]}{dw_k'} = [x_k'^{p-1}] + \Phi$$

and, consequently,

$$\mathsf{S} n_i'^1 \frac{d[S_{p-1}]}{dw_i'} = \Phi + \text{const.},$$

which is an equation yielding $[S_{p-1}]$ and thus achieving the determination of S_{p-1}.

Equation (15.10e) and the analogous equation

$$\frac{dS_{p-1}}{dw_k} = x_k^{p-1} + \Phi$$

thus achieve the determination of x_k^{p-1} and $x_k'^{p-1}$.

Next, let us equate, on both sides of Eq. (15.1a), the terms of degree p, so that

$$\sum n_k^0 \frac{dy_i^p}{dw_k} + \sum n_k^1 \frac{dy_i^{p-1}}{dw_k} + n_i^p = \Phi + A + B, \tag{15.12a}$$

where A represents the coefficient of μ^p in $-dF_0/dx_i$ while B is the coefficient of μ^{p-1} in $-dF_1/dx_i$.

The quantity F_0 depends only on the x_i which now are wholly known, up to and including x_i^{p-1}. Thus we can write

$$A = \Phi - \mathsf{S} \frac{d^2 F_0}{dx_i^0 dx_k^0} x_k^p.$$

Similarly, since the x_i^{p-1} are wholly known, we will have

$$B = \Phi - \sum_k \frac{d^2 F_1}{dy_k^0 dx_i^0} y_k^{p-1},$$

or even, since we know

$$y_k^{p-1} - [y_k^{p-1}],$$

$$B = \Phi - \sum_k \frac{d^2 F_1}{dy_k^0 dx_i^0} [y_k^{p-1}].$$

On the other hand, the quantities $n_k'^0$ are zero and, since y_i^{p-1} is known to within an arbitrary function of w', we will have

$$\sum n_k^0 \frac{dy_i^p}{dw_k} = \mathsf{S} n_k^0 \frac{dy_i^p}{dw_k},$$

$$\sum n_k^1 \frac{dy_i^{p-1}}{dw_k} = \Phi + \mathsf{S} n_k'^1 \frac{dy_i^{p-1}}{dw_k'},$$

whence

$$\begin{aligned} &\mathsf{S} n_k^0 \frac{dy_i^p}{dw_k} + \mathsf{S} n_k'^1 \frac{dy_i^{p-1}}{dw_k'} + n_i^p \\ &\quad = \Phi - \sum \frac{d^2 F_0}{dx_i^0 dx_k^0} x_k^p - \sum \frac{d^2 F_1}{dy_k^0 dx_i^0} [y_k^{p-1}]. \end{aligned} \tag{15.12b}$$

Let us now take the mean value of both sides and note that

$$\left[\frac{d^2 F_1}{dy_k^0 dx_i^0}\right] = \frac{d^2 R}{dy_k^0 dx_i^0} = 0;$$

it follows that

$$\mathsf{S} n_k'^1 \frac{d[y_i^{p-1}]}{dw_k'} = \Phi - \sum \frac{d^2 F_0}{dx_i^0 dx_k^0} [x_k^p] - n_i^p. \tag{15.12c}$$

We have found above

$$[x_k^p] = \Phi + \text{arb. const.}, \tag{15.10d}$$

which means that, since the constant on the right-hand side can be arbitrarily chosen, $[x_k^p]$ is known. We thus have

$$\mathsf{S} n_k'^1 \frac{d[y_i^{p-1}]}{dw_k'} = \Phi - n_i^p. \tag{15.12d}$$

The term n_i^p can be used for canceling the mean value of the right-hand side, after which Eq. (15.12d) is readily integrated, yielding $[y_i^{p-1}]$.

In a similar manner $n_i'^p$ and $[y_i'^{p-1}]$ is calculated, so that $x_i^{p-1}, x_i'^{p-1}, y_i^{p-1}, y_i'^{p-1}$ will now be wholly known.

Then, Eqs. (15.9a) and (15.10a) can be written as

$$\sum n_k^0 x_k^p = \mathrm{S} n_k^0 x_k^p = \Phi + \text{const.}, \tag{15.9f}$$

$$\frac{dS_p}{dw_k} = x_k^p + \Phi, \tag{15.10f}$$

$$\frac{dS_p}{dw_k'} = x_k'^p + \Phi, \tag{15.10g}$$

from which we obtain the equation

$$\mathrm{S} n_k^0 \frac{dS_p}{dw_k} = \Phi + \text{arb. const.},$$

which determines S_p to within an unknown function of w' (since, above, we selected $[S_{p-1}]$ in such a manner that the mean value of the right-hand side reduced to a constant); that is, it determines

$$S_p - [S_p].$$

Equations (15.10f) and (15.10g) will then yield x_k^p and $x_k'^p$ to within functions of w', i.e., they will determine

$$x_k^p - [x_k^p], \quad x_k'^p - [x_k'^p].$$

We should add that, since Eq. (15.10d) already yields $[x_k^p]$, the quantity x_k^p is wholly known but not the quantity $x_k'^p$.

This changes Eq. (15.12b) into

$$\mathrm{S} n_k^0 \frac{dy_i^p}{dw_k} = \Phi. \tag{15.12e}$$

The mean value of the right-hand side is zero because of Eq. (15.12c). From this, we derive

$$y_i^p - [y_i^p],$$

and, similarly,

$$y_i'^p - [y_i'^p].$$

Three-Body Problem

162. As independent variables we will take

$$\Lambda, \quad \Lambda', \quad \sigma_i,$$
$$\lambda_1, \quad \lambda'_1, \quad \tau_i$$

and, as in no. 152, by eliminating the indices that now have become useless, we will use λ and λ' instead of λ_1 and λ'_1.

Next we will attempt to satisfy the equations of the problem by substituting each of these variables by series (14.20) of no. 152 and the series (14.36) of no. 155; we will proceed in powers of μ and of certain constants denoted in nos. 152 and 155 by x_i^0 and $x_i'^0$ and denoted here by α_i, in analogy with the notations of no. 159 to avoid confusion.

In addition, we have

$$\Lambda_{0.0} = \text{const.}, \quad \Lambda'_{0.0} = \text{const.}; \quad \lambda_{0.0} = w_1, \quad \lambda'_{0.0} = w_2;$$
$$\Lambda_{0.q} = \Lambda'_{0.q} = \lambda_{0.q} = \lambda'_{0.q} = 0 \quad (q > 0);$$
$$\sigma_i^{0.0} = \tau_i^{0.0} = 0;$$
$$\sigma_i^{0.1} = \alpha_i \cos w'_i; \quad \tau_i^{0.1} = \alpha_i \sin w'_i.$$

The next step is to form the equation that is to be analogous to Eq. (15.6) and to Eq. (15.18).

This equation will be

$$\frac{dS}{dw_k} = (\Lambda - \Lambda_{0.0}) \frac{d\lambda}{dw_k} + (\Lambda' - \Lambda'_{0.0}) \frac{d\lambda'}{dw_k}$$
$$+ \sum \sigma_i \frac{d\tau_i}{dw_k} - \sum \frac{d(\sigma_i^{0.1}\tau_i)}{dw_k} \tag{15.6a}$$

with an analogous equation in which w_k is replaced by w'_k.

To Eq. (15.6a) and to the kinetic energy equation

$$F = \text{const.} \tag{15.4}$$

we will add the following equations: first,

$$\frac{d\lambda}{dt} = \sum_k n_k \frac{d\lambda}{dw_k} = -\frac{dF}{d\Lambda}, \tag{15.1b}$$

to which still another equation of the same form should be added, in which Λ and λ are replaced by Λ' and λ'. Further, let us agree once and for all, without having to repeat it again, that to each equation that is not symmetric in Λ and Λ', λ and λ', etc., another equation where these symbols are interchanged is to be added. The symbols Σ and S retain the meaning they had in the preceding number. Secondly, we will again have equations analogous to Eqs. (15.16) of no. 159.

For this, as in no. 159, let us put

$$x_i = \sigma_i \cos w_i' + \tau_i \sin w_i',$$

$$y_i = \sigma_i \sin w_i' - \tau_i \cos w_i',$$

$$x_i^{p.q} = \sigma_i^{p.q} \cos w_i' + \tau_i^{p.q} \sin w_i', \quad \ldots,$$

whence

$$x_i^{0.1} = \alpha_i, \quad y_i^{0.1} = 0.$$

We then will have

$$\sum_k n_k \frac{dx_i}{dw_k} = \frac{dF}{d\tau_i} \cos w_i' - \frac{dF}{d\sigma_i} \sin w_i' - n_i' y_i, \tag{15.27}$$

$$\sum_k n_k \frac{dy_i}{dw_k} = \frac{dF}{d\tau_i} \sin w_i' + \frac{dF}{d\sigma_i} \cos w_i' + n_i' x_i. \tag{15.28}$$

It is obvious, as in the preceding numbers, that the equation

$$\frac{d\Lambda}{dt} = \frac{dF}{d\lambda}$$

as well as Eqs. (15.27) are a necessary consequence of Eqs. (15.6a), (15.4), (15.16), and (15.28). Thus these latter are sufficient for solving the problem.

The problem, posed in this manner, presents a combination of all the difficulties that we had solved separately in the first number of this chapter. The same procedures are applicable here.

We will use the following notation for abbreviating some of the notation; let us write

$$\sigma_i^p = \sum_q \sigma_i^{p.q},$$

$$\sigma_i^{(q)} = \sum_p \sigma_i^{p.q}$$

[see series (14.36)]. Similar notations will be used for symbols other than σ_i.

After this, we can start by canceling μ in all our equations; Eq. (15.4) will yield

$$F_0(\Lambda_0, \Lambda_0') = \text{const.} \tag{15.4a}$$

Since the constant on the right-hand side is arbitrary, we will satisfy this equation by giving Λ_0 and Λ_0' arbitrary constant values. We can assume, as we did above, that these constants are independent of α_i, i.e., that

$$\Lambda_{0.q} = 0 \quad \text{for } q > 0.$$

We will then have, starting from Eq. (15.6a),

$$\frac{dS_0}{dw_k} = \sum \sigma_i^0 \frac{d\sigma_i^0}{dw_k} - \sum \frac{d(\sigma_i^{0.1}\tau_i^0)}{dw_k}. \qquad (15.6b)$$

So far as Eq. (15.1b) is concerned, it will reduce to

$$n_1^0 = -\frac{dF_0}{d\Lambda_0}$$

and, similarly,

$$n_2^0 = -\frac{dF_0}{d\Lambda_0'}.$$

Equations (15.27) and (15.28) will yield

$$\sum n_k^0 \frac{dx_i^0}{dw_k} = -n_i'^0 y_i^0, \qquad (15.27a)$$

$$\sum n_k^0 \frac{dy_i^0}{dw_k} = n_i'^0 x_i^0. \qquad (15.28a)$$

These equations are satisfied by assuming that the terms $n_i'^0$ are zero and that the terms x_i^0, y_i^0, σ_i^0, and τ_i^0 do not depend on w but only on w'.

Let us now determine S_0 or, rather,

$$S_0 - [S_0].$$

Since σ_i^0 and τ_i^0 are independent of w, Eq. (6.1a) will yield

$$\frac{dS_0}{dw_1} = \frac{dS_0}{dw_2} = 0,$$

which means that S_0 does not depend on w but only on w'.

Let us now consider all terms of the first degree in μ in our equations. Then Eq. (15.4) will give

$$\mathsf{S}n_1^0\Lambda_1 = n_i^0\Lambda_1 + n_2^0\Lambda_1' = F_1 + \text{const.} \qquad (15.4b)$$

Equation (15.6a) will furnish

$$\frac{dS_1}{dw_k} = \mathsf{S}\Lambda_1 \frac{d\lambda_0}{dw_k} + \sum \sigma_i^1 \frac{d\tau_i^0}{dw_k} + \sum \sigma_i^0 \frac{d\tau_i^1}{dw_k} - \sum \frac{d(\sigma_i^{0.1}\tau_i^1)}{dw_k}. \quad (15.6c)$$

The first term on the right-hand side obviously reduces to Λ_1 for $k = 1$ and to Λ_1' for $k = 2$, since we know that

$$\lambda_0 = w_1, \quad \lambda_0' = w_2.$$

For the same reason, on transforming w_k into w_k', we obtain

$$\frac{dS_1}{dw_k'} = \sum \left[\sigma_i^1 \frac{d\tau_i^0}{dw_k'} + \sigma_i^0 \frac{d\tau_i^1}{dw_k'} - \frac{d(\sigma_i^{0.1}\tau_i^1)}{dw_k'} \right]. \qquad (15.6d)$$

It has been shown above that τ_i^0 depends neither on w_1 nor on w_2, and the same holds for σ_i^0; thus it follows that

$$\frac{d\,[S_1]}{dw_k} = \text{const.}, \quad \frac{d\tau_i^0}{dw_k} = 0, \quad \left[\frac{d(\sigma_i^{0.1}\tau_i^1)}{dw_k}\right] = 0,$$

$$\left[\sigma_i^0 \frac{d\tau_i^1}{dw_k}\right] = [\sigma_i^0]\left[\frac{d\tau_i^1}{dw_k}\right] = 0.$$

If we take the mean values of both sides in Eq. (15.6c), and if we successively set $k = 1$ or 2, we will have

$$[\Lambda_1] = \left[\frac{dS_1}{dw_1}\right] = \text{const.},$$

$$[\Lambda_1'] = \left[\frac{dS_1}{dw_2}\right] = \text{const.}$$

Taking the mean values of both sides in Eq. (15.4b), the left-hand side will become an arbitrary constant with which the constant of the right-hand side can be interchanged, with the result that

$$[F_1] = R = \text{const.} \tag{15.4c}$$

In F_1, the variables $\Lambda, \lambda, \sigma_i$, and τ_i are assumed to be replaced by Λ_0, λ_0, σ_i^0, and τ_i^0. Since the variables λ_0 and λ_0' have disappeared from R, R will remain a function of Λ_0, σ_i^0, and τ_i^0. This function can be expanded in powers of σ_i^0 and of τ_i^0. The lowest-degree terms are of the second degree and are written as

$$\sum A_i(\sigma_i^0)^2 + \sum A_i(\tau_i^0)^2.$$

Let us now pass to Eq. (15.1b); the first-degree terms in μ will yield

$$\mathsf{S}n_k^0 \frac{d\lambda_1}{dw_k} + \mathsf{S}n_k^1 \frac{d\lambda_0}{dw_k} = -\frac{dF_1}{d\Lambda_0} - \frac{d^2F_0}{d\Lambda_0^2}\Lambda_1 - \frac{d^2F_0}{d\Lambda_0\,d\Lambda_0'}\Lambda_1'. \tag{15.1c}$$

Taking the mean values of both sides, we obtain

$$n_1^1 = -\frac{dR}{d\Lambda_0} - \frac{d^2F_0}{d\Lambda_0^2}[\Lambda_1] - \frac{d^2F_0}{d\Lambda_0 d\Lambda_0'}[\Lambda_1']. \tag{15.1d}$$

This equation will be used below for determining n_1^1.

Let us now turn to Eqs. (15.27) and (15.28), which yield

$$\sum n_k^0 \frac{dx_i^1}{dw_k} + \sum n_k^1 \frac{dx_i^0}{dw_k} = \frac{dF_1}{d\tau_i^0}\cos w_i' - \frac{dF_1}{d\sigma_i^0}\sin w_i' - n_i'^1 y_i^0, \tag{15.27b}$$

$$\sum n_k^0 \frac{dy_i^1}{dw_k} + \sum n_k^1 \frac{dy_i^0}{dw_k} = \frac{dF_1}{d\tau_i^0}\sin w_i' - \frac{dF_1}{d\sigma_i^0}\cos w_i' + n_i'^1 x_i^0 \tag{15.28b}$$

or, taking the mean values of both sides and noting that x_i^0, y_i^0, σ_i^0, τ_i^0 depend only on w',

$$\mathrm{S}n_k'^1 \frac{dx_i^0}{dw_k'} = \frac{dR}{d\tau_i^0} \cos w_i' - \frac{dR}{d\sigma_i^0} \sin w_i' - n_i'^1 y_i^0, \qquad (15.27\text{c})$$

$$\mathrm{S}n_k'^1 \frac{dy_i^0}{dw_k'} = \frac{dR}{d\tau_i^0} \sin w_i' - \frac{dR}{d\sigma_i^0} \cos w_i' - n_i'^1 y_i^0. \qquad (15.28\text{c})$$

We are now in a position to determine the quantities S_0, σ_i^0, τ_i^0, and $n_k'^1$. The analogy with the problem in no. 159 is quite obvious.

We pass from the present problem to that of no. 159 by changing, respectively,

$$\sigma_i^0, \quad \tau_i^0, \quad x_i^0, \quad y_i^0, \quad n_k'^1, \quad w_k', \quad R, \quad S_0$$

into

$$\xi_i, \quad \eta_i, \quad x_i, \quad y_i, \quad n_k, \quad w_k, \quad F, \quad S.$$

In that case, Eqs. (15.4c), (15.27c), and (15.28c) are, respectively, equivalent to Eqs. (15.17), (15.15), and (15.16) of no. 159. Similarly, Eq. (15.6b′), obtained on replacing w_k in Eq. (15.6b) by w_k', is equivalent to Eq. (15.18) of no. 159.

It is true that R depends not only on σ_i^0 and on τ_i^0 but also on Λ_0 and Λ_0'. However, these quantities—as we have demonstrated above—must reduce to constants.

Consequently, the procedures of no. 159 become applicable and will furnish

$$\sigma_i^0, \quad \tau_i^0, \quad n_k'^1, \quad S_0.$$

According to Eq. (15.4c), the quantity R reduces to a constant which latter will depend on α_i, Λ_0, and Λ_0' which are our integration constants.

From this it results that $dR/d\Lambda_0$ is still a constant. Since $[\Lambda_1]$, $[\Lambda_1']$, and the derivatives of F_0 are still constants, the right-hand side of Eq. (15.1d) will thus also be a constant, which makes it possible to set it equal to n_1^1.

The quantity n_2^1 is calculated in exactly the same manner.

The right-hand side of Eqs. (15.27b) and (15.28b) are now completely known, which makes it possible to write these equations in the form

$$\mathrm{S}n_k^0 \frac{dx_i^1}{dw_k} = \Phi,$$

$$\mathrm{S}n_k^0 \frac{dy_i^1}{dw_k} = \Phi.$$

The mean value of the right-hand side is zero because of Eqs. (15.27c) and (15.28c); thus these equations permit calculating

$$x_i^1 - [x_i^1], \quad y_i^1 - [y_i^1],$$

and, consequently,

$$\sigma_i^1 - [\sigma_i^1], \quad \tau_i^1 - [\tau_i^1], \quad \frac{d\tau_i^1}{dw_k}, \quad \ldots .$$

However, it is preferable to operate differently.

Equating on both sides of

$$\frac{d\tau_i}{dt} = -\frac{dF}{d\sigma_i} \tag{15.29}$$

the terms in μ, we obtain

$$\mathsf{S}n_k^0 \frac{d\tau_i^1}{dw_k} = \Phi, \tag{15.30}$$

which furnishes the value of

$$\tau_i^1 - [\tau_i^1], \quad \frac{d\tau_i^1}{dw_k}.$$

Equation (15.6c), for $w_k = w_1$, will then give

$$\frac{dS_1}{dw_1} = \Lambda_1 + \sum (\sigma_i^0 - \sigma_i^{0.1}) \frac{d\tau_i^1}{dw_1},$$

whence

$$\Lambda_1 = \frac{dS_1}{dw_1} + \Phi, \quad \Lambda_1' = \frac{dS}{dw_2} + \Phi.$$

Then, since F_1 is known and since the constant on the right-hand side of Eq. (15.4b) had above been selected in an arbitrary manner, Eq. (15.4b) will become

$$\mathsf{S}n_k^0 \frac{dS_1}{dw_k} = \Phi,$$

an equation which determines

$$S_1 - [S_1]$$

and, consequently,

$$\Lambda_1 - [\Lambda_1], \quad \Lambda_1' - [\Lambda_1'].$$

Since, as stated above, $[\Lambda_1]$ and $[\Lambda_1']$ are constants that can be chosen arbitrarily, Λ_1 and Λ_1' are known.

Equation (15.6d) then gives us

$$\sum \sigma_i^1 \frac{d\tau_i^0}{dw_k'} = \frac{dS_1}{dw_k'} + \Phi, \tag{15.31}$$

or, taking the mean values and noting that $d\tau_i^0/dw_k'$ does not depend on w_k,

$$\sum [\sigma_i^1] \frac{d\tau_i^0}{dw_k'} = \frac{d[S_1]}{dw_k'} + \Phi$$

or else, by subtracting and noting that $S_1 - [S_1]$ is known,

$$\sum \frac{d\tau_i^0}{dw_k'} (\sigma_i^1 - [\sigma_i^1]) = \Phi. \tag{15.32}$$

This yields a sequence of linear equations, from which we can derive

$$\sigma_i^1 - [\sigma_i^1].$$

Let us note that the equation

$$\mathsf{S} n_k^0 \frac{d\sigma_i^1}{dw_k} = \Phi, \tag{15.33}$$

deduced from

$$\frac{d\sigma_i}{dt} = \frac{dF}{d\tau_i}, \tag{15.34}$$

by equating the terms in μ, is a consequence of Eqs. (15.29), (15.30), (15.32), and of the previously satisfied equations (15.4a), (15.4b), (15.1b), (15.27a), (15.28a), (15.6b), and (15.6c).

This is practically obvious and we will return to this point later. It is then easy to derive Eqs. (15.27b) and (15.28b) from this.

Since, on the other hand,

$$\mathsf{S} n_k^1 \frac{d\lambda_0}{dw_k} = n_1^1$$

is known through Eq. (15.1d), we can write Eq. (15.1c) in the form

$$\mathsf{S} n_k^0 \frac{d\lambda_1}{dw_k} = \Phi.$$

Since the mean value of Φ is zero according to Eq. (15.1d), this equation will yield

$$\lambda_1 - [\lambda_1].$$

Similarly, we obtain

$$\lambda_1' - [\lambda_1'].$$

Let us now consider, in our equations, all second-degree terms in μ. First, Eq. (15.4a) will furnish

$$Sn_1^0 \Lambda_2 = \sum \frac{dF_1}{d\sigma_i^0}\sigma_i^1 + \sum \frac{dF_1}{d\tau_i^0}\tau_i^1 + \sum \frac{dF_1}{d\lambda_0}\lambda_1 + \Phi + \text{const.} \qquad (15.4\text{d})$$

Similarly, Eq. (15.6a) yields

$$\frac{dS_1}{dw_k} = S\Lambda_2 \frac{d\lambda_0}{dw_k} + S\Lambda_1 \frac{d\lambda_1}{dw_k}$$
$$+ \sum \left[\sigma_i^2 \frac{d\tau_i^0}{dw_k} + \sigma_i^1 \frac{d\tau_i^1}{dw_k} + \sigma_i^0 \frac{d\tau_i^2}{dw_k} - \frac{d(\sigma_i^{0.1}\tau_i^2)}{dw_k}\right]. \qquad (15.6\text{e})$$

Let us take the mean values of the two sides. I say that the mean value of the right-hand side reduces to

$$[\Lambda_2] + \Phi.$$

In fact, Λ_1 and $\lambda_1 - [\lambda_1]$ are known so that also $d\lambda_1/dw_k$ is known. It will be shown that, as in the treatment of Eq. (15.6c),

$$\left[\sigma_i^2 \frac{d\tau_i^0}{dw_k}\right] = \left[\sigma_i^0 \frac{d\tau_i^2}{dw_k}\right] = \left[\frac{d(\sigma_i^{0.1}\tau_i^2)}{dw_k}\right] = 0.$$

On the other hand,

$$\left[\sigma_i^1 \frac{d\tau_i^1}{dw_k}\right] = \left[(\sigma_i^1 - [\sigma_i^1])\frac{d\tau_i^1}{dw_k}\right] + \left[[\sigma_i^1]\frac{d\tau_i^1}{dw_k}\right].$$

The first term on the right-hand side is known since σ_i^1 and τ_i^1 are known to within a function of w'. The second term is zero since

$$\left[[\sigma_i^1]\frac{d\tau_i^1}{dw_k}\right] = [\sigma_i^1]\left[\frac{d\tau_i^1}{dw_k}\right] = 0.$$

Thus we finally obtain

$$[\Lambda_2] = \left[\frac{dS_2}{dw_k}\right] + \Phi = \Phi + \text{arb. const.}$$

We will now take the mean value of both sides of Eq. (15.4d). We have just obtained the mean value of $[\Lambda_2]$; let us now consider a term on the right-hand side, for example,

$$\frac{dF}{d\sigma_i^0}\sigma_i^1.$$

We obtain

$$\left[\frac{dF_1}{d\sigma_i^0}\sigma_i^1\right] = \left[\frac{dF_1}{d\sigma_i^0}(\sigma_i^1 - [\sigma_i^1])\right] + \left[\frac{dF_1}{d\sigma_i^0}[\sigma_i^1]\right]$$

$$= \Phi + \left[\frac{dF_1}{d\sigma_i^0}\right][\sigma_i^1] = \Phi + \frac{dR}{d\sigma_i^0}[\sigma_i^1].$$

Operating in the same manner on the other terms of Eq. (15.4d), by combining into a single function the known functions Φ and the arbitrary constants, we find

$$\sum\left(\frac{dR}{d\sigma_i^0}[\sigma_i^1] + \frac{dR}{d\tau_i^0}[\tau_i^1]\right) = \Phi + \text{const.} \tag{15.4e}$$

It is known in fact that

$$\frac{dR}{d\lambda_0} = 0.$$

Let us now pass to Eq. (15.27) and let us see what this equation will furnish. First, the left-hand side will yield

$$\sum n_k^0 \frac{dx_i^2}{dw_k} + \sum n_k^1 \frac{dx_i^1}{dw_k} + \sum n_k^2 \frac{dx_i^0}{dw_k}.$$

If we take the mean value of this expression, recalling that $n_k'^0$ is zero and that the mean value of a derivative taken with respect to w_1 or w_2 is also zero, we will find

$$\mathrm{S}n_k'^1\left[\frac{dx_i^1}{dw_k'}\right] + \mathrm{S}n_k^2\left[\frac{dx_i^0}{dw_k'}\right].$$

On the right-hand side, let us first consider the term in $dF/d\tau_i$ which gives

$$\frac{dF_2}{d\tau_i^0} + \sum\left(\frac{d^2F_1}{d\tau_i^0 d\Lambda_0}\Lambda_1 + \frac{d^2F_1}{d\tau_i^0 d\lambda_0}\lambda_1 + \frac{d^2F_1}{d\tau_i^0 d\tau_k^0}\tau_k^1 + \frac{d^2F_1}{d\tau_i^0 d\sigma_k^0}\sigma_k^1\right)$$

that is,

$$\Phi + \sum\left(\frac{d^2F_1}{d\tau_i^0 d\lambda_0}[\lambda_1] + \frac{d^2F_1}{d\tau_i^0 d\tau_k^0}[\tau_k^1] + \frac{d^2F_1}{d\tau_i^0 d\sigma_k^0}[\sigma_k^1]\right),$$

so that the mean value will be

$$\Phi + \sum\left(\frac{d^2R}{d\tau_i^0 d\tau_k^1}[\tau_k^1] + \frac{d^2R}{d\tau_i^0 d\sigma_k^0}[\sigma_k^1]\right).$$

We can operate in the same manner for $dF_1/d\sigma_i^0$. This will permit us to write what becomes of Eqs. (15.27) and (15.28) if we take the second-degree terms in μ on both sides. This furnishes

$$S n_k'^1 \frac{d\,[x_i^1]}{dw_k'} = A \cos w_i' - B \sin w_i' - n_i'^1 [y_i^1] - n_i'^2 y_i^0, \qquad (15.27e)$$

$$S n_k'^1 \frac{d\,[y_i^1]}{dw_k'} = A \sin w_i' - B \cos w_i' - n_i'^1 [y_i^1] - n_i'^2 y_i^0, \qquad (15.28e)$$

where A and $-B$ are the right-hand sides of Eqs. (14.41) from which Eqs. (15.27e) and (15.28e) are readily deduced. To Eqs. (15.27e) and (15.28e) we can add the following, obtained by taking the mean values in Eq. (15.6d):

$$\frac{d\,[S_1]}{dw_k'} = \sum \left([\sigma_i^1] \frac{d\tau_i^0}{dw_k'} + \sigma_i^0 \frac{d\,[\tau_i^1]}{dw_k'} - \frac{d(\sigma_i^{0.1}[\tau_i^1])}{dw_k'} \right). \qquad (15.6f)$$

With the aid of Eqs. (15.4e), (15.27e), (15.28e), and (15.6f) we will now determine

$$[\sigma_i^1], \quad [\tau_i^1], \quad n_i'^2 \quad [S_1].$$

The equations are not distinct, and we have demonstrated in Chap. 14 that these quantities can be determined solely from Eqs. (14.41) of no. 155, equivalent to Eqs. (15.27e) and (15.28e).

However, we wish to outline another procedure in which only Eqs. (15.4e), (15.6f), and (15.28e) are used and which more closely resembles the method I have always used in the present chapter.

It might thus be of interest to demonstrate that Eq. (15.27e) can be derived from Eqs. (15.24e), (15.6f), and (15.28e). However, for this it is necessary to examine in more detail the manner in which Eq. (15.27) can be deduced from Eqs. (15.4), (15.6a), and (15.28), and thus make a digression which will occupy the following numbers.

163. Let us return to the problem and the notations of no. 158; the references, unless stated differently, will always pertain to that number. At the beginning of that number, we demonstrated that Eqs. (15.2) are a consequence of Eqs. (15.1), (15.3), (15.4), and (15.6). However, one can also raise the following question: Let us assume that all equations derived from Eqs. (15.1), (15.3), (15.4), and (15.6) have been satisfied by equating, on both sides, all terms independent of μ, all terms in μ, in μ^2, and so on up to terms in μ^p inclusive. Does it follow from this that the equations derived from Eq. (15.2) have simultaneously been satisfied by equating on both sides all wholly known terms as well as the terms in $\mu, \mu^2, \ldots, \mu^p$? In other words, we assume that Eqs. (15.1), (15.3), (15.4), and (15.6) have been satisfied up to terms in μ^{p+1}, i.e., in such a manner that, after substitution of our approximate solution, the difference of the two sides becomes divisible by μ^{p+1}; does it follow from this that Eqs. (15.2) are also satisfied up to terms in μ^{p+1}? If Eqs. (15.1), (15.3), (15.4), and (15.6) are satisfied up to terms in μ^{p+1}, this will also be the case for the equations obtained therefrom by differentiation, addition, or multiplication, such as—for example—Eqs. (15.7) and (15.8). Thus Eqs. (15.7) and (15.8) are still valid but with the one difference that, on the right-hand side, the zero will have to be replaced by a function expandable in powers of μ and divisible by μ^{p+1}.

We will thus have

$$\sum \frac{dy_i}{dw_k}\left(\frac{dF}{dx_i}-\frac{dy_i}{dt}\right)=H,$$

where H is divisible by μ^{p+1}. This then permits the conclusion that

$$\frac{dF}{dx_i}-\frac{dy_i}{dt}$$

is equal to a function of the same form, provided that the determinant of dy_i/dw_k is not divisible by μ. However, this is exactly what happens since the determinant reduces to unity for $\mu=0$.

Consequently, Eqs. (15.2) are satisfied to within terms in μ^{p+1}.

Q.E.D.

Let us now pass to the problem in no. 161. The preceding reasoning applies here without change, but one more question has to be raised in this respect.

Besides the equations derived from Eqs. (15.1a), (15.2), (15.4), and (15.6) by equating on both sides the coefficients of μ^p, we also must consider the equations obtained by equating the mean values on both sides.

We will assume that Eqs. (15.1a), (15.4), and (15.6) are satisfied up to terms in μ^p. As we have just seen, it follows from this that the same holds for Eq. (15.2).

We will assume, in addition, that the equations obtained in the following manner are also satisfied: In Eqs. (15.1a), (15.4), and (15.6), let us equate the coefficients of μ^p and let us then take the mean values of the two sides. Does it follow from this that the equation derived from Eq. (15.2) by the same procedure will also be satisfied?

Our hypotheses can be expressed as follows: Equations (15.1a), (15.4), and (15.6) are not exactly satisfied, but the difference of the two sides is a periodic function of w and w' which can be expanded in powers of μ and is divisible by μ^p and whose mean value, taken with respect to w, is divisible by μ^{p+1}.

We will designate by H any function satisfying these conditions. It results from this that the sum of the two functions H is a function H and that the derivative of H with respect to w_k or w'_k is a function H. If, finally, we multiply H by a function K periodic in w and w' and expandable in powers of μ, then the product will still be a function H provided that, for $\mu = 0$, the function K does not depend on w but only on w'. Then, we will have

$$\sum \left(\frac{dx_i}{dt} \frac{dy_i}{dw_k} - \frac{dx_i}{dw_k} \frac{dy_i}{dt} \right) = H$$

and

$$\frac{dy_i}{dt} = -\frac{dF}{dx_i} + H;$$

$$\frac{dx_i}{dw_k} \left(\frac{dy_i}{dt} + \frac{dF}{dx_i} \right) = H \frac{dx_i}{dw_k} = H,$$

since dx_i/dw_k or dx_i/dw'_k reduces to zero for $\mu = 0$ and, consequently, is independent of w.

It results from this that the right-hand side of Eq. (15.8) will again be a function H. Since the differentiation of Eq. (15.4) yields

$$\sum \left(\frac{dF}{dx_i} \frac{dx_i}{dw_k} + \frac{dF}{dy_i} \frac{dy_i}{dw_k} \right) = H,$$

it follows that

$$\sum \frac{dy_i}{dw_k} \left(\frac{dx_i}{dt} - \frac{dF}{dy_i} \right) = H_k,$$

where H_k is a function H; from this,

$$\frac{dx_i}{dt} - \frac{dF}{dy_i} = \sum H_k \frac{\Delta_{ik}}{\Delta},$$

where Δ is the determinant of the dy_i/dw_k, including therein, of course, there the quantities dy'_i/dw_k, dy_i/dw'_k, and dy'_i/dw'_k. As for Δ_{ik}, it is one of the minors of Δ.

For $\mu = 0$, the determinant Δ reduces to unity, Δ_{ik} to one or to zero; thus Δ_{ik}/Δ is independent of w. Consequently, we have

$$\frac{dx_i}{dt} - \frac{dF}{dy_i} = H.$$

Q.E.D.

164. Let us now return to the hypotheses of no. 159. Let us adopt the notations given there and let us agree that all references will pertain to the equations of no. 159. It is now a question to establish that

(a) Equations (15.15) can be deduced from Eqs. (15.16)–(15.18). This

is a point which we had postulated above without proof but of which we will now give a demonstration which will be useful later.
(b) If Eqs. (15.17) and (15.18) are satisfied up to terms of order $p+2$ with respect to α_i and Eqs. (15.16) up to terms of order $p+1$, then Eqs. (15.15) will also be satisfied up to terms of the order $p+1$ or, in other words, Eqs. (15.25) and (15.26) will be a consequence of Eqs. (15.19)–(15.21).

Equations (15.18), which express that dS is an exact differential, will yield

$$\sum_i \left(\frac{d\xi_i}{dw_k} \frac{d\eta_i}{dw_q} - \frac{d\xi_i}{dw_q} \frac{d\eta_i}{dw_k} \right) = 0, \tag{15.35}$$

from which, as in no. 158, we can derive

$$\sum_i \left(\frac{d\xi_i}{dw_k} \frac{d\eta_i}{dt} - \frac{d\xi_i}{dt} \frac{d\eta_i}{dw_k} \right) = 0. \tag{15.36}$$

Moreover, Eq. (15.17), when differentiated with respect to w_k, yields

$$\sum \left(\frac{dF}{d\xi_i} \frac{d\xi_i}{dw_k} + \frac{dF}{d\eta_i} \frac{d\eta_i}{dw_k} \right) = 0. \tag{15.37}$$

Let us now put

$$\frac{d\xi_i}{dt} \cos w_i + \frac{d\eta_i}{dt} \sin w_i = X_i,$$

$$\frac{d\xi_i}{dt} \sin w_i - \frac{d\eta_i}{dt} \cos w_i = Y_i,$$

$$\frac{d\xi_i}{dw_k} \cos w_i + \frac{d\eta_i}{dw_k} \sin w_i = X_i^k,$$

$$\frac{d\xi_i}{dw_k} \sin w_i - \frac{d\eta_i}{dw_k} \cos w_i = Y_i^k,$$

$$\frac{dF}{d\eta_i} \cos w_i - \frac{dF}{d\xi_i} \sin w_i = A_i,$$

$$\frac{dF}{d\eta_i} \sin w_i + \frac{dF}{d\xi_i} \cos w_i = B_i.$$

Indeed, with these new notations, Eqs. (15.15) and (15.16) will read, respectively,

$$X_i = A_i, \tag{15.38}$$

$$Y_i = B_i. \tag{15.39}$$

Equation (15.36) becomes

$$\sum (X_i^k Y_i - Y_i^k X_i) = 0 \tag{15.36a}$$

and Eq. (15.37) becomes

$$\sum (X_i^k B_i - Y_i^k A_i) = 0. \tag{15.37a}$$

I say that from Eqs. (15.39), (15.36a), and (15.37a) we can derive Eq. (15.38); and, indeed, that from Eqs. (15.36a) and (15.39) we can deduce

$$\sum (X_i^k B_i - Y_i^k X_i) = 0 \tag{15.40}$$

or, finally,

$$\sum_i Y_i^k (X_i - A_i) = 0 \quad (k = 1, 2, \ldots, n). \tag{15.41}$$

Since the determinant of the Y_i^k is not zero, one can conclude from this that

$$X_i = A_i.$$

Q.E.D.

Let us now assume that Eqs. (15.16) are valid up to terms of order $p + 1$ with respect to α_i and that Eqs. (15.17) and (15.18) are valid up to terms of order $p + 2$.

Then, Eqs. (15.35), (15.36), (15.37), and (15.37a) will be valid up to terms of order $p + 2$, and Eq. (15.39) up to terms of order $p + 1$. Since the expansion of X_i^k starts with terms of the first order, a multiplication of Eq. (15.39) by X_i^k will yield an equation valid up to terms of order $p + 2$.

It follows from this that Eqs. (15.40) and (15.41) are satisfied to within terms of the order $p + 2$. I say that, consequently, Eq. (15.38) will also be valid up to terms of order $p + 1$.

In fact, let us put for the moment

$$\alpha_i = \lambda \alpha_i'$$

in such a manner that the terms of order p with respect to α_i become divisible by λ^p.

We will then put

$$X_i - A_i = \lambda^{p+1} C_i,$$
$$Y_i^k = \lambda Z_i^k.$$

What I propose to establish here is that C_i remains finite for $\lambda = 0$.

Since Eq. (15.41) is satisfied up to terms of the order $p + 2$, we will have

$$\sum Y_i^k(X_i - A_i) = \lambda^{p+2} H_k,$$

where H_k remains finite for $\lambda = 0$; hence

$$\sum Z_i^k C_i = H_k.$$

It follows from this that C_i remains finite for $\lambda = 0$ provided that the determinant of Z_i^k does not vanish for $\lambda = 0$.

However, this determinant reduces, for $\lambda = 0$, to

$$\pm \alpha_1' \cdot \alpha_2' \cdots \alpha_n'$$

and thus is not zero. Q.E.D.

165. We will return now to the problem of no. 162. We will prove that Eq. (15.27e) is a consequence of Eqs. (15.4e), (15.6f), and (15.28e), naturally by assuming, as had been done above, that Eqs. (15.4a), (15.4b), (15.6b), (15.6c), (15.28a), (15.28b), (15.1b), and (15.1c) had been satisfied beforehand.

These hypotheses can be formulated in the following manner:

To state that Eqs. (15.4a), (15.4b), and (15.4e) are satisfied means that we have

$$F = \text{const.} + \mu^2 H_0.$$

We will denote by H any function expandable in ascending powers of μ and periodic with respect to w and w', while we will call H_0 any such function whose mean value vanishes for $\mu = 0$.

From this, we can derive

$$\sum \left(\frac{dF}{d\lambda}\frac{d\lambda}{dw_k} + \frac{dF}{d\Lambda}\frac{d\Lambda}{dw_k} + \frac{dF}{d\tau_i}\frac{d\tau_i}{dw_k} + \frac{dF}{d\sigma_i}\frac{d\sigma_i}{dw_k} \right) = \mu^2 H_0. \quad (15.42)$$

To state that Eqs. (15.1b) and (15.1c) are satisfied, means that

$$\frac{d\lambda}{dt} - \frac{dF}{d\Lambda} = \mu^2 H, \quad (15.43)$$

whence, since $d\Lambda/dw_k$ vanishes for $\mu = 0$,

$$\frac{d\Lambda}{dw_k}\left(\frac{d\lambda}{dt} - \frac{dF}{d\Lambda}\right) = \mu^3 H. \quad (15.43a)$$

Let us now pass to the equations deduced from Eq. (15.6a).

We will assume that Eqs. (15.6b), (15.6c), (15.6d) are satisfied, but this is not all. In fact, to establish Eq. (15.4e), we have made use of Eq. (15.6e) or, rather, of an equation, which we might call Eq. (15.6g), that is derived from the former by equating the mean values on both sides.

Consequently, this equation (15.6g) is assumed as satisfied. However, this is not the case for Eq. (15.6g′) which is derived from the former by changing there w_k into w'_k.

How can all this be expressed in our new parlance?

Since Eqs. (15.6b), (15.6c), and (15.6g) are satisfied, we will have

$$\frac{dS}{dw_k} = C_k + \mu^2 H_0,$$

where C_k, for the moment, is to denote the right-hand side of Eq. (15.6a). If w_k is transformed into w'_q, we will obtain, when denoting by C'_q what becomes of C_k,

$$\frac{dS}{dw'_q} = C'_q + \mu^2 H.$$

The mean value of H does not vanish for $\mu = 0$ since Eq. (15.28e′) is not assumed to be satisfied.

On differentiating the first of these equations with respect to w'_q and the second with respect to w_k and subtracting, we have

$$\frac{dC_k}{dw'_q} - \frac{dC'_q}{dw_k} = \mu^2 H_0.$$

Similarly, we would obtain

$$\frac{dC_k}{dw_q} - \frac{dC_q}{dw_k} = \mu^2 H_0.$$

However, we only would have

$$\frac{dC'_k}{dw'_q} - \frac{dC'_q}{dw'_k} = \mu^2 H,$$

without having the mean value of H vanish for $\mu = 0$. Nevertheless, if the equation is multiplied by n'_q which vanishes for $\mu = 0$, we get

$$n'_q \left(\frac{dC'_k}{dw'_q} - \frac{dC'_q}{dw'_k} \right) = \mu^3 H.$$

Thus we will have

$$\sum_q \left[n_q \left(\frac{dC_k}{dw_q} - \frac{dC_q}{dw_k} \right) + n'_q \left(\frac{dC_k}{dw'_q} - \frac{dC'_q}{dw_k} \right) \right] = \mu^2 H_0,$$

with the analogous equation that is derived from this by transforming w_k into w'_k.

This permits us to write

$$\sum \left(\frac{d\Lambda}{dt} \frac{d\lambda}{dw_k} - \frac{d\lambda}{dt} \frac{d\Lambda}{dw_k} + \frac{d\sigma_i}{dt} \frac{d\tau_i}{dw_k} - \frac{d\tau_i}{dt} \frac{d\sigma_i}{dw_k} \right) = \mu^2 H_0 \quad (15.44)$$

together with the equation derived from this by transforming w_k into w'_k.

Let us put, as in the preceding number,

$$\frac{d\sigma_i}{dt}\cos w'_i + \frac{d\tau_i}{dt}\sin w'_i = X_i,$$

$$\frac{d\sigma_i}{dt}\sin w'_i - \frac{d\tau_i}{dt}\cos w'_i = Y_i,$$

$$\frac{d\sigma_i}{dw_k}\cos w'_i + \frac{d\tau_i}{dw_k}\sin w'_i = X_i^k,$$

$$\frac{d\sigma_i}{dw_k}\sin w'_i - \frac{d\tau_i}{dw_k}\cos w'_i = Y_i^k,$$

$$\frac{dF}{d\tau_i}\cos w'_i - \frac{dF}{d\sigma_i}\sin w'_i = A_i,$$

$$\frac{dF}{d\tau_i}\sin w'_i + \frac{dF}{d\sigma_i}\cos w'_i = B_i$$

with other analogous equations where w_k, X_i^k, Y_i^k are replaced by the same primed symbols.

Equations (15.42) and (15.44) then become

$$\sum\left(\frac{dF}{d\lambda}\frac{d\lambda}{dw_k} + \frac{dF}{d\Lambda}\frac{d\Lambda}{dw_k} + X_i^k B_i - Y_i^k A_i\right) = \mu^2 H_0, \qquad (15.42a)$$

$$\sum\left(\frac{d\Lambda}{dt}\frac{d\lambda}{dw_k} + \frac{d\lambda}{dt}\frac{d\Lambda}{dw_k} + X_i^k Y_i - Y_i^k X_i\right) = \mu^2 H_0, \qquad (15.44a)$$

with other analogous equations in which w_k, X_i^k, Y_i^k are replaced by the same primed symbols.

On the other hand, since Eqs. (15.28a), (15.28b), and (15.28c) were assumed as satisfied, we obtain

$$Y_i - B_i = \mu^2 H_0.$$

A combination of all our equations will then yield

$$\sum \frac{d\lambda}{dw_k}\left(\frac{d\Lambda}{dt} - \frac{dF}{d\lambda}\right) + Y_i^k(X_i - A_i) = \mu^2 H_0$$

with another equation in which the terms w_k and Y_i^k are replaced by the same primed symbols.

This constitutes a system of linear equations from which one can find

$$\frac{d\Lambda}{dt} - \frac{dF}{d\lambda} \quad \text{and} \quad X_i - A_i.$$

For $\mu = 0$, what becomes of the coefficients of these equations and of their determinant?

The derivatives of $d\lambda/dw_k$ vanish except for $d\lambda/dw_1$ and $d\lambda'/dw_2$ which reduce to one. The terms Y_i^k vanish. With respect to

$$Y_i'^k = \frac{d\sigma_i^0}{dw_k'} \sin w_i' - \frac{d\tau_i^0}{dw_k'} \cos w_i'$$

this expression is independent of w_1 and w_2.

The determinant and its minors are thus independent of w for $\mu = 0$. In addition, this determinant does not vanish.

From this it follows that

$$X_i - A_i = \mu^2 H_0,$$

which means that Eqs. (15.27a), (15.27b), and (15.27e) are satisfied.

Q.E.D.

I must still establish that, as announced above, Eq. (15.33) of no. 162 is a consequence of Eqs. (15.29), (15.30), (15.32), (15.4a), (15.4b), (15.1b), (15.27a), (15.28a), (15.6b), and (15.6c).

From Eqs. (15.4a) and (15.4b), we deduce

$$A = \mu^2 H, \tag{15.42b}$$

where A is the left side of Eq. (15.42).

From Eq. (15.1b) we obtain

$$\frac{d\lambda}{dt} - \frac{dF}{d\Lambda} = \mu H$$

and

$$\frac{d\Lambda}{dw_k}\left(\frac{d\lambda}{dt} - \frac{dF}{d\Lambda}\right) = \mu^2 H. \tag{15.43b}$$

Let us now pass to the equations derived from Eqs. (15.6a). Since Eqs. (15.6b) and (15.6c) are satisfied, it follows that

$$\frac{dS}{dw_k} = C_k + \mu^2 H.$$

Similarly, Eq. (15.6b′) is satisfied but Eq. (15.6d) is satisfied only to within a function of w'. In fact, we have deduced Eq. (15.32) from Eq. (15.31), which is equivalent to Eq. (15.6d), by subtracting from it another equation whose two sides are unknown functions of w'. Therefore, we can now write

$$\frac{dS}{dw_q'} = C_q' + \mu^2 H + \mu K,$$

where K is independent of the w.

One can deduce from this

$$\frac{dC_k}{dw'_q} - \frac{dC'_q}{dw_k} = \mu^2 H,$$

$$\frac{dC_k}{dw_q} - \frac{dC_q}{dw_k} = \mu^2 H,$$

$$\frac{dC'_k}{dw'_q} - \frac{dC'_q}{dw'_k} = \mu H,$$

or, since n'_q is divisible by μ,

$$n'_q \left(\frac{dC'_k}{dw'_q} - \frac{dC'_q}{dw'_k} \right) = \mu^2 H,$$

or, finally

$$C = \mu^2 H, \qquad (15.44b)$$

where C is the left-hand side of Eq. (15.44) or else this same side with w_k replaced by w'_k.

Equations (15.27a), (15.28a), and (15.32) will yield

$$\frac{d\tau_i}{dt} + \frac{dF}{d\sigma_i} = \mu^2 H.$$

A combination of all our equations will then give

$$\sum \frac{d\lambda}{dw_k} \left(\frac{d\Lambda}{dt} - \frac{dF}{d\lambda} \right) + \frac{d\tau_i}{dw_k} \left(\frac{d\sigma_i}{dt} - \frac{dF}{d\tau_i} \right) = \mu^2 H,$$

which are linear equations from which, as above, we can derive

$$\frac{d\sigma_i}{dt} - \frac{dF}{d\tau_i} = \mu^2 H.$$

Q.E.D.

166. After this long digression, let us return to the problem of no. 162 where we left off. At that point, it was a question of determining $[\sigma_i^1]$ and $[\tau_i^1]$ by means of Eqs. (15.4e), (15.28e), and (15.6f).

For this, we will assume the two terms of our equations as being expanded in powers of α_i and equate all terms of the same degree on both sides.

Equation (15.4e) will start with first-degree terms and, equating all terms of the first degree, we will obtain

$$\sum 2A_i(\sigma_i^{0.1}[\sigma_i^{1.0}] + \tau_i^{0.1}[\tau_i^{1.0}]) = \Phi + \text{const.} \qquad (15.4f)$$

Since the right-hand side of Eq. (15.6f) starts with first-degree terms, we will first find

$$[S_{1.0}] = \text{const.}$$

After this, equating all terms of the first degree, we obtain

$$\frac{d[S_{1.1}]}{dw'_k} = \sum \left[[\sigma_i^{1.0}] \frac{d\tau_i^{0.1}}{dw'_k} + \sigma_i^{0.1} \frac{d[\tau_i^{1.0}]}{dw'_k} - \frac{d(\sigma_i^{0.1}[\tau_i^{1.0}])}{dw'_k} \right]$$

or else

$$\frac{d[S_{1.1}]}{dw'_k} = \sum \left([\sigma_i^{1.0}] \frac{d\tau_i^{0.1}}{dw'_k} - [\tau_i^{1.0}] \frac{d\sigma_i^{0.1}}{dw'_k} \right)$$

$$= [\sigma_k^{1.0}] \frac{d\tau_k^{0.1}}{dw'_k} - [\tau_k^{1.0}] \frac{d\sigma_k^{0.1}}{dw'_k}$$

$$= -\sigma_k^{0.1}[\sigma_k^{1.0}] - \tau_k^{0.1}[\tau_k^{1.0}] = -x_k^{0.1}[x_k^{1.0}]. \qquad (15.6\text{h})$$

In this manner, Eq. (15.4f) will become

$$\sum 2A_i \frac{d[S_{1.1}]}{dw'_i} = \Phi + \text{const.},$$

which yields $[S_{1.1}]$ and, consequently, the terms $[x_k^{1.0}]$.

It then remains to determine the quantities $[y_k^{1.0}]$ and to satisfy Eq. (15.28f), obtained by equating all terms of zero degree in Eq. (15.28e) with respect to α_i. Rigorously, Eq. (15.4f) is sufficient for this if it is recalled that the terms $[\sigma_k^{1.0}]$ and $[\tau_k^{1.0}]$ must be constants since the terms σ_k and τ_k are expandable in powers of $\alpha_i \cos w'_i$ and $\alpha_i \sin w'_i$ so that the terms of zero degree with respect to α_i must be independent of w'_i.

What is now the function Φ on the right-hand side of Eq. (15.4f)? To obtain this function, it is obviously necessary to do the following: Take the function $-F_2$; replace there $\Lambda, \lambda, \sigma_i$, and τ_i by Λ_0, w, σ_i^0, and τ_i^0; take their mean value; consider, in this mean value, the terms of first degree with respect to σ_i^0 and to τ_i^0; replace there σ_i^0 and τ_i^0 by $\sigma_i^{0.1}$ and $\tau_i^{0.1}$. Thus Φ will have the form

$$\sum B_i \sigma_i^{0.1} + \sum C_i \tau_i^{0.1},$$

where B_i and C_i are constants. Then, Eq. (15.4f) is written as follows:

$$\sum 2A_i(\sigma_i^{0.1}[\sigma_i^{1.0}] + \tau_i^{0.1}[\tau_i^{1.0}]) = \sum B_i \tau_i^{0.1} + \sum C_i \tau_i^0 + \text{const.}$$

If the quantities $[\sigma_i^{1.0}]$ and $[\tau_i^{1.0}]$ must be constants, the equation can be satisfied only by canceling the constant and by setting

$$[\sigma_i^{1.0}] = \frac{B_i}{2A_i}, \quad [\tau_i^{1.0}] = \frac{C_i}{2A_i}.$$

I say that, furthermore, Eq. (15.28f) is satisfied in this manner, since thus Eqs. (14.42) of no. 155 are also satisfied, and Eq. (15.28f) is only a simple combination obtained by adding these equations after having multiplied them by $+\sin w_i'$ and $-\cos w_i'$.

Let us now equate the second-degree terms in Eq. (15.4e), so that

$$\sum 2A_i(\sigma_i^{0.1}[\sigma_i^{1.1}] + \tau_i^{0.1}[\tau_i^{1.1}]) = \Phi + \text{const.} \qquad (15.4g)$$

Similarly, on equating the second-degree terms in Eq. (15.6f) we obtain

$$\Phi + \frac{d[S_{1.2}]}{dw_k'} = -(\sigma_k^{0.1}[\sigma_k^{1.1}] - \tau_k^{0.1}[\tau_k^{1.1}]) = -x_k^{0.1}[x_k^{1.1}]. \quad (15.6i)$$

Then, Eq. (15.4g) becomes

$$\sum 2A_i \frac{d[S_{1.2}]}{dw_i'} = \Phi + \text{const.},$$

which yields $[S_{1.2}]$ and, consequently, also the $[x_k^{1.1}]$.

Let us now consider Eq. (15.28g) which is obtained by equating, in Eq. (15.28e), all first-degree terms. This could also be obtained by setting $q = 1$ in Eqs. (14.44) of no. 155, multiplying the first equation by $\sin w_i'$ and the second by $-\cos w_i'$ and then adding them. Let us perform this operation by recalling that the constant, denoted by $x_i^{1.0}$ in no. 155, is now represented by α_i. This yields

$$\mathsf{S} n_k'^{1.0} \frac{d[y_i^{1.1}]}{dw_k'} = \Delta''[y_i^{1.1}] = \Phi - n_i^{1.0}[x_i^{1.1}] + n_i'^{2.1}\alpha_i. \quad (15.2g)$$

We now know $[x_i^{1.1}]$, so that the equation reduces to

$$\Delta''[y_i^{1.1}] = \Phi + n_i'^{2.1}\alpha_i.$$

The quantity $n_i'^{2.1}$ is determined such that the mean value of the right-hand side becomes zero, after which the equation readily will furnish $[y_i^{1.1}]$ and thus $[\sigma_i^{1.1}]$ and $[\tau_i^{1.1}]$.

Continuing in this manner, one would similarly determine the quantities $n_i'^{2.q-1}$, $[\sigma_i^{1.q}]$, and $[\tau_i^{1.q}]$.

After the terms n_i^2, $[\sigma_i^1]$, and $[\tau_i^1]$ have been determined in this fashion, the other quantities can be calculated by the methods given in no. 162.

Each quantity would have to be determined by a procedure differing from this only by the fact that its index (relative to the degree in μ) is less by one unit.

Naturally, it is necessary to observe the same sequence as in no. 162.

The methods of Chap. 15 thus permit reaching the same goal as those of Chap. 14. Several of the calculations are somewhat simplified. In addition, these new methods have an advantage worth mentioning and not exhibited by the methods of the preceding chapter, namely, that of inherently containing the proof of their own feasibility. Thus these methods can be developed without having to go over the intermediary of Chaps. 9 to 13, not to mention the numerous changes of variables that had to be made in those chapters and that are useful only for demonstration purposes but not for the calculations themselves.

CHAPTER 16
Gyldén Methods

167. The methods to be discussed here possess considerable originality. Most of them, despite appearance to the contrary, are based on the methods discussed in the foregoing chapters. However, some of the new methods go beyond the former and permit treatment of problems to which the procedures given in Chaps. 9 and 15 are no longer applicable; thus they have a more intimate relationship with the methods to be discussed later in the text.

Naturally, the manner of approach will differ greatly from that employed by Gyldén.

The Gyldén methods are actually a composite of several artifices that have no necessary interconnection and that should preferably be studied separately, it then being only necessary to effect the synthesis, which can be done by the reader himself without any difficulty.

The first of these artifices is the use of a particular independent variable.

Let us first assume that the three bodies move in the same plane. In this plane, let us consider the motion of one of the planets, subject to the action of a central body whose position we will take as the origin, and subject to the perturbing action of another planet.

Let r and v be the polar coordinates of the planet under study, μ the mass of the central body, and Ω the perturbing function. The equations of motion will be

$$\frac{d\left(r^2\dfrac{dv}{dt}\right)}{dt}=\frac{d\Omega}{dv},\quad \frac{d^2r}{dt^2}-r\left(\frac{dv}{dt}\right)^2+\frac{\mu}{r^2}=\frac{d\Omega}{dr}. \tag{16.1}$$

In the case in which $\Omega=0$, the motion becomes Keplerian. In that case, the first of the equations of system (16.1) is directly integrated and yields (with c a constant)

$$r^2\frac{dv}{dt}=\sqrt{c}. \tag{16.2}$$

If then the quantity v is used as the independent variable and if we put $r=-1/u$, the second equation of system (16.1) becomes

$$\frac{d^2u}{dv^2} + u + \frac{\mu}{c} = 0, \tag{16.3}$$

which directly demonstrates the elliptical shape of the trajectory.

Let us return to the general case in which Ω is not zero. Glydén suggested adopting an independent variable such that the equations of motion assume a form analogous to that of Eqs. (16.2) and (16.3).

For this, let us put

$$\frac{dv_0}{dt} = \frac{\sqrt{c_0}}{r^2}, \tag{16.4}$$

where c_0 is a new constant.

When taking v_0 as an independent variable, the first part of system will become

$$\frac{d^2v}{dv_0^2} = \frac{r^2}{c_0}\frac{d\Omega}{dv} \tag{16.5}$$

and the second part of Eq. (16.1), again putting $r = -1/u$,

$$\frac{d^2u}{dv_0^2} + u\left(\frac{dv}{dv_0}\right)^2 + \frac{\mu}{c_0} = \frac{r^2}{c_0}\frac{d\Omega}{dr}.$$

The analogy with Eq. (16.3) becomes even more evident if it is noted that, in the calculations given below, v will differ very little from v_0 and if, introducing into the right-hand side of the equation a very small term being of the same order of magnitude as Ω, we write

$$\frac{d^2u}{dv_0^2} + u + \frac{\mu}{c_0} = \frac{r^2}{c_0}\frac{d\Omega}{dr} + u\left[1 - \left(\frac{dv}{dv_0}\right)^2\right]. \tag{16.6}$$

The choice of the variable v_0, despite the fact that it offers obvious advantages, is not without drawback.

In fact, the three-body problem presents itself in two quite different forms, depending on whether one has to do with two planets whose masses are comparable or with planets one of which is much smaller than the other.

In the former case, it is necessary to refer one of the planets to the independent variable v_0 and the other to the independent variable v_0' which is analogous but different and is defined by the equation

$$\frac{dv_0'}{dt} = \frac{\sqrt{c_0'}}{r'^2},$$

where r' is the radius vector of the second planet.

This constitutes a source of complications. Therefore, the Gyldén method, in its original form, is more suitable for the second case, for example in studying perturbations of the minor planets produced by Jupiter.

However, here again difficulties arise.

The motion of Jupiter is known, but as a function of t rather than of v_0. To change from the expression as a function of t to the expression as a function of v_0, it is necessary to substitute for t its value as a function of v_0, derived from Eq. (16.4). This expression of t as a function of v_0 will differ in each approximation; therefore, each time it will be necessary to correct the coordinates of Jupiter. These drawbacks are partly compensated by significant advantages. Another disadvantage is the fact that our equations have lost the form of Lagrange equations; however, we will be quick in restoring these.

168. Below, we give the form under which the equations of motion appear.

The coordinates u and v of the first planet are expressed as a function of v_0 by Eqs. (16.5) and (16.6), whose left-hand sides have the simple form

$$\frac{d^2v}{dv_0^2} \quad \text{and} \quad \frac{d^2u}{dv_0^2} + u + \frac{\mu}{c_0},$$

and whose right-hand sides depend not only on u and on v but also on the corresponding coordinates u' and v' of the perturbing planet.

The variable v_0 will be correlated with t via Eq. (16.4).

The coordinates u' and v' of the second planet will similarly be expressed as a function of a new variable v_0' by Eqs. (16.5′) and (16.6′) analogous to Eqs. (16.5) and (16.6).

The variable v_0', for its part, will be defined as a function of t by an equation (16.4′) analogous to Eq. (16.4).

Let us assume now that one wishes to apply to these equations procedures analogous to those of the old methods of celestial mechanics; then the following is necessary: Let us imagine that the approximate values of u and v as well as of u' and v' are known as a function of v_0 and as a function of v_0'.

On the right-hand side of Eq. (16.5) or Eq. (16.6), let us replace u, v, u', and v' by their approximate values as a function of v_0. The right-hand sides will become known functions of v_0 and our equations are then easy to integrate by quadrature.

This furnishes the closest values of u and v as a function of v_0.

In operating similarly for Eqs. (16.5′) and (16.6′), we will obtain closer values of u' and v' as a function of v_0'.

By quadrature, Eq. (16.4) will yield t as a function of v_0 and Eq. (16.4′)

will furnish t as a function of v_0'. Consequently, by comparing these two results, we will obtain v_0 as a function of v_0' and vice versa.

It will then be possible to express, in a closer approximation, u and v as a function of v_0' or u' and v' as a function of v_0. Since we now have closer values of u, v, u', and v' as functions of v_0 and as functions of v_0', we can use this second approximation for operating as we had done with the first approximation, and so on.

It now remains to select the first approximation. For the moment, let us attempt to see what mathematicians, imbued with the spirit of the older methods, would have done for a better understanding of the improvements that Gyldén had found necessary to introduce. It is clear that the choice conforming best to this spirit would be that of taking, as first approximation, the Keplerian motion.

This will yield

$$v = v_0, \qquad u = -\frac{\mu}{c_0} + \alpha \cos v_0 + \beta \sin v_0,$$

$$v' = v_0', \qquad u' = -\frac{\mu'}{c_0'} + \alpha' \cos v_0' + \beta' \sin v_0',$$

where α, β, α', and β' are constants of integration.

So far as the relation between v_0 and v_0' is concerned, its form is rather complicated. We obtain

$$\frac{dv_0}{u^2\sqrt{c_0}} = \frac{dv_0'}{u'^2\sqrt{c_0'}},$$

which is an equation that can be integrated by quadrature. From this, v_0' can be derived as a function of v_0. On expanding v_0' in ascending powers of the constants α, β, α', and β' which generally are very small, the first term of the series which is independent of these four constants will reduce to a linear function of v_0.

Thus the correlation between v_0' and v_0 is complicated from the first approximation on. This represents a somewhat artificial difficulty of an entirely new type. It has to do with the choice of independent variables and will disappear only in abandoning the procedures inspired by the old methods for the actual Gyldén methods.

We encountered nothing similar in studying the Newcomb methods; nevertheless, the importance of the fact must not be exaggerated. The expansion of the perturbing function will always require tedious and lengthy calculations. However, the series is obtained more rapidly as a function of the true anomalies than as a function of the mean anomalies. In the Newcomb method, we had assumed the perturbing function as being expressed over osculating elements of the two planets and their mean anomalies. To obtain this function, excessive efforts would be necessary, but were it actually to be derived, all obstacles would immediately be leveled. Here, conversely, we have expressed Ω as a function of u, v, u', and v' which is incomparably easier. However, the difficulty that we have thus avoided for the moment will of necessity reappear. The complex relation that connects v_0 to v_0' is the first form under which we encounter it. The inconvenience here is the fact that one has to start anew at each approximation and that this will have to be repeated several times.

Let us now define the pitfalls to be feared in employing these procedures that mimic the old methods and let us define the artifices used by Gyldén for avoiding them.

Equations (16.5) and (16.6), after replacing on the right-hand side the terms u, v, u', and v' as a function of v_0, will become linear on the right-hand side and will be easy to integrate.

In the first approximation, these right sides present themselves in the form of trigonometric series whose terms will depend on the sines and cosines of

$$(n + mk)v_0,$$

where n and m are integers and k is the ratio of the mean motions. If the right-hand side of Eq. (16.5) did not contain known terms or if the right-hand side of Eq. (16.6) did not contain terms in $\sin v_0$ or in $\cos v_0$, then the values of u and v derived from Eqs. (16.5) and (16.6) would still have the same form. However, the right-hand side of Eqs. (16.5) and (16.6) do contain wholly known terms, terms in $\sin v_0$ and in $\cos v_0$; consequently, there will appear in the expression for v, a term in

$$v_0^2$$

and in the expression for u, terms in

$$v_0 \cos v_0 \quad \text{and} \quad v_0 \sin v_0,$$

where the independent variable v_0 comes out from under the trigonometric functions.

In the following approximations, it is obvious that still higher powers of v_0 will be encountered outside these trigonometric functions. Thus, as was easy to predict, the use of the variable v_0 has changed nothing in the essential character of the old methods; therefore, one must resort to another artifice to prevent the variable from leaving the trigonometric functions.

The only advantage of the choice of v_0, aside from the drawbacks mentioned above, is thus the fact that it imparts a linear form to the equations.

169. To avoid secular terms, i.e., terms in which v_0 is no longer under the sine or cosine functional sign, Gyldén had to invent a new artifice.

Let us consider one of Eqs. (16.5) or (16.6). Let us transfer, to the left-hand side, that or those terms of the right-hand side whose influence seems greatest. On the right-hand side, let us replace u, v, u', and v' by their approximate values in such a manner that the unknown quantities v, u', and v' will appear only on the left-hand side. This will yield new equations which can be integrated with the aid of the new artifices.

Obviously, this constitutes a high degree of arbitrariness. In fact, depending on the individual case, one can center attention on either one or the other term and transfer it to the left-hand side. This is the prime reason for the flexibility of the method. Although this can be varied *ad infinitum*, so to say, we will enumerate here the forms of equations most often considered by Gyldén.

Let u_1, v_1, u_1', and v_1' be the approximate values of u, v, u', and v'. Let us put

$$u = u_1 + \rho, \quad v = v_1 + \chi, \quad u' = u_1' + \rho', \quad v' = v_1' + \chi'.$$

The quantity ρ will be what Gyldén called "evection" and the quantity χ will be called "variation." Ordinarily, one is content to take

$$v_1 = v_0; \quad v = v_0 + \chi.$$

With these new unknowns, Eqs. (16.5) and (16.6) will assume the form

$$\frac{d^2\chi}{dv_0^2} = A, \tag{16.5a}$$

$$\frac{d^2\rho}{dv_0^2} + \rho = B, \tag{16.6a}$$

where A and B are functions expanded in ascending powers of ρ, ρ', χ, and χ' and in addition—at least so far as B is concerned—in ascending powers of $d\chi/dv_0$. The coefficients of the series will be known functions of v_0. Then we will transfer to the left-hand side some of the terms of A and of B; retaining the unknowns ρ and χ on the left-hand side, we will replace these quantities by zero on the right-hand side for a first approximation.

The function B, among other notable terms, will contain also terms of the form

$$C\rho'', \quad C\rho \sin(\lambda v_0 + k),$$

where C, λ, and k are constants.

(i) If the second of these terms is transferred to the left-hand side of Eq. (16.6a), we will find

$$\frac{d^2\rho}{dv_0^2} + \rho[1 - C\sin(\lambda v_0 + k)] = B', \tag{16.6b}$$

where B' is what becomes of B when eliminating the term which thus has been transferred to the left-hand side.

In B', we will then set

$$\rho = \chi = \rho' = \chi' = 0.$$

Equation (16.6b) will still be an equation linear on the right-hand side but will no longer be an equation with constant coefficients.

It is now obvious that one can just as well write

$$\frac{d^2\rho}{dv_0^2} + \rho[(1+\alpha) - C(1+\beta)\sin(\lambda v_0 + k)] = B'', \quad (16.6c)$$

where α and b are any two very small quantities, and

$$B'' = B' + \alpha\rho + C\beta\rho\sin(\lambda v_0 + k)$$

and where, in addition, we finally will have in first approximation

$$B'' = B' = B,$$

since it has been agreed to cancel ρ, ρ', χ, and χ' on the right-hand side.

This makes it possible to profit in various ways from the indeterminacy of α and β.

(ii) It is also possible to transfer, to the left-hand side, a term in ρ^3 and to write

$$\frac{d^2\rho}{dv_0^2} + \rho - C\rho^3 = B - C\rho^3$$

or

$$\frac{d^2\rho}{dv_0^2} + \rho(1+\alpha) - C\rho^3 = B + \alpha\rho - C\rho^3, \quad (16.6d)$$

and to then set

$$\rho = \chi = 0$$

on the right-hand side.

(iii) It is clear that A will be a function expandable in the sines and cosines of multiples of v and v'. Let then

$$C\sin(mv + nv' + k)$$

be a term of A; m and n are integers and k is a constant. Let us replace there v by $v_1 + \chi$ and v' by its approximate value v_1', expressed as a function of v_0, so that we obviously will have

$$v_1 = v_0 + \varphi,$$
$$v_1' = \mu v_0 + \varphi',$$

where φ and φ' are series expandable in the sines and cosines of multiples of v_0 and μv_0, while μ is the ratio of the mean motions. The complementary terms φ and φ' are incidentally much smaller than the leading terms v_0 and μv_0.

Then, the expression

$$C \sin(m v_1 + n v_1' + k)$$

can also be expanded in a series of the trigonometric functions of multiples of v_0 and μv_0, while the main term of the series will be

$$C \sin(m v_0 + n \mu v_0 + k).$$

Similarly, if in the expression

$$C \sin(m v + n v_1' + k),$$

we replace v and v_1' by

$$v_0 + \chi + \varphi \quad \text{and} \quad \mu v_0 + \varphi',$$

then this expression will be expandable in trigonometric functions of

$$v_0, \quad v_0 + \chi, \quad \text{and} \quad \mu v_0$$

and the main term of the series will become

$$C \sin(m v_0 + n \mu v_0 + m\chi + k).$$

This is the term which we will transfer to the left-hand side of Eq. (16.5), which then can be written as

$$\frac{d^2\chi}{dv_0^2} - C \sin(m v_0 + n \mu v_0 + m\chi + k) = A' \qquad (16.5b)$$

where

$$A' = A - C \sin(m v_0 + n \mu v_0 + m\chi + k).$$

Next, we will set in A',

$$\rho = \chi = \rho' = \chi' = 0,$$

in such a manner that A' can be considered a known function of v_0.

Most often, one will be content to take

$$v_1 = v_0, \quad v_1' = \mu v_0$$

which somewhat simplifies the discussion of the above procedure.

Equations (16.6c), (16.6d), and (16.5b) are those most frequently used by Gyldén.

Let us note that all have the form

$$\frac{d^2\rho}{dv_0^2} = f(\rho, v_0) \qquad (16.7)$$

or

$$\frac{d^2\chi}{dv_0^2} = f(\chi, v_0). \tag{16.8}$$

Consequently, one can reduce them to the canonical form according to what we said in Vol. I, in the discussion following Eq. (1.6) of no. 2.

We have assumed that, on the right sides of our equations, we had set

$$\rho = \chi = \rho' = \chi' = 0.$$

This is actually what is done in first approximation. However, in second approximation, it becomes necessary to replace on the right sides the quantities ρ, χ, ρ', and χ' by their values derived from the first approximation, and so on.

Consequently, these right-hand sides will always be known functions of v_0 and the equations will retain the same form.

Reduction of the Equations

170. Equations (16.5b), (16.6c), and (16.6d) are of the second order. This is due to the fact that we made certain that only terms depending on χ alone were transferred to the left-hand side of Eq. (16.5) and only terms depending on ρ along to the left-hand side of Eq. (16.6).

If then, on the right-hand side, we set

$$\rho = \chi = \rho' = \chi' = 0,$$

then Eq. (16.5) will contain only a single unknown χ and Eq. (16.6) will also contain a single unknown, namely, ρ.

However, this cannot be always legitimate. It might happen for example that, on the right-hand side of Eq. (16.5) certain terms depending on ρ are just as important as the most influential terms depending on χ and that this also has to be transferred to the left-hand side.

It is the same for Eq. (16.6). Thus after canceling the terms ρ and χ on the right-hand side, the two equations (16.5) and (16.6) will still contain the two unknowns ρ and χ so that, after elimination of one of these, the resultant equation will no longer be of the second order but will now be of the fourth order.

The order will even be higher if one had been forced to transfer to the left-hand side all terms depending on ρ' and χ'.

In these cases, Gyldén—to return the equations to the second order—used a procedure of which we will give the essential principle.

Let us first consider a fourth-order equation, for example, having the form

$$\frac{d^2\rho}{dv^2} + \rho = \alpha \left(A + B_4 \frac{d^4\rho}{dv^4} + B_3 \frac{d^3\rho}{dv^3} + B_2 \frac{d^2\rho}{dv^2} + B_1 \frac{d\rho}{dv} + B_0 \rho \right), \tag{16.9}$$

where A and B are known functions of v which are assumed to be finite, while α is a very small coefficient.

The equation demonstrates primarily that, if the initial values of ρ and of $d\rho/dv$ are of the order of α, which we will assume here, then ρ will remain of the order of α.

Thus neglecting all terms of the order of α^2, we could write

$$\frac{d^2\rho}{dv^2} + \rho = \alpha A$$

which would return the equation to the second order.

However, we wish to take terms of the order of α^2 into consideration, neglecting those of the order of α^3. With this degree of approximation, we obtain

$$\begin{aligned} \alpha \frac{d^4\rho}{dv^4} &= -\alpha \frac{d^2\rho}{dv^2} + \alpha^2 \frac{d^2A}{dv^2}, \\ \alpha \frac{d^3\rho}{dv^3} &= -\alpha \frac{d\rho}{dv} + \alpha^2 \frac{dA}{dv}, \\ \alpha \frac{d^2\rho}{dv^2} &= -\alpha\rho + \alpha^2 A. \end{aligned} \tag{16.10}$$

This result is obtained on multiplying Eq. (16.9) by α and there neglecting all terms that have become of the order of α^3 by this multiplication.

Then, Eq. (16.9) becomes

$$\frac{d^2\rho}{dv^2} + \rho = \alpha C + \alpha D \frac{d\rho}{dv} + \alpha E\rho, \tag{16.11}$$

where

$$C = A + B_4 \left(\frac{d^2A}{dv^2} - A \right) + B_3 \frac{dA}{dv} + B_2 A,$$
$$D = B_1 - B_3,$$
$$E = B_4 - B_2 + B_0.$$

Thus the equation has been returned to the second order.

Equation (16.11) is valid for quantities up to the third order, namely, to the order of α^3. This will yield, to within quantities of the fourth order,

$$\alpha \frac{d^{2+i}\rho}{dv^{2+i}} = -\alpha \frac{d^i\rho}{dv^i} + \alpha^2 \frac{d^i}{dv^i}\left(C + D\frac{d\rho}{dv} + E\frac{d^2\rho}{dv^2}\right). \qquad (16.12)$$

If then on the right-hand side of Eq. (16.9), we replace

$$\alpha \frac{d^4\rho}{dv^4}, \quad \alpha \frac{d^3\rho}{dv^3}, \quad \alpha \frac{d^2\rho}{dv^2}$$

by their values from Eq. (16.12), an equation will be obtained that is valid to within quantities of the fourth order and which itself will be of the second order.

And so on.

It is obvious that the same method is applicable to any equation of the form

$$\frac{d^2\rho}{dv^2} + f_1 = \alpha f_2, \qquad (16.13)$$

where α is a very small coefficient, and where f_1 can be expanded in powers of ρ and of $d\rho/dv$, and f_2 can be expanded in powers of

$$\rho, \quad \frac{d\rho}{dv}, \quad \ldots, \quad \frac{d^{n-1}\rho}{dv^{n-1}}, \quad \frac{d^n\rho}{dv^n}.$$

Consequently, Eq. (16.13) is no longer linear. However, the only difference resulting from this is the possibility that there are terms present of a degree higher than ρ and its derivatives, and that these would be allowed for only after the second and third approximation.

171. Let us now consider the following equation:

$$\frac{d^2\rho}{dv^2} + \rho = \alpha\left(A + B\int \rho C\, dv\right), \qquad (16.14)$$

where α is again a very small number while A, B, and C are known functions of v. This equation, when differentiated so as to eliminate the integral sign, would become of the third order. However, Gyldén reduced this expression to the second order by profiting from the smallness of the number α and by using a process analogous in essence to that employed by us in simpler examples.

In fact, ρ and α are of the first order so that the term

$$\alpha B \int \rho C\, dv$$

can be considered as being of the second order. In that case, neglecting all quantities of the third order, we obtain

$$\alpha \frac{d^2\rho}{dv^2} + \alpha\rho = \alpha^2 A,$$

whence

$$\alpha B \int \rho C \, dv = \alpha^2 B \int AC \, dv - \alpha B \int \frac{d^2\rho}{dv^2} C \, dv, \qquad (16.15)$$

and, integrating by parts and denoting by C' and C'' the derivatives of C with respect to v,

$$\int \frac{d^2\rho}{dv^2} C \, dv = C \frac{d\rho}{dv} - C'\rho + \int \rho C'' dv.$$

In general, the quadrature $\int AC \, dv$ can be performed readily, so that

$$B \int AC \, dv = D$$

can be considered a known function of v and that the integral $\int \rho C \, dv$ will be reduced to the integral $\int \rho C'' dv$ which has the same form.

In general, in the examples that Gyldén had to treat, C has the form

$$C = \beta \sin(\lambda v + \mu),$$

where β, λ, and μ are constants. It results from this that

$$C'' = -\lambda^2 C,$$

whence

$$\alpha B \int \rho C \, dv = \alpha^2 D - \alpha BC \frac{d\rho}{dv} + \alpha BC'\rho + \lambda^2 \alpha B \int \rho C \, dv,$$

from which it follows that

$$\alpha B \int \rho C \, dv = \frac{\alpha^2 D}{1-\lambda^2} - \frac{\alpha BC}{1-\lambda^2} \frac{d\rho}{dv} + \frac{\alpha BC'}{1-\lambda^2} \rho. \qquad (16.16)$$

If, in Eq. (16.14), we replace

$$\alpha B \int \rho C \, dv$$

by its value (16.16), which can be done by neglecting all quantities of the third order, the equation will be reduced to the second order.

If C is a sum of terms of the form

$$\beta \sin(\lambda v + \mu),$$

then the expression

$$\alpha B \int \rho C \, dv$$

will be a sum of terms of the form

$$\alpha \beta B \int \rho \sin(\lambda v + \mu) dv,$$

and each of these terms can then be transformed by a formula analogous to Eq. (16.16). Thus Eq. (16.14) will again be reduced to the second order.

Equation (16.15) is valid only for quantities up to the third order. If one does not wish to neglect these quantities, one must write

$$\alpha B \int \rho C\, dv = \alpha^2 D - \alpha B \int \frac{d^2\rho}{dv^2} C\, dv + \alpha^2 \sigma,$$

by putting, for abbreviation,

$$\sigma = B \int dv\left(CB \int \rho C\, dv\right).$$

Again assuming that C reduces to a single term

$$C = \beta \sin(\lambda v + \mu),$$

we can derive from this that

$$\alpha B \int \rho C\, dv = \frac{\alpha^2 D}{1-\lambda^2} - \frac{\alpha BC}{1-\lambda^2}\frac{d\rho}{dv} + \frac{\alpha BC'}{1-\lambda^2}\rho + \frac{\alpha^2\sigma}{1-\lambda^2},$$

such that Eq. (16.14), on transferring certain terms to the left-hand side, will become

$$\frac{d^2\rho}{dv^2} + \rho\left(1 - \frac{\alpha BC'}{1-\lambda^2}\right) + \frac{d\rho}{dv}\frac{\alpha BC}{1-\lambda^2} = \alpha A + \frac{\alpha^2 D}{1-\lambda^2} + \frac{\alpha^2\sigma}{1-\lambda^2}.$$

In general, one does not retain on the left-hand side all terms that we have transposed to there but only the most important of these. When bringing the other terms back to the right-hand side, one obtains an equation of the form

$$\frac{d^2\rho}{dv^2} + E\frac{d\rho}{dv} + F\rho = G + H\rho + K\frac{d\rho}{dv} + L\sigma, \qquad (16.17)$$

where E, F, G, H, K, and L are known functions of v.

This permits operating as follows: On the right-hand side, let us first set $\rho = \sigma = 0$. This will yield an equation linear on the right-hand side which can be integrated and will yield a first approximate value of ρ and, consequently, of σ. These values are then substituted on the right-hand side, yielding a new equation linear on the right side which will yield a second approximation for ρ and σ, and so on.

It is clear that we could also operate in the same manner if Eq. (16.14) were not linear and, for example, contained powers higher than ρ. From this it would merely result that the right-hand side of Eq. (16.17) would contain terms of the form

$$B\rho^n \quad \text{and} \quad B\int C\rho^n dv,$$

where B and C are known functions of v ($n > 1$). In these terms, which are of the order $n+1$ at least with respect to α, it is possible without inconvenience—as in the other terms on the right-hand side of Eq. (16.17)—to first replace ρ by 0 and then by its approximate first value, followed by the second, and so on.

So as to render application of this method useful, it is necessary that λ be very close to 1, such that the expression

$$\frac{\alpha}{1-\lambda^2}$$

will definitely be small but much less so than α. One then admits that the various terms on the right-hand side of Eq. (16.16) will be sufficiently large for not being neglected, even in the first approximation.

Otherwise, it would be rather simple to leave the term

$$\alpha B \int \rho C \, dv$$

on the right-hand side and to first assign a value of zero to the term ρ, followed by its various approximate values.

This means that, in most cases, it is a question of transposing only a small number of terms (and frequently only a single term) to the left-hand side, having the form

$$\alpha B \int \rho\beta \sin(\lambda v + \mu) dv.$$

The method of reduction to the second order, discussed above, is of advantage only if A contains no term in $\sin \lambda v$ or in $\cos \lambda v$. Without this, the integral

$$\int AC \, dv$$

would contain a term in v, and in the expression of D the variable would be removed from under the trigonometric functions.

This particular fact never occurred in the applications that Gyldén himself made of his method; besides, there always is some means of avoiding it.

Let us write Eq. (16.14) in the form of

$$\frac{d^2\rho}{dv^2} + \rho - \alpha B \int \rho C \, dv = \alpha A. \tag{16.14a}$$

Until now, we have considered A as being a known function of v. However, it can also be assumed that A depends not only on v but also on ρ in any manner, linearly or not, directly or through the intermediary of its derivatives or integrals of the form $\int \rho D\, dv$. Only the terms of A that depend on ρ are supposed to be smaller than the term

$$\alpha B \int \rho C\, dv,$$

which had been transferred to the left-hand side.

Then, the value ρ in A will first be replaced by zero and then by its successive approximate values. Thus at each approximation, A can be considered as being a known function of v.

Under these conditions, Eq. (16.14a) is an equation linear on the right side. Indeed, if we set

$$\int \rho C\, dv = \tau,$$

then ρ and $d^2\rho/dv^2$ can be expressed linearly using the derivatives of τ.

For integrating the equation with a right-hand side, it will be sufficient to be able to integrate the equation without a right-hand side

$$\frac{d^2\rho}{dv^2} + \rho - \alpha B \int \rho C\, dv = 0.$$

This equation has the same form as Eq. (16.14) and can be subjected to exactly the same method of reduction. Only, since A is zero, the difficulty mentioned above need no longer be feared.

172. To recapitulate, this is what we have done: Let us suppose that a term containing a single integral

$$\int \rho C\, dv$$

will be sufficiently important to have it of necessity brought over to the left-hand side. Because of the above-discussed transformation, this term can be replaced by a sum of terms independent of ρ and of $d\rho/dv$ to within terms that are sufficiently small for being brought back to the right-hand side.

Let us assume now that one had been obliged to transpose to the left-hand side a term containing a double integral, namely, a term of the form

$$M = \alpha A \int dv B\left(\int \rho C\, dv\right) = A \int dv \left(\alpha B \int \rho C\, dv\right),$$

where A, B, and C are known functions of v. To within terms that can be brought back to the right-hand side, we will then have

$$\alpha B \int \rho C\, dv = D\rho + E\frac{d\rho}{dv},$$

where D and E are known functions of v from which it follows that

$$M = A\int \rho D\,dv + A\int \frac{d\rho}{dv} E\,dv = AE\rho + A\int \rho\left(D - \frac{dE}{dv}\right) dv.$$

Thus M is reduced to terms depending only on a single integral and treatable as in the preceding number.

It then remains to explain how these terms, containing single or double integrals, can be introduced into our equations.

These equations can be written as

$$\frac{d^2\chi}{dv_0^2} = A + B\chi + C\rho + D,$$

$$\frac{d^2\rho}{dv_0^2} + \rho = E + F\rho + G\frac{d\chi}{dv_0} + H\chi + K,$$

where A, B, C, E, F, G, and H represent known functions of v_0 while D and K depend on ρ' and χ' or on higher powers of ρ, χ, and $d\chi/dv_0$.

From this, we derive

$$\frac{d^2\rho}{dv_0^2} + \rho = E' + F\rho + G\int B\chi\,dv_0 + G\int C\rho\,dv_0$$
$$+ H\iint B\chi\,dv_0\,dv_0 + H\iint C\rho\,dv_0\,dv_0 + K',$$

where E' is a new known function of v_0, easy to form, while

$$K' = K + G\int D\,dv_0 + H\iint D\,dv_0\,dv_0$$

depends on ρ', χ', and on higher powers ρ, χ, ...

It is obvious that we thus have introduced terms of the form

$$G\int C\rho\,dv_0, \quad H\iint C\rho\,dv_0\,dv_0,$$

which one could transpose to the left-hand side and transform as stated above.

173. We have reduced our equation to the form

$$\frac{d^2\rho}{dv_0^2} + A\frac{d\rho}{dv_0} + B\rho = C,$$

where A and B are known functions of v_0 while C contains the unknown functions and, in particular, ρ. However, we have agreed to replace there these quantities first by zero and then by their successive approximate values such that C can also be considered a known function of v_0.

This is an equation linear on the right-hand side which can be further simplified by causing the term in $d\rho/dv_0$ to vanish. For this, it is sufficient—as is known—to put

$$\rho = \sigma \exp\left(- \int \frac{A}{2}\, dt \right),$$

so that the equation becomes

$$\frac{d^2\sigma}{dv_0^2} + B'\sigma = C',$$

where B' and C' are new known functions of v_0.

In general, it is sufficient to retain a single term in B' and to transpose all others to the right-hand side so that the equation will be reduced to the form of Eq. (16.6b) in no. 169.

174. Up to now, we had assumed that the three-body motion takes place in a plane. Very little has to be changed in the case in which the inclinations of the orbits must be allowed for.

In that case, let r, v, and θ be the radius vector, the longitude, and the latitude of the first planet so that the equations of motion, returning in all other respects to the notations of no. 167, will be

$$\frac{d}{dt}\left(r^2 \cos^2\theta \frac{dv}{dt}\right) = \frac{d\Omega}{dt},$$

$$\frac{d^2r}{dt^2} - r\cos^2\theta \frac{dv^2}{dt^2} - r\frac{d\theta^2}{dt^2} + \frac{\mu}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) + r^2 \sin\theta \cos\theta \frac{dv^2}{dt^2} = \frac{d\Omega}{d\theta}.$$

Let us then put

$$u = \frac{1}{r\cos\theta}, \qquad s = \tan\theta$$

and let us introduce, as in no. 167, an auxiliary variable v_0 defined by the equation

$$\frac{dv_0}{dt} = u^2\sqrt{c_0}.$$

From this, we deduce first

$$\frac{d^2v}{dv_0^2} = \frac{1}{c_0u^2}\frac{d\Omega}{dv}, \tag{16.18}$$

an equation analogous to Eq. (16.5) of no. 167, and similarly we find

$$\frac{d^2u}{dv_0^2} + u = A - \frac{\mu}{c_0}\cos^3\theta, \tag{16.19}$$

$$\frac{d^2s}{dv_0^2} + s = B, \tag{16.20}$$

where A and B are combinations of the derivatives of the perturbing function.

Then, in B, A, and $d\Omega/dv$, one can replace the coordinates of the planets by their approximate values. The right-hand sides of our equations (16.18) and (16.20) are then known and we can calculate v and s. If s and thus also θ are known, the right-hand side of Eq. (16.19) will also be known and u can be calculated.

Operating in this manner means that we remain within the essence of the old methods. However, Gyldén, as indicated above, transposed some of the most important terms of the right-hand side to the left-hand side of Eqs. (16.18), (16.19), and (16.20), applying, as necessary, the reduction processes of nos. 170–173 and thus obtaining equations of the same form as those of no. 169.

Intermediate Orbit

175. We have put above

$$u = u_1 + \rho, \quad v = v_1 + \chi,$$

where u_1 and v_1 are approximate values of u and v.

The choice of these approximate values, which remains arbitrary to a certain extent, obviously is of great importance. To keep in line with the old methods, it would be necessary to take, for u_1 and v_1, the values corresponding to the Keplerian motion.

Gyldén preferred to come closer to the real orbit from the very first approximation. However, it is obvious that the subsequent approximations will be more rapid and in addition—as shown in no. 133—the case in which the motion is Keplerian in first approximation presents a special difficulty which one should try to avoid.

Below, we will show what Gyldén did in this respect.

He assumed first that $v_1 = v_0$ and that u_1 is determined as a function of v_0 in the following manner: We have the equation

$$\frac{d^2u}{dv_0^2} + u\left(\frac{dv}{dv_0}\right)^2 + \frac{\mu}{c_0} = \frac{r^2}{c_0}\frac{d\Omega}{dr}.$$

Let us first replace $d\Omega/dr$ by a function $c_0u^2\varphi(u)$ depending only on u and differing very little from the mean of the values taken by $d\Omega/dr$ when varying (while keeping u constant) v from 0 to 2π and varying u' and v' such that the second of the planets (whose coordinates are u' and v') assumes all possible positions on its Keplerian orbit. Next, let us replace u and v by u_1 and v_1 such that dv/dv_0 reduces to

$$\frac{dv_1}{dv_0} = \frac{dv_0}{dv_0} = 1.$$

Thus our equation becomes

$$\frac{d^2u_1}{dv_0^2} + u_1 + \frac{\mu}{c_0} = \varphi(u_1). \tag{16.21}$$

This equation is easily integrated by quadrature. Interpretation of this approximate solution is readily possible.

Let us add the following equation to Eq. (16.21):

$$\frac{dv_0}{d\tau} = u_1^2\sqrt{c_0}. \tag{16.22}$$

It is obvious that, if we consider a fictitious star that has radius vector $-1/u_1$ and longitude v_0, at the epoch τ_0, this star will have a same motion as though it were attracted by a fixed mass located at the origin, obeying a certain law different from the Newtonian law. Nevertheless, this attraction depends only on the distance, since it obviously is equal to

$$\mu u_1^2 - c_0 u_1^2 \varphi(u_1),$$

and since $1/u_1$ exactly represents the distance of our fictitious star from the origin, i.e., from the fictitious attractive mass.

The variables t and τ, corresponding to the same value of v_0, are connected by the relation

$$\frac{dt}{d\tau} = \frac{u_1^2}{u^2}.$$

The variable τ, which is only rarely used outside this interpretation, has been given the name "reduced time."

The orbit described by our fictitious star has been denoted as "intermediate orbit" since, so to say, it keeps the middle between the real orbit and the Keplerian orbit.

For

$$\varphi(u) = hu^{-3} \quad \text{or} \quad hu^2,$$

where h is a constant, the integration of Eq. (16.21) can be done with elliptic functions.

Absolute Orbit

176. If one reflects on the spirit of Gyldén's theories, one will understand that the choice of the variable v_0 plays no major role here, and that fully analogous results can be obtained by using an independent variable.

The simplest and, in most cases, the most advantageous variable would be the time t. This has been done by Gyldén himself in one of his papers.[9]

Numerous other choices are also possible. Among others, Gyldén has used in some of his investigations a variable whose definition is much more complicated and which also is denoted by v_0. We will discuss this briefly.

The famous astronomer undertook to determine an orbit deviating very little from the true orbit and called an "absolute orbit." This orbit in turn keeps, so to say, the middle between the intermediate and the real orbit.

Let us return to Eqs. (16.1) of no. 167.

On the right-hand side of these equations, let us consider the most important terms. Let Q be the set of the most important terms of $r^2\, d\Omega/dv$ and let P be the set of the most important terms of $r^2\, d\Omega/dr$ such that, to a first approximation, we can neglect the differences

$$r^2 \frac{d\Omega}{dv} - Q, \quad r^2 \frac{d\Omega}{dr} - P.$$

Let Q_0 be what becomes of Q on replacing there u, u', and v' by their approximations as functions of v, and then replacing v by v_0.

Let us introduce an auxiliary function c_1 by putting

$$r^2 \frac{d\sqrt{c_1}}{dt} = Q_0 \tag{16.23}$$

and let us then put

$$r^2 \frac{dv_0}{dt} = \sqrt{c_1}. \tag{16.24}$$

The function v_0, defined in this manner, will be our new independent variable; it will differ little from v since it will satisfy the equation

$$\frac{d\left(r^2 \dfrac{dv_0}{dt}\right)}{dt} = \frac{Q_0}{r^2},$$

whereas v satisfied the equation

$$\frac{d\left(r^2 \dfrac{dv}{dt}\right)}{dt} = \frac{d\Omega}{dv}$$

which differed from the former only by the addition of certain terms which we assumed to be very small. Thus we can put

$$v = v_0 + \chi.$$

From Eqs. (16.23) and (16.24), we derive

$$\sqrt{c_1}\, d\sqrt{c_1} = \frac{dc_1}{2} = Q_0\, dv_0. \tag{16.25}$$

Now, we have assumed that, to obtain Q_0, the coordinates u, u', and v' in Q were replaced by their approximate values as a function of v and then v by v_0.

As a result, Q_0 is a function of v_0 alone and Eq. (16.25) can be readily integrated by quadrature.

The second part of system (16.1) will then become

$$\frac{d^2u}{dv_0^2} + \frac{1}{2}\frac{d\log c_1}{dv_0}\frac{du}{dv_0} + u\left(\frac{dv}{dv_0}\right)^2 + \frac{\mu}{c_1} = \frac{r^2}{c_1}\frac{d\Omega}{dr}.$$

It is obvious, since χ is very small, that dv/dv_0 will be close to one. Consequently, the coefficient $(dv/dv_0)^2$ can be replaced by 1 in first approximation. Thus, provided that u_1 is the approximate value of u and if we put

$$u = u_1 + \rho,$$

we could define u_1 by the equation

$$\frac{d^2u_1}{dv_0^2} + \frac{1}{2}\frac{d\log c_1}{dv_0}\frac{du_1}{dv_0} + u_1 + \frac{\mu}{c_1} = \frac{P_0}{c_1},$$

where P_0 is what becomes of P upon replacing in it v by v_0, u by u_1, and u' and v' by their approximate values as a function of v_0.

Since P_0 depends only on u_1 and v_0 and since c_1 is a function of v_0, known through Eq. (16.25), we will have a differential equation of the second order between u_1 and v_0.

On transforming this expression by means of the process given in no. 173, i.e., by putting

$$u_1 = \sigma(c_1)^{-1/4},$$

we obtain

$$\frac{d^2\sigma}{dv_0^2} = F(v_0, \sigma),$$

which has the same form as Eqs. (16.7) and (16.8) of no. 169.

Having thus determined the coordinates $1/u_1$ and v_0 of the fictitious star which describes the absolute orbit, we can calculate the corrections ρ and χ by similar procedures, thus yielding the coordinates of the real planet.

CHAPTER 17

Case of Linear Equations

177. We still must do the following:

(i) Perform an integration of Eqs. (16.6b), (16.6c), (16.6d), (16.5b), (16.7), and (16.8) of no. 169.

(ii) Establish how one can, in forming these equations, distinguish the terms that must be transposed to the left-hand side from those that must remain on the right-hand side.

I will first investigate the integration of Eqs. (16.6b) and (16.6c), and I will devote this chapter to it.

Equation (16.6c) which is the most general, is written as

$$\frac{d^2\rho}{dv_0^2} + \rho[(1+\alpha) - C(1+\beta)\sin(\lambda v_0 + k)] = B'',$$

where B'' is considered a known function of v_0. This is an equation linear on the right-hand side, whose integration reduces to that of an equation with zero right-hand side:

$$\frac{d^2\rho}{dv_0^2} + \rho[(1+\alpha) - C(1+\beta)\sin(\lambda v_0 + k)] = 0.$$

If this equation is transformed by changing the notation and by putting

$$\rho = x, \quad \lambda v_0 + k = 2t + \frac{\pi}{2},$$

$$\frac{4}{\lambda^2}(1+\alpha) = q^2, \quad \frac{4}{\lambda^2}C(1+\beta) = q_1,$$

it will become

$$\frac{d^2x}{dt^2} = x(-q^2 + q_1\cos 2t).$$

Study of the Gyldén Equation

178. Let us consider the equation[R13]

$$\frac{d^2x}{dt^2} = x(-q^2 + q_1\cos 2t). \tag{17.1}$$

It was shown above that Gyldén, in his research, had been induced to consider the following equation [see no. 169, Eqs. (16.7) and (16.8)]:

$$\frac{d^2x}{dt^2}=f(x,t), \tag{17.2}$$

where $f(x,t)$ is a function that can be expanded in powers of x and which is periodic with respect to t.

Now, it so happens that, in the applications of this equation made by Gyldén, the most important terms of $f(x,t)$ have the form

$$\varphi(t)+x(-q^2+q_1\cos 2t),$$

with $\varphi(t)$ being a periodic function of t alone, and that all other terms can be neglected in first approximation.

Equation (17.2) can then be substituted by

$$\frac{d^2x}{dt^2}=\varphi(t)+x(-q^2+q_1\cos 2t). \tag{17.3}$$

This is an equation linear on the right-hand side whose integration, as known, readily reduces to that of the corresponding equation with zero on the right-hand side which is nothing else but Eq. (17.1).

Let us now study Eq. (17.1), and let us first recall what the general results given in Vol. I on the subject of linear equations (Chap. 2, no. 29, and Chap. 4, *passim*) allow us to conclude.

Primarily, these results show that Eq. (17.1) admits of two particular integrals of the form

$$x=e^{iht}\varphi_1(t),\quad x=e^{-iht}\varphi_2(t),$$

where φ_1 and φ_2 are two periodic functions of t with period π, and where the two characteristic exponents $h\sqrt{-1}$ and $-h\sqrt{-1}$ are equal and of opposite sign.

To go beyond this, we will make use of a general theorem demonstrated earlier in a paper on groups of linear equations.[10]

Let there be a linear equation of the form

$$\frac{d^py}{dx^p}=\varphi_{p-1}(x)\frac{d^{p-1}y}{dx^{p-1}}+\varphi_{p-2}(x)\frac{d^{p-2}y}{dx^{p-2}}+\cdots$$
$$+\varphi_1(x)\frac{dy}{dx}+\varphi_0(x)y. \tag{17.4}$$

The coefficients $\varphi_i(x)$ are functions not only of x but also of a certain number of parameters on which they depend linearly.

Let us assume, for example, that there are three parameters, and let us denote these by A, B, and C. Then, the function $\varphi_i(x)$ will have the form

$$\varphi_i(x)=A\varphi_i'(x)+B\varphi_i''(x)+C\varphi_i'''(x).$$

The functions $\varphi_i'(x)$, $\varphi_i''(x)$, and $\varphi_i'''(x)$ will be continuous, as will be all their derivatives, on the interior of a domain from which we have not removed x.

After this, let us use the initial values of y and of its $p-1$ first derivatives at the point $x=0$ and let us vary x from zero to a certain value x_1, by following a predetermined path. Let y_1 be the value assumed by y as soon as x arrives at the point x_1. It is clear that y_1 will depend on:
(a) the initial values of y and its derivatives (on which it will depend linearly);
(b) the parameters A, B, C.

The theorem in question postulates that y_1 can be expanded in a series in ascending powers of A, B, and C and that this series will converge no matter what the values of these three quantities might be; in other words, y_1 will be an integral function of A, B, and C.

Let us apply this theorem to Eq. (17.1).

Let $F(t)$ be a particular integral of this equation such that

$$F(0)=1, \quad F'(0)=0;$$

[for abbreviation, we will denote dF/dt by $F'(t)$].

Let, similarly, $f(t)$ be a second particular integral such that

$$f(0)=0, \quad f'(0)=1.$$

Then, if x_0 and x_0' are the initial values of x and of dx/dt for $t=0$, we will have

$$x=x_0F(t)+x_0'f(t).$$

Our theorem thus is that $F(t)$ and $f(t)$ will be entire functions of q^2 and of q_1. This is also the case for $F'(t)$ and $f'(t)$.

Let us assume, specifically, that

$$x=e^{iht}\varphi_1(t),$$

so that

$$\varphi_1(0)=x_0, \quad \varphi_1'(0)+ih\varphi_1(0)=x_0'$$

and

$$e^{ih\pi}\varphi_1(\pi)=x_0F(\pi)+x_0'f(\pi),$$

$$e^{ih\pi}[\varphi_1'(\pi)+ih\varphi_1(\pi)]=x_0F'(\pi)+x_0'f'(\pi).$$

However, the function φ_1 is periodic so that we will have

$$\varphi_1(0)=\varphi_1(\pi), \quad \varphi_1'(0)=\varphi_1'(\pi),$$

whence

$$e^{ih\pi}x_0 = x_0F(\pi) + x_0'f(\pi),$$
$$e^{ih\pi}x_0' = x_0F'(\pi) + x_0'f'(\pi).$$

Hence

$$[F(\pi) - e^{ih\pi}][f'(\pi) - e^{ih\pi}] - f(\pi)F'(\pi) = 0.$$

Thus $e^{ih\pi}$ is a root of the equation in S,

$$[F(\pi) - S][f'(\pi) - S] - f(\pi)F'(\pi) = 0.$$

In the same manner, it can be demonstrated that the other root is $e^{-ih\pi}$. Consequently, the sum of the roots is equal to 2 cos $h\pi$, so that we have

$$2 \cos h\pi = F(\pi) + f'(\pi).$$

It results from this that cos $h\pi$ is an entire function of q^2 and of q_1, i.e., that cos $h\pi$ can be expanded in integral powers of q^2 and of q_1 and that the series is always convergent.

We state now that this series contains only even powers of q_1.

If, in fact, we change t into $t + \pi/2$, the solutions

$$x = e^{iht}\varphi_1(t), \quad x = e^{-iht}\varphi_2(t)$$

will become

$$x = e^{iht}\psi_1(t), \quad x = e^{-iht}\psi_2(t)$$

where

$$\psi_1(t) = e^{ih\pi/2}\varphi_1\left(t + \frac{\pi}{2}\right), \quad \psi_2(t) = e^{-ih\pi/2}\varphi_2\left(t + \frac{\pi}{2}\right)$$

are periodic functions in t. Consequently, the characteristic exponents do not change.

At the same time, since

$$\cos 2\left(t + \frac{\pi}{2}\right) = -\cos 2t,$$

Eq. (17.1) becomes

$$\frac{d^2x}{dt^2} = x(-q^2 - q_1 \cos 2t),$$

which means that the characteristic exponents and thus also cos $h\pi$ do not change when q_1 is changed into $-q_1$. However, this can take place only if the expansion of cos $h\pi$ contains only even powers of q_1.

Let us note now that Eq. (17.1) will not change when one changes t into $-t$. It results from this that $F(t)$ is an even function of t and that $f(t)$ is an odd function, i.e., that

$$F(t) = F(-t), \quad f(t) = -f(-t).$$

However, the solutions of Eq. (17.1) can be expanded in cosines and sines of $(h + 2m)t$, with m being a positive and negative integer. It results from this that F will contain only cosines while f will contain only sines. We will then have

$$F(t) = \sum A_m \cos(h + 2m)t,$$

$$f(t) = \sum B_m \sin(h + 2m)t,$$

where m varies from $-\infty$ to $+\infty$. This will yield

$$F(0) = \sum A_m = 1,$$

$$F(\pi) = \sum A_m \cos(h\pi + 2m\pi) = \sum A_m \cos h\pi = \cos h\pi,$$

$$f'(t) = \sum B_m (h + 2m)\cos(h + 2m)t,$$

$$f'(0) = \sum B_m (h + 2m) = 1,$$

$$f'(\pi) = \sum B_m (h + 2m)\cos h\pi = \cos h\pi.$$

We thus have

$$F(\pi) = f'(\pi) = \cos h\pi.$$

179. Let us now see how one can obtain the expansion of $F(\pi)$ in ascending powers of q_1.

Let us assume that we are searching more generally for the expansion of $F(t)$ and let us put

$$F(t) = F_0(t) + q_1 F_1(t) + q_1^2 F_2(t) + \cdots .$$

For determining $F_0, F_1, F_2, \ldots$, we will have the following sequence of equations:

$$\begin{aligned} \frac{d^2F_0}{dt^2} + q^2F_0 &= 0, \\ \frac{d^2F_1}{dt^2} + q^2F_1 &= F_0 \cos 2t, \\ \frac{d^2F_2}{dt^2} + q^2F_2 &= F_1 \cos 2t, \\ &\vdots \end{aligned} \tag{17.5}$$

In addition, the functions F_i must be even; F_0 must reduce to 1 and the other functions F_i must reduce to 0 for $t = 0$.

One can directly conclude from this that

$$F_0(t) = \cos qt,$$

$$\frac{d^2F_1}{dt^2} + q^2F_1 = \frac{\cos(q+2)t}{2} + \frac{\cos(q-2)t}{2},$$

and

$$F_1(t) = -\frac{\cos(q+2)t - \cos qt}{8(q+1)} + \frac{\cos(q-2)t - \cos qt}{8(q-1)}.$$

We next obtain

$$\frac{d^2F}{dt^2} + q^2F_2 = \alpha_0 \cos(q+4)t + \alpha_1 \cos(q+2)t$$
$$+ \alpha_2 \cos qt + \alpha_3 \cos(q-2)t + \alpha_4 \cos(q-4)t,$$

where α_0, α_1, α_2, α_3, and α_4 are coefficients easy to calculate, from which we can derive

$$F_2 = \frac{\alpha_0 \cos(q+4)t}{8(q+2)} - \frac{\alpha_1 \cos(q+2)t}{4(q+1)}$$
$$+ \frac{\alpha_1 \cos(q-2)t}{4(q-1)} + \frac{\alpha_1 \cos(q-4)t}{8(q-2)}$$
$$+ \frac{\alpha_0 \cos qt}{8(q+2)} + \frac{\alpha_1 \cos qt}{4(q+1)} - \frac{\alpha_3 \cos qt}{4(q-1)} - \frac{\alpha_4 \cos qt}{8(q-2)} + \frac{\alpha_2 t \sin qt}{2q}.$$

One can also see that α_2 is equal to $1/8(q^2-1)$. The general rule is clear:

$$F_i(t) = \sum \beta^0_{in}[\cos(q+2n)t - \cos qt] + t\sum \beta^1_{in} \sin(q+2n)t$$
$$+ t^2\sum \beta^2_{in} \cos(q+2n)t + \cdots + t^k \sum \beta^k_{in} \begin{matrix}\sin\\ \cos\end{matrix}(q+2n)t.$$

Since the function $F_i(t)$ must be even, the coefficient of t^k

$$\sum \beta^k_{in} \begin{matrix}\sin\\ \cos\end{matrix}(q+2n)t$$

will contain only sines if k is odd and only cosines if k is even.

What are the values that the integer n can take?

In the first term

$$\sum \beta^0_{in}[\cos(q+2n)t - \cos qt],$$

n will vary from $-2i$ to $+2i$; in the coefficient of t, n can vary from $-2(i-2)$ to $+2(i-2)$; in the coefficient of t^2, n can vary from $-2(i-4)$ to $+2(i-4)$; and so on, such that k cannot exceed $i/2$.

By means of Eqs. (17.5), it is possible to obtain recurrence relations between the coefficients β^k_{in}. For the time being, we will not discuss this further.

When we set $t = \pi$, we will have

$$\cos(q+2n)\pi - \cos q\pi = 0$$

and the first term of $F_i(t)$ will vanish, such that

$$F_i(\pi) = \pi \sum \beta^1_{in} \sin q\pi + \pi^2 \sum \beta^2_{in} \cos q\pi + \cdots .$$

Furthermore, we know that $F_i(\pi)$ will be zero if i is odd since we already know *a priori* that the expansion of $F(\pi)$ must contain only even powers of q_1.

This was the manner used by Tisserand in calculation $F(\pi)$ and thus also $\cos h\pi$. He found

$$\begin{aligned}\cos h\pi = \cos q\pi &\left(1 - \frac{\pi^2}{512q^2(1-q^2)^2} q_1^4 + \cdots\right)\\ &+ \sin q\pi \left(-\frac{\pi}{16q(1-q^2)} q_1^2\right.\\ &\left.+ \frac{(15q^4 - 35q^2 + 8)\pi}{1024q^3(1-q^2)(2^2-q^2)} q_1^4 + \cdots\right),\end{aligned}$$

which we will write here as

$$\cos h\pi = \varphi(q,q_1)\cos q\pi + \varphi_1(q,q_1)\sin q\pi,$$

where $\varphi(q,q_1)$ and $\varphi_1(q,q_1)$ will be series developed in ascending powers of q_1^2 whose coefficients will be rational in q.

The first question to settle is to know whether h is real or imaginary. If

$$|\cos h\pi| = |F(\pi)| < 1,$$

then h is real and the solution of our differential equation will be stable, and $F(t)$ as well as $f(t)$ will remain within finite limits. If, conversely

$$|F(\pi)| > 1,$$

then h is imaginary and the two functions $F(t)$ and $f(t)$ will have the form

$$F(t) = e^{\alpha t}\psi(t) + e^{-\alpha t}\psi(-t)$$

$$f(t) = Ae^{\alpha t}\psi(t) - Ae^{-\alpha t}\psi(-t)$$

where A and α are real constants and $\psi(t)$ is a periodic function of t with period π. It results from this that $F(t)$ and $f(t)$ can increase beyond all bounds and that the solution of our differential equation is unstable.

If we consider, for the moment, that q and q_1 are coordinates of a point in a plane, then this plane will be subdivided into two regions: one in which $|F(\pi)|$ will be smaller than 1 and h will be real and another in which $|F(\pi)|$ will be larger than 1 and h imaginary. These two regions are separated by the various segments of the two curves

$$\cos h\pi = +1, \quad \cos h\pi = -1.$$

Therefore, it is of interest to plot these two curves, at least within the portion of the plane that corresponds to the low values of q_1.

For $q_1 = 0$, we have

$$\cos h\pi = \cos q\pi.$$

Thus the curve $\cos h\pi = +1$, which I will call C, intersects the q-axis at points whose abscissas are even integers, while the curve $\cos h\pi = -1$, which I will call C', intersects the q-axis at points whose abscissas are odd integers.

All other points of the q axis belong to the first region, namely, the region in which h is real.

Let us then return to the equation

$$\cos h\pi - \cos q\pi - q_1^2 F_2(\pi) - q_1^4 F_4(\pi) - \cdots = 0,$$

which connects h with q and with q_1. The left-hand side vanishes for $h = q$, $q_1 = 0$; the equation can be expanded in ascending powers of $h - q$ and also of q_1^2; finally, its derivative with respect to h reduces to $-\pi \sin q\pi$ for $h = q, q_1 = 0$ and thus does not vanish unless q is an integer. Consequently, if we assume that q is not an integer, the theorem of no. 30 demonstrates that h can be expanded in ascending powers of q_1^2 and that the series is convergent provided that q_1 is sufficiently small.

Let us now see what happens when q is an integer. In applying his formula, Tisserand found:

for $|q| \geqslant 3$,

$$\cos h\pi = (-1)^q \left(1 - \frac{\pi^2}{512q^2(1-q^2)^2} q_1 + \cdots\right),$$

for $|q| = 2$,

$$\cos h\pi = 1 + \frac{5\pi^2 q_1^4}{73728} + \cdots$$

for $|q| = 1$,

$$\cos h\pi = -1 - \frac{\pi^2 q_1^2}{32} - \cdots,$$

and, finally, for $|q| = 0$,

$$\cos h\pi = 1 - \frac{\pi^2 q_1^2}{16} + \cdots .$$

In fact, when q is an integer, $\cos q\pi$ becomes equal to ± 1 and $\sin q\pi$ vanishes. However, it happens at the same time that $\varphi_1\ (q,q_1)$ becomes infinite so that the product

$$\sin q\pi\, \varphi_1(q,q_1),$$

tends to a finite value as q tends to a whole number. Let us then consider the limit

$$L = \lim \sin q\pi\, \varphi_1(q,q_1),$$

when q tends to an integral value.

This limit can be expanded in powers of q_1^2; however, in the expansion of $\varphi_1(q,q_1)$, the coefficient of q_1^2 becomes infinite for $|q| = 0$ or 1 while the coefficient of q_1^4 becomes infinite for $|q| = 0$, 1, or 2, and that of q_1^6 becomes infinite for $|q| = 0, 1, 2$, or 3. It results from this that, if q tends to an integer n_1, the expansion of L will start with a term in q_1^{2n}. On the other hand, the expansion of $\varphi(q,q_1) - 1$ starts with a term in q_1^4. It is for this reason that, in the expansions of

$$\cos h\pi - (-1)^q,$$

obtained by Tisserand, the first term is in q_1^2 for $|q| = 0$ or 1 and in q_1^4 for $|q| > 1$.

Let us thus consider the equation of the curve C, which can be written as

$$1 - \cos q\pi - q_1^2 F_2(\pi) - q_1^4 F_4(\pi) - \cdots = 0.$$

Since the curve passes through the point

$$q = 2n, \quad q_1 = 0 \quad (n \text{ integer})$$

the left-hand side vanishes for $q = 2n$, $q_1 = 0$; it is also expandable in ascending powers of $q - 2n$ and of q_1. It is easy to prove that this expansion contains no terms of degree 0 or 1, but starts with second-degree terms

$$\frac{\pi^2}{2}(q - 2n)^2 + Aq_1^2,$$

where A is equal to

$$\frac{\pi^2}{16} \text{ for } n = 0,$$

and to zero for $n \lessgtr 0$.

It follows from this that the point $q = 2n$, $q_1 = 0$ is a double point for the curve C; however, two cases must be differentiated here:
(i) If $n = 0$, the second-degree terms reduce to the sum of two squares and the two segments of the curve passing through the double point are imaginary. Consequently, the origin for the curve C is an isolated point.
(ii) If $n \lessgtr 0$, then A is zero. The two segments of the curve passing through the double point are mutually tangent and intersect the q-axis at a right angle. To determine whether these two segments are real or imaginary, it is necessary to consider terms in $(q - 2n)q_1^2$ and in q_1^4.

The coefficient of q_1^4, as demonstrated above, is

$$\frac{+\pi^2}{512q^2(1-q^2)^2} \quad \text{or} \quad \frac{-5\pi^2}{73728},$$

depending on whether $|n| > 1$ or $= 1$.

The coefficient of $(q - 2n)q_1^2$ will be obtained by taking derivatives of

$$\frac{\pi \sin q\pi}{16q(1-q^2)},$$

with respect to q and by setting there $q = 2n$. This will yield

$$\frac{\pi^2}{16q(1-q^2)}.$$

So as to have the segments of the curves be real (assuming $|n| > 1$), it is necessary and sufficient that the quadratic form

$$\frac{x^2}{2} + \frac{xy}{16q(1-q^2)} + \frac{y^2}{512q^2(1-q^2)^2}$$

be indefinite. This is beyond doubt only if this form is reduced to a perfect square, however, this is exactly what happens; thus we see that our two curve segments are not only mutually tangent but actually osculate. Nevertheless, to determine if they are real, we would have to calculate the terms of higher order unless we were fortunate enough to have an indirect means for deciding this question, a means that I will discuss below.

In the case of $|n| = 1$, $|q| = 2$, our quadratic form becomes

$$\frac{x^2}{2} - \frac{xy}{96} - \frac{5q^2}{73728}$$

and is indefinite. The two curve segments are definitely real.

Let us now plot the curve C' whose equation is

$$-1 - \cos q\pi - q_1^2 F_2(\pi) \cdots = 0.$$

The left-hand side vanishes for

$$q = n, \quad q_1 = 0 \quad (n \text{ odd integer}),$$

and its expansion in powers of $q-n$ and q_1 starts with second-degree terms

$$A(q-n)^2+Bq_1^2.$$

If $|n|=1$, then A and B are of opposite sign and the two curve segments passing through the cusp are real.

If $|n|>1$, then B is zero and the two curve segments are mutually tangent (and probably osculate). To decide whether they are real, it is necessary to use the indirect process mentioned above.

The process consists in the following:

One can ask what will happen when we have

$$F(\pi)=\cos h\pi=\pm 1.$$

According to what we have seen in no. 29, the most general solution of Eq. (17.1) will have the form

$$\psi(t)+t\psi_1(t),$$

where $\psi(t)$ and $\psi_1(t)$ are periodic functions of t with period π if $F(\pi)=+1$ and with period 2π if $F(\pi)=-1$ (thus they change sign when t changes into $t+\pi$). Consequently, we have

$$F(t)=\psi(t)+t\psi_1(t).$$

If $\psi_1(t)$ is not zero, this constitutes a solution of Eq. (17.1). However, $F(t)$ is an even function; consequently $\psi(t)$ is even and $\psi_1(t)$ is odd. Thus $\psi_1(t)$ reduces to $f(t)$ to within a constant factor, and $f(t)$ is thus periodic.[R14]

If $\psi_1(t)$ is identically zero, $F(t)$ is periodic.

Three different cases may occur:

(a) Either $F(t)$ is periodic, so that

$$F'(\pi)=0.$$

(b) Or else $f(t)$ is periodic, so that

$$f(\pi)=0.$$

(c) Or else these two functions are both periodic, so that

$$F(\pi)=f(\pi)=0.$$

The same result can be obtained in the following manner. We have identically

$$F(t)f'(t)-F'(t)f(t)=1.$$

If, then

$$F(\pi)=f'(\pi)=\pm 1,$$

we obtain

$$F'(\pi)f(\pi) = 0.$$

Thus at least one of the two quantities $F'(\pi)$ and $f(\pi)$ will be zero.

Similarly, if

$$F'(\pi) = 0 \quad \text{or} \quad f(\pi) = 0,$$

we will have

$$F(\pi)f'(\pi) = 1,$$

and, since

$$F(\pi) = f'(\pi) = \cos h\pi,$$

we obtain

$$\cos h\pi = \pm 1.$$

The various points of the curves C and C' belong thus to the two curves

$$F'(\pi) = 0, \quad f(\pi) = 0,$$

and vice versa.

Let us note that $F'(\pi)$ and $f(\pi)$ are entire functions of q and of q_1. For $q_1 = 0$, these functions reduce to

$$F'(\pi) = -q \sin q\pi, \quad f(\pi) = \frac{\sin q\pi}{q}.$$

Thus if q passes through an integer value different from zero, then $F'(\pi)$ and $f(\pi)$ will vanish and change sign, and these values of q will be single zeros for these two functions. It results from this that the points

$$q = n, \quad q_1 = 0 \quad (n \text{ integer}, \ n \lessgtr 0),$$

which are double points, sometimes for C and sometimes for C', are single points for each of the two curves

$$F'(\pi) = 0, \quad f(\pi) = 0.$$

If q passes through zero, then $F'(\pi)$ vanishes without change in sign (double zero) and $f(\pi)$ will not vanish. Thus the origin will be a double point for $F'(\pi) = 0$ but $f(\pi)$ will not vanish at the origin.

Consequently, four analytically distinct curves exist:

$$F(\pi) = 1, \quad F'(\pi) = 0, \quad F(t + \pi) = F(t); \tag{17.6}$$

$$F(\pi) = 1, \quad f(\pi) = 0, \quad f(t + \pi) = f(t); \tag{17.7}$$

$$F(\pi) = -1, \quad F'(\pi) = 0, \quad F(t + \pi) = -F(t); \tag{17.8}$$

$$F(\pi) = -1, \quad f(\pi) = 0, \quad f(t + \pi) = -f(t). \tag{17.9}$$

The curve C is then formed from the union of the two curves (17.6) and (17.7). Each of these has a single point in

$$q = 2n, \quad q_1 = 0,$$

and it is for this reason that the point is a double point of C. However, the two segments of C passing through this point thus belong to two analytically distinct curves and can be nothing else but real.

There is one exception, for the origin:

$$q = 0, \quad q_1 = 0.$$

This point is a double point of (17.6) but does not belong to (17.7). According to our above statements, the preceding reasoning thus does not apply; furthermore, we have seen that the two curve segments are then imaginary.

Similarly, the curve C' is formed from the union of the two curves (17.6) and (17.7); each of these has a single point at

$$q = 2n + 1, \quad q_1 = 0.$$

The two segments of C' passing through this point belong to two analytically distinct curves and consequently are real.

It has been demonstrated above that changing q_1 into $-q_1$ amounts to exactly the same thing as changing t into $t + \pi/2$.

Let us first consider the curve (17.6):

$$F(t + \pi) = F(t).$$

We have

$$F(t) = F(-t),$$

from which it follows that

$$F\left(t + \frac{\pi}{2}\right) = F\left(-t - \frac{\pi}{2}\right) = F\left(\frac{\pi}{2} - t\right).$$

The function $F(t + \pi/2)$ is thus even and periodic, with period π. Consequently, if q_1 is changed into $-q_1$, then $F(t)$ will change into $F(t + \pi/2)$ which again is even and periodic. Thus if the point (q,q_1) belongs to the curve (17.6) it must be the same for the point $(q, -q_1)$. Therefore, the curve (17.6) is symmetric with respect to the axis of q.

Since the curve C is symmetric on the whole and is composed of (17.6) and of (17.7), we can conclude (which is easy to verify) that the curve (17.7) is also symmetric with respect to the axis of q.

From this, one can conclude that the two curves (17.6) and (17.7), at the point

$$q = 2n, \quad q_1 = 0$$

can only have a contact of odd order.

Let us now consider the curve (17.8):

$$F(t + \pi) = -F(t).$$

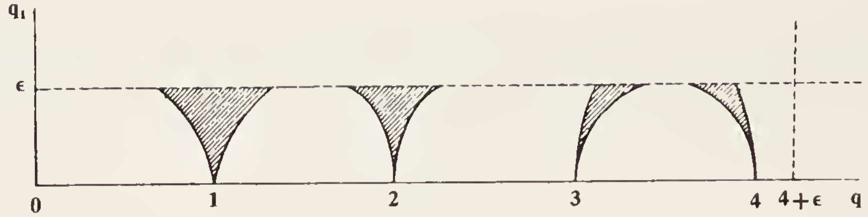

Figure 6

We obtain from this

$$F\left(t+\frac{\pi}{2}\right)=F\left(-\frac{\pi}{2}-t\right)=-F\left(\frac{\pi}{2}-t\right).$$

Consequently, the function $F(t+\pi/2)$ is odd and periodic. Thus if the point (q,q_1) belongs to (17.8), then the point $(q,-q_1)$ will belong to (17.9). The two curves (17.8) and (17.9) are thus symmetric with respect to each other and with respect to the axis of q.

From this it follows that these two curves, at the point

$$q=2n+1, \quad q_1=0$$

can have only a contact of even order.

Thus, the contact of the two curve segments in $q=n, q_1=0$, is of zero order for $n=1$, of first order for $n=2$, of second order (at least) for $n=3$, and of the third order (at least) for $n=4$. After this, it will alternatively be of even and odd order and will always be at least of the second order.

This might induce one to believe that the osculation always is of the order $n-1$; however, we have never been able to prove this.

Figure 6, contained in the rectangle

$$q=0, \quad q_1=0, \quad q_1=\epsilon, \quad q=4+\epsilon,$$

can be used for summarizing the above discussion. The hatched region is that in which h is imaginary.

From the equation yielding $\cos h\pi$ as a function of q and q_1, we can derive various series whose convergence is more or less rapid and which yield h, arranged in powers of q_1. However, we believe it preferable to calculate $\cos h\pi$ by means of the preceding formulas and to derive h from this via trigonometric tables.

180. Once h is determined, it will be a question of finding the coefficients A_n of the expansion

$$F(t) = \sum A_n \cos(h + 2n)t. \ .$$

According to the definition of $F(t)$, we must have

$$\sum A_n = 1.$$

On the other hand, it follows that

$$F(t) = \sum A_n \frac{e^{i(h+2n)t}}{2} + \sum A_n \frac{e^{-i(h+2n)t}}{2}.$$

However, according to what we have seen in no. 29, Eq. (17.1) must admit of two solutions of the form

$$\sum B_n \frac{e^{i(h-2n)t}}{2}, \quad \sum C_n \frac{e^{-i(h+2n)t}}{2},$$

and $F(t)$ must be a linear combination; this can take place only if

$$A_n = B_n = C_n.$$

It results from this that

$$\sum A_n \sin(h + 2n)t = \sum \frac{A_n}{2i} e^{i(h+2n)t} - \sum \frac{A_n}{2i} e^{-i(h+2n)t}$$

will also satisfy Eq. (17.1) and, consequently, that

$$f(t) = \frac{\sum A_n \sin(h + 2n)t}{\sum A_n (h + 2n)}.$$

It is obvious that A_n is a function of q and q_1 but that it is no longer an entire function of these two variables as had been the case for cos $h\pi$. It is not even a uniform function. Evidently the only singular points of this function are the points of the curves

$$\cos h\pi = \pm 1,$$

for which the functions $F(t)$ and $f(t)$ can no longer be put in the form given to them before.

How does the function A_n behave in the neighborhood of one of these singular points?

Let us assume that the point (q,q_1) indefinitely approaches a point M belonging to the curve

$$F'(\pi) = 0,$$

and that h tends toward an integral value p. Then, $F(t)$ is still periodic at the limit. For abbreviation, let us put

$$B = \sum A_n (h + 2n),$$

so that, on combining in $F(t)$ and $f(t)$ all terms in $(h+2n)t$ and in $(h-2n-2p)t$, we obtain

$$F(t) = \sum [A_n \cos(h+2n)t + A_{-n-p} \cos(h-2n-2p)t],$$

$$Bf(t) = \sum [A_n \sin(h+2n)t + A_{-n-p} \sin(h-2n-2p)t].$$

If we then set

$$A_n + A_{-n-p} = C, \quad A_n - A_{n-p} = D;$$

we obtain

$$F(t) = \sum [C \cos(h-p)t \cos(2n+p)t$$
$$- D \sin(h-p)t \sin(2n+p)t],$$

$$Bf(t) = \sum [C \sin(h-p)t \cos(2n+p)t$$
$$+ D \cos(h-p)t \sin(2n+p)t].$$

When h tends to p, then $\cos(h-p)t$ will tend to 1 and $\sin(h-p)t$ to zero. However, if D tends to infinity in such a manner that $D(h-p)$ tends to a finite limit, then the product $D \sin(h-p)t$ will tend to $D_0 t$ where D_0 is a constant.

If, then, the point M belongs to the curve $F'(\pi) = 0$, the expansion of $F(t)$ must contain only terms in

$$\cos(2n+p)t$$

and the expansion of $f(t)$, terms in $\sin(2n+p)t$ and in $t \cos(2n+p)t$.

Consequently, it is necessary that C tends to a finite limit and D and B, to zero. Thus A_n and A_{-n-p} will tend to finite limits which are mutually equal. If p is even, then $A_{p/2}$ must tend to zero. It is easy to prove that, if

$$\lim A_n = \lim A_{-n-p},$$

we will have, as agreed,

$$\lim B = \lim \sum A_n (h+2n) = 0.$$

If, conversely, the point M belongs to the curve $f(\pi) = 0$, then the expansion of $F(t)$ must contain terms in

$$\cos(2n+p)t \quad \text{and} \quad t \sin(2n+p)t,$$

and the expansion of $f(t)$, terms in $\sin(2n+p)t$.

Thus it is necessary that C tends to a finite limit while D and B tend to infinity.

Consequently, A_n and A_{-n-p} tend to infinity but their algebraic sum will remain finite.

However, no matter whether the point M belongs to the curve $F'(\pi) = 0$ or to the curve $f(\pi) = 0$, it still will be a singular point for the function A_n. In fact, when the point (q,q_1) rotates about M, the function A_n will be exchanged for the function A_{-n-p}, as is done by two determinations of one and the same algebraic function.

It follows from this that, if q is not an integer, A_n could be expanded in ascending powers of q_1 and that the radius of convergence of this series will be the modulus of the closest singular point, while the singular points themselves will be the points of the curves C and C' that correspond to the value of q being considered.

This leaves the coefficients of the series to be determined. Let us assume that the problem is solved and let

$$F(t) = \sum A_n \cos(h + 2n)t = \sum A_n \cos(q + 2n + h - q)t,$$

or, expanding in powers of $(h - q)$, let

$$F(t) = \sum A_n \cos(q + 2n)t - t\sum A_n(h - q)\sin(q + 2n)t$$

$$-\frac{t^2}{1\cdot 2}\sum A_n(h - q)^2 \cos(q + 2n)t + \cdots .$$

This series, containing the trigonometric lines of $(q + 2n)t$ multiplied by powers of t, must be identical to that obtained above:

$$F(t) = \sum q_1^i F_i(t),$$

$$F_i(t) = \sum \beta^0_{in}[\cos(q + 2n)t - \cos qt] + t\sum \beta^1_{in} \sin(q + 2n)t + \cdots .$$

Let us note that A_n, $A_n(h - q)$, can be expanded in ascending powers of q_1.

Identifying the two expansions, we then obtain

$$A_n = \sum q_1^i \beta^0_{in}$$

and

$$A_0 = 1 - \sum_i q_1^i\Big(\sum_n \beta^0_{in}\Big).$$

This gives us the means for calculating the coefficients of the series. The convergence is usually sufficient, provided that q is not close to a whole number. If q is close to an integer p, the convergence can be improved in the following manner:

Since A_n and A_{-n-p} are interchanged as one turns about the closest singular point, the two functions

$$(A_n + A_{-n-p}) \quad \text{and} \quad (A_n - A_{-n-p})^2$$

remain uniform in the vicinity of this singular point, but the first of these functions will remain finite while the second can become infinite of the first order if the singular point belongs to $f(\pi) = 0$. However, in that case

$$(A_n - A_n)^2 f(\pi)$$

will remain finite. Consequently, the expansions of

$$A_n + A_{-n-p} \quad \text{and} \quad (A_n - A_{-n-p})^2 f(\pi)$$

will converge much better than those of A_n and A_{-n-p}. Therefore, it is of advantage to use these and to derive A_n and A_{-n-p} from this by an equation of the second degree.

Finally, let us note that a discussion of the slopes of the curves C and C' in the vicinity of the points

$$q = n, \quad q_1 = 0$$

will be greatly facilitated if the expansion of $F'(\pi)$ and of $f(\pi)$ is used instead of that of $F(\pi)$.

What we have stated above constitutes the complete theory of Eq. (17.1). However, it is necessary to discuss the various methods suggested for integration of this equation and based on an application of the theorems by Jacobi, Gyldén, Bruns, Hill, and Lindstedt.

Jacobi Method

181. The method discussed in detail in Chap. 9 can be applied to Eq. (17.1), with the only difference that the series will definitely be convergent. In fact, Eq. (17.1) appears as a particular case of Eq. (1.6) of no. 2. However, we have seen that this equation from no. 2 can be reduced to the canonical form of the Jacobi equations.

Consequently, if we put

$$x = \sqrt{\frac{2x_1}{q}} \sin y_1, \quad \frac{dx}{dt} = \sqrt{2qx_1} \cos y_1, \quad y_2 = t, \quad q = \mu q_1,$$

and

$$F = -qx_1 - x_2 + \mu x_1 \sin^2 y_1 \cos 2y_2,$$

then the equation

$$\frac{d^2x}{dt^2} + q^2 x = q_1 x \cos 2t \tag{17.10}$$

can be replaced by the canonical equations

$$\frac{dy_2}{dt} = -\frac{dF}{dx_2} = 1, \quad \frac{dx_2}{dt} = \frac{dF}{dy_2}, \quad \frac{dx_1}{dt} = \frac{dF}{dy_1} = \mu \sin 2y_1 \cos 2y_2,$$

$$\frac{dy_1}{dt} = -\frac{dF}{dx_1} = q - \mu \sin^2 y_1 \cos 2y_2.$$

In that case, the problem reduces to an integration of the partial differential equation

$$q\frac{dS}{dy_1} + \frac{dS}{dy_2} = \mu \frac{dS}{dy_1} \sin^2 y_1 \cos 2y_2, \tag{17.11}$$

to which the method of successive approximations of no. 125 is directly applicable.

However, there is no great advantage in using it unless Eq. (17.10) is only an approximate expression of the postulated problem and unless, after integrating this equation, one wishes to continue this approximation further by using the method variation of constants or unless it is to be used as verification.

Let us note, in passing, that an integration of Eq. (17.11) reduces to that of a differential equation of the first order

$$\frac{dy_2}{dy_1} = q + \mu \sin^2 y_1 \cos 2y_2.$$

Be that as it may, let us see what relation might exist between the function defined by Eq. (17.11) and the functions $F(t)$ and $f(t)$ defined in the foregoing.

For the general solution we will find canonical equations derived from Eq. (17.10) by the change of variables that precedes the subsequent expression. We recall that we had put

$$F(t) = \sum A_n \cos(h + 2n)t.$$

Using $x_1^0, x_2^0, y_1^0, y_2^0$ to denote constants of integration, our expression will become

$$\sqrt{\frac{2x_1}{q}} \sin y_1 = x_1^0 \sum A_n \cos(ht + hy_1^0 + 2nt + 2ny_2^0),$$

$$\sqrt{2qx_1} \cos y_1 = -x_1^0 \sum A_n (h + 2n) \sin(ht + hy_1^0 + 2nt + 2ny_2^0),$$

$$y_2 = t + y_2^0, \quad x_2 = x_2^0 + \int \frac{dF}{dy_2} dt.$$

If one eliminates the two constants y_1^0 and y_2^0 between these equations and solves them for x_1 and x_2 one obtains x_1 and x_2 as a function of y_1, y_2, x_1^0, and x_2^0. Furthermore,

$$x_1 dy_1 + x_2 dy_2 = dS$$

will be an exact differential (see no. 19, *in fine*).

For abbreviation, let us then put

$$ht + hy_1^0 = \varphi,$$

so that

$$\sqrt{\frac{2x_1}{q}} \sin y_1 = x_1^0 \sum A_n \cos(\varphi + 2ny_2),$$

$$\sqrt{2qx_1} \cos y_1 = -x_1^0 \sum A_n (h + 2n)\sin(\varphi + 2ny_2),$$

and it is now a question of eliminating φ between these two equations.

To perform this elimination, let us note that these two equations can be written as follows:

$$\sqrt{\frac{2x_1}{q}} \frac{\sin y_1}{x_1^0} = \theta_1(y_2)\cos\varphi + \theta_2(y_2)\sin\varphi,$$

$$\sqrt{2qx_1} \frac{\cos y_1}{x_1^0} = \theta_3(y_2)\cos\varphi + \theta_4(y_2)\sin\varphi,$$

where θ are periodic functions of y_2 with period π that are readily expressed by means of $F(t)$ and $f(t)$. Solving these equations for $\cos\varphi$ and $\sin\varphi$, we obtain

$$\frac{x_1^0}{\sqrt{x_1}} \cos\varphi = \eta_1(y_2)\cos y_1 + \eta_2(y_2)\sin y_1,$$

$$\frac{x_1^0}{\sqrt{x_1}} \sin\varphi = \eta_3(y_2)\cos y_1 + \eta_2(y_2)\sin y_1,$$

where the four functions $\eta(y_2)$ are periodic, with period π, and are readily expressed in terms of θ and thus also in terms of $F(t)$ and $f(t)$. On forming the sum of squares, we will obtain, if we note that x_1 must be an even function in y_1,

$$\frac{(x_1^0)^2}{x_1} = \zeta_0(y_2) + \zeta_1(y_2)\cos 2y_1 + \cdots,$$

where the two functions ζ are still periodic with period π, and are readily expressed by means of $F(t)$ and $f(t)$.

Now, we have

$$x_1 = \frac{dS}{dy_1},$$

from which we can draw the following conclusion:

The quantity dS/dy_1 is a periodic function, with period π both with respect to y_1 and with respect to y_2; its expansion, as one can see on applying the method of no. 125, contains terms in $\cos(2my_1 + 2ny_2)$ where m and n can assume all possible integral values. However, the reciprocal function

$$\left(\frac{dS}{dy_1}\right)^{-1},$$

which is also periodic in y_1 and y_2, can obtain only terms in

$$\cos 2ny_2 \quad \text{or} \quad \cos(2y_1 + 2ny_2),$$

since obviously dS/dy_1 is an even function with respect to y_1 as well as with respect to y_2.

If we put

$$u = \left(\frac{dS}{dy_1}\right)^{-1},$$

then Eq. (17.11) will yield

$$q\frac{du}{dy_1} + \frac{du}{dy_2} = \mu\left(\frac{du}{dy_1}\sin^2 y_1 \cos 2y_2 - 2u \sin y_1 \cos y_1 \cos 2y_2\right).$$

The procedures of no. 125 are applicable to this equation despite the fact that it contains not only the derivatives of u but also the function u itself.

We then find

$$u = u_0 + \mu u_1 + \mu^2 u_2 + \cdots.$$

Here, u_0 is a constant and it is easy to prove that $u_1, u_2, \ldots$ have exactly the indicated form, i.e., that

$$u_i = \sum A_n^i \cos 2ny_2 + \sum B_n^i \cos(2y_1 + 2ny_2).$$

It is also easy to derive the recurrence relations that permit determining the constants

$$A_n^{i+1} \quad \text{and} \quad B_n^{i+1},$$

if A_n^i and B_n^i are known.

Gyldén's Method

182. Picard demonstrated the following theorem:

If a linear equation has doubly periodic functions as coefficients and if its general integral has no singularity other than poles, then this integral can be expressed by means of "doubly periodic functions of the second kind," i.e., by functions that recur multiplied by a constant factor when the variable increases by one period.

The significance of this theorem is based on the following two facts:
(a) It is always easy to see from the equation itself whether the general integral has no singularity other than poles.
(b) Any doubly periodic function of the second kind is expressed simply by means of Jacobi θ functions or of Weierstrass σ functions.

Gyldén had the ingenious idea of applying this theorem to the integration of Eq. (17.10). However, it would be unjust to present matters in this form, without mentioning the name of Hermite. What Gyldén applied is in reality a theorem by Hermite on the Lamé equation which, in turn, is only a particular case of the Picard theorem but predates it by quite some time.

Our equation (17.10) can then be written as

$$\frac{d^2x}{dt^2} = (a + b\cos^2 t)x, \tag{17.12}$$

by putting

$$a = -q^2 + \frac{q_1}{2}, \quad b = \frac{q_1}{2}.$$

Let us next consider the function

$$\cos \operatorname{am} t = \operatorname{cn} t \quad (\operatorname{mod} k).$$

It is obvious, from the definition of this function, that cn t will tend to cos t as k tends to zero. Thus, if k is very small, Eq. (17.12) can be replaced by

$$\frac{d^2x}{dt^2} = x(a + b\operatorname{cn}^2 t) \tag{17.13}$$

and the approximation will be better the smaller is k.

After this, let us define the conditions for which the general integral of Eq. (17.13) will have no singularity other than poles. The only singular point of Eq. (17.13) is the point

$$\frac{\omega' i}{2},$$

where ω and $\omega' i$ denote the periods of $\operatorname{cn}^2 t$. In fact, for

$$t = \frac{\omega' t}{2},$$

cn t becomes infinite. It is known that the residue of cn t is $-\sqrt{-1}/k$ so that, on expanding $\operatorname{cn}^2 t$ in powers of

$$t - \frac{\omega' i}{2} = u,$$

we will have a series of the form

$$-\frac{1}{k^2u^2}+\alpha_0+\alpha_1u^2+\cdots$$

containing only even powers of u.

The condition stipulating that the expansion of x in ascending powers of u start by a term in u^{-n} is readily obtained by equating, on both sides of Eq. (17.13), all terms in u^{-n-2}, which then are terms of the lowest degree. This condition is written as

$$n(n+1)=+\frac{b}{k^2},$$

whence

$$k^2=+\frac{q_1}{2n(n+1)}. \qquad (17.14)$$

If this condition is fulfilled and if n is an integer, then Eq. (17.13) admits of a particular integral which will have a pole for $t=\omega'i/2$.

How does the other integral behave? We learn from the theory of linear equations that this integral can have no singularity other than a pole in $t=\omega'/2$ or else a logarithmic point. However, a study of the expansion of x in powers of u readily shows that, from the instant at which $a+b\,\mathrm{cn}^2\,t$ becomes an even function of u, one need no longer fear that the expansion of the integrals contains a logarithm. For further details, the reader is referred to the well-known work by Fuchs on linear equations[11] and to the thesis by Tannery[12] where this is continued. Thus if the condition (17.14) is satisfied, Eq. (17.13) will admit of two particular integrals of the form

$$x=\frac{\theta(t-a_1)\theta(t-a_2)\cdots\theta(t-a_n)}{\theta^n(t)},$$

where θ is that one of the four Jacobi functions θ which vanishes for $t=\omega'i/2$.

The n quantities $a_1, a_2, \ldots, a_n$ can be readily determined, as demonstrated in Hermite's research on the Lamé equation, which has completely exhausted the question.

We can now select an integer n sufficiently large that the value of k which satisfies the condition (17.14) is as small as desired and, consequently, that Eqs. (17.12) and (17.13) differ as little as desired.

However, since q_1 generally is very small, Gyldén estimated that one could be content in applications with the first approximation and set $n=1$.

Bruns Method

183. Let us return to the equation

$$\frac{d^2x}{dt^2} + q^2x = q_1x \cos 2t \tag{17.15}$$

and let us set there

$$x = \exp \int z\, dt.$$

The equation then becomes

$$\frac{dz}{dt} + z^2 + q^2 = q_1 \cos 2t. \tag{17.16}$$

Let us assume now that z is expanded in ascending powers of q_1 and that we have

$$z = z_0 + q_1z_1 + q_1^2z_2 + \cdots .$$

We will then determine successively

$$z_0, \quad z_1, \quad z_2, \quad z_3, \quad \ldots$$

for the sequence of equations

$$\begin{aligned}
z_0 &= \pm iq, \\
\frac{dz_1}{dt} + 2z_0z_1 &= \cos 2t, \\
\frac{dz_2}{dt} + 2z_0z_2 &= -z_1^2, \\
\frac{dz_3}{dt} + 2z_0z_3 &= -2z_1z_2, \\
\frac{dz_4}{dt} + 2z_0z_4 &= -2z_1z_3 - z_2^2, \\
\frac{dz_5}{dt} + 2z_0z_5 &= -2z_1z_4 - 2z_2z_3, \\
&\vdots
\end{aligned} \tag{17.17}$$

Equations (17.17) permit calculating the quantities z_k by recurrence. If, in fact, the first k of these equations have been integrated and if thus

$$z_0, \quad z_1, \quad z_2, \quad \ldots, \quad z_{k-1},$$

is known, then the $(k+1)$th term (for example, using $z_0 = +iq$) will be written as

$$\frac{dz_k}{dt} + 2iqz_k = U_k,$$

where U_k is a known function of t.

If $z_0, z_1, z_2, \ldots, z_{k-1}$ are periodic functions of t, with period π, this will be the same for U_k so that we can write

$$U_k = \sum a_n e^{2nit},$$

whence

$$z_k = \sum \frac{a_n e^{2nit}}{2i(n+q)}.$$

Thus, unless q is an integer, it is possible to equate z_k with a periodic function of t.

Then z is a periodic function of t, and we can write

$$z = ih + \frac{du}{dt},$$

where ih is the mean value of this periodic function z, while u is another periodic function. From this, for a particular integral of Eq. (17.15), we derive

$$x = e^{iht+u}.$$

What we have denoted by $F(t)$ in no. 178 will then be the real part of

$$e^{iht}e^{u}.$$

This method is the simplest procedure if one wishes to expand h in powers of q_1.

Lindstedt Method

184. Let us consider the equation

$$\frac{d^2x}{dt^2} + q^2x - q_1 x \cos 2t = 0 \tag{17.18}$$

and its even solution

$$x = F(t) = \sum A_n \cos(h+2n)t.$$

It is clear that we will have

$$A_n[q^2 - (h+2n)^2] = \frac{q_1}{2}(A_{n-1} + A_{n+1}). \tag{17.19}$$

Now, the problem consists in determining h and A_n in such a manner that Eqs. (17.19) are satisfied and that the series $F(t)$ converges. We can also consider the inhomogeneous equation

$$\frac{d^2x}{dt^2} + q^2x - q_1 x \cos 2t = \beta \cos \lambda t. \tag{17.20}$$

This equation admits a solution of the form

$$x = \sum B_n \cos(\lambda + 2n)t.$$

It will also be easy (using the ordinary methods of integration for inhomogeneous equations) to calculate the coefficients B_n once h and A_n are known. However, if they are to be calculated directly, the following equations analogous to Eqs. (17.19) will obtain

$$B_n[q^2 - (\lambda + 2n)^2] = \frac{q_1}{2}(B_{n-1} + B_{n+1}). \tag{17.21}$$

For $n = 0$, this equation should be replaced by

$$B_0(q^2 - \lambda^2) = \frac{q_1}{2}(B_{-1} + B_1) + \beta, \tag{17.21a}$$

which again reduces to Eqs. (17.19) when one sets $\lambda = h, \beta = 0$. Let us then put

$$\frac{q_1}{2[q^2 - (\lambda + 2n)^2]} = M_n,$$

where M_n will be a function of λ.

For $n > 0$, let us put

$$\alpha_n = \frac{B_n}{B_{n-1}}$$

and, conversely, for $n < 0$,

$$\alpha_n = \frac{B_n}{B_{n+1}},$$

in such a manner that

$$\alpha_1 = \frac{B_1}{B_0}, \quad \alpha_2 = \frac{B_2}{B_1}, \quad \ldots, \quad \alpha_{-1} = \frac{B_{-1}}{B_0}, \quad \alpha_{-2} = \frac{B_{-2}}{B_{-1}}, \quad \ldots.$$

Assuming $n > 0$, Eqs. (17.21) will then become

$$\frac{1}{M_n} = \alpha_{n+1} + \frac{1}{\alpha_n},$$

whence

$$\alpha_n = \frac{M_n}{1 - M_n\alpha_{n+1}} = M_n\left(1 - \frac{M_nM_{n+1}}{1 - M_{n+1}\alpha_{n+2}}\right)^{-1} = \cdots.$$

We are thus led to express α_1 by the continued fraction

$$\cfrac{M_1}{1 - \cfrac{M_1M_2}{1 - \cfrac{M_2M_3}{1 - \cfrac{M_3M_4}{1 - \cdots}}}}.$$

Is this continued fraction convergent? Let P_n/Q_n be its nth convergent, so that we will have

$$P_n = P_{n-1} - P_{n-2}M_nM_{n-1}, \quad Q_n = Q_{n-1} - Q_{n-2}M_nM_{n-1} \quad (17.22)$$

and, moreover,

$$P_nQ_{n-1} - P_{n-1}Q_n = M_1^2M_2^2M_3^2 \cdots M_{n-1}^2M_n.$$

I note first that, when n increases indefinitely, M_n will tend to zero and the series

$$M_1 + M_1M_2 + M_2M_3 + M_3M_4 + \cdots \quad (17.23)$$

will be absolutely convergent (except in the case in which one of the quantities M_n is infinite, i.e., in which λ is equal to $\pm q$ to within an integer; this case must be excluded from the discussion which follows). In addition, starting from a certain order, all terms of this series will be positive.

I now state that P_n tends to a finite limit and that this will be the same for Q_n.

In fact, P_n and Q_n are defined by the recurrence equations (17.22). Let us determine, by the same equations, two quantities R_n and R'_n such that

$$R_n = R_{n-1} - R_{n-2}M_nM_{n-1},$$
$$R'_n = R'_{n-1} - R'_{n-2}M_nM_{n-1}.$$

We can arbitrarily determine any two of the quantities R_n and also any two of the quantities R'_n.

Let us consider, in series (17.23) the first p terms that follow the nth term:

$$M_nM_{n+1} + M_{n+1}M_{n+2} + \cdots + M_{n+p-1}M_{n+p}.$$

Let S_{np} be the sum of these p terms; we can always choose n sufficiently large to have S_{np} be positive and smaller than one.

Let us next consider the recurrence formula

$$R_{n+p} = R_{n+p-1} - R_{n+p-2}M_{n+p-1}M_{n+p}.$$

This equation demonstrates that, if we have

$$1 > R_{n+p-1} > (1 - S_{np-1}),$$
$$1 > R_{n+p-2} < (1 - S_{np-2}) > 0,$$

we will also have

$$1 > R_{n+p} > (1 - S_{np}). \quad (17.24)$$

Consequently, it is sufficient to choose R_{n+1} and R_{n+2} such as to satisfy the inequality (17.24), to have all terms R_{n+p} satisfy it. Thus R_{n+p} is always larger than $1 - S_{np}$ and therefore positive. In addition, the recurrence equation shows that R_{n+p} constantly decreases with increasing index $n+p$. Therefore, R_{n+p} tends to a finite and determined limit. For this reason, let us choose R_{n+1} and R_{n+2}, R'_{n+1} and R'_{n+2} such as to satisfy the inequality (17.24) and such that the determinant

$$R_{n+1}R'_{n+2} - R_{n+2}R'_{n+1}$$

will not be zero

Then, R_{n+p} and R'_{n+p} will tend to two finite limits, determined and different from zero, namely, R and R'.

Since P_n and Q_n satisfy the same recurrence relations as R_n and R'_n and since these relations are linear, we will have

$$P_n = \mu R_n + \mu' R'_n,$$

$$Q_n = \mu_1 R_n + \mu'_1 R'_n,$$

where μ, μ', μ_1, and μ'_1 are constant coefficients and where the limit of our continued fraction will be

$$\frac{\mu R + \mu' R'}{\mu_1 R + \mu'_1 R'}.$$

For certain values of q_1, q, and λ and thus also for certain values of the coefficients μ, it may happen that this fraction will be zero or infinite; however, it will never occur in the indeterminate form 0/0.

For the case in which $q = \pm(\lambda + 2n)$ and in which consequently $M_n = \infty$, almost nothing need be changed in the above statements. For example, if we have $M_2 = \infty$, our continued fraction would become

$$\cfrac{M_1}{1 + \cfrac{M_1/M_3}{1 - \cfrac{M_3 M_4}{1 - \cdots}}}.$$

Since the limit of our continued fraction is a function of λ, we can denote it by $\psi(\lambda)$ and write

$$\alpha_1 = \psi(\lambda).$$

Similarly, we would find

$$\alpha_{-1} = \psi(-\lambda), \quad \alpha_2 = \psi(\lambda+2), \quad \alpha_3 = \psi(\lambda+4),$$
$$\alpha_n = \psi(\lambda + 2n - 2), \quad \alpha_{-2} = \psi(2-\lambda), \quad \alpha_{-n} = \psi(2n - 2 - \lambda),$$

which demonstrates the characteristic property of the function $\psi(\lambda)$, namely, that

$$\psi(\lambda) + \frac{1}{\psi(\lambda-2)} = \frac{2(q^2 - \lambda^2)}{q_1}.$$

After having calculated $\psi(\lambda)$ and $\psi(-\lambda)$, it is easy to calculate all ratios α_n and α_{-n}. If we then had the value of B_0, it would be easy to deduce from it the value of all coefficients B_n. However, it is obvious that B_0 will satisfy the equation

$$B_0(q^2-\lambda^2)=\frac{B_0 q_1}{2}[\psi(\lambda)+\psi(-\lambda)]+\beta,$$

which determines B_0.

For $\lambda=h$, the coefficient B_n must reduce to A_n and β must reduce to zero. From this, we obtain the following equation which determines h:

$$q^2-h^2=\frac{q_1}{2}[\psi(h)+\psi(-h)].$$

Once h is determined (which is generally easier by using one of the methods discussed above), the values of α_n and $A_n=B_n$ can be calculated as we have just described.

Hill's Method

185. Let us return to Eqs. (17.18), (17.19), (17.20), (17.21), and (17.21a) of the foregoing number. These equations are linear and, although they are infinite in number, Hill has had the boldness to treat them by the conventional processes of solving linear equations of finite number, i.e., by determinants.

Was this unorthodox behavior justified? I tried to prove this in a discussion published earlier[13] whose main results I will recall here.

Let us consider a matrix of indefinite size:

$$\begin{array}{ccccccc}
1 & a_{2\,1} & a_{3\,1} & a_{4\,1} & \cdots & a_{n\,1} & \cdots, \\
a_{1\,2} & 1 & a_{3\,2} & a_{4\,2} & \cdots & a_{n\,2} & \cdots, \\
a_{1\,3} & a_{2\,3} & 1 & a_{4\,3} & \cdots & a_{n\,3} & \cdots, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\
a_{1\,n} & a_{2\,n} & \cdots & \cdots & a_{n-1\,n} & 1 & a_{n+1\,n}, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots .
\end{array} \qquad (17.25)$$

In this matrix, the terms of the main diagonal are all equal to one.

Let Δ_n be the determinant formed by taking the first n rows and the first n columns of the matrix (17.25). I will say that the matrix (17.25) is a determinant of infinite order, and that this determinant converges if Δ_n tends to a finite and determined limit Δ when n increases indefinitely.

To find the conditions of convergence of a determinant, let us use the following mode of generation, which is nothing else but the operation known as "algebraic keys."

Consider the expansion of the determinant

$$D = \begin{vmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n\,1} & a_{n\,2} & \cdots & a_{n\,n} \end{vmatrix}.$$

Let us expand the product

$$\prod_p \Big(\sum_n a_{p\,n} \Big),$$

and then let us attach to each term of the expanded product, depending on the individual case, one of the coefficients $+1$, -1, or 0; this will yield D.

It is easy to derive from this the following inequality: On forming the product

$$\Pi = \prod_p \Big(\sum_n |a_{p\,n}| \Big),$$

we obtain

$$|D| < \Pi. \tag{17.26}$$

Let us now assume that, in the determinant D, a certain number of elements is replaced by zero so that the determinant D will become D' and Π will become Π'. A certain number of terms will vanish in the expansion of Π, and the corresponding terms will then also vanish in the expansion of D. We will then have

$$|D - D'| < \Pi - \Pi'. \tag{17.27}$$

These are the two very simple inequalities which will serve us as a starting point.

In order that the determinant Δ of infinite order converges, it is sufficient that the corresponding product Π, which is written as

$$\begin{gathered} (1 + |a_{2\,1}| + |a_{3\,1}| + \cdots + |a_{n\,1}| + \cdots) \\ \times (1 + |a_{1\,2}| + |a_{3\,2}| + \cdots + |a_{n\,2}| + \cdots) \\ (1 + |a_{1\,3}| + a_{2\,3}| + \cdots) \ldots , \end{gathered} \tag{17.28}$$

also converges or, according to a well-known theorem, that the series

$$\begin{gathered} |a_{2\,1}| + |a_{3\,1}| + |a_{4\,1}| + \cdots + |a_{n\,1}| \\ + \cdots + |a_{1\,2}| + |a_{3\,2}| + \cdots + |a_{1\,3}| + \cdots \end{gathered}$$

itself converges.

In fact, let Δ_n and Δ_{n+p} be the determinants obtained by taking, in the matrix (17.15), the n first rows and then the $n+p$ first rows and columns. Let Π_n and Π_{n+p} be the corresponding values of the above-defined product Π.

Since, in the matrix (17.15), the terms of the main diagonal are equal to one, we will pass from Δ_{n+p} to Δ_n by canceling a certain number of elements of this determinant Δ_{n+p}. This will yield

$$|\Delta_{n+p} - \Delta_n| < \Pi_{n+p} - \Pi_n.$$

However, if the product (17.28) converges, then the right-hand side of this inequality will tend to zero as soon as n and p increase indefinitely. This must then also be the case for the left-hand side, which proves that Δ_n tends to a finite and determined limit.

Thus, if the determinant Δ is to converge, it is sufficient that the series obtained by taking all elements that do not belong to the main diagonal of the determinant be absolutely convergent.

We will now demonstrate that the determinant converges absolutely, i.e., that the order of the columns or rows can be modified without changing the limiting value of the determinant.

Let there be two matrices analogous to the matrix (17.25), differing only by the order of the columns and rows. However, it will be assumed that in both of the matrices, the elements equal to unity will occupy the main diagonal. Let Δ_n be the determinant obtained by taking the n first rows and columns of the first table. Let Δ'_p be the determinant obtained by taking the p first rows and columns of the second table, where p is sufficiently large for having all elements of Δ_n be located in Δ'_p. Let Π_n and Π'_p be the products Π corresponding to Δ_n and Δ'_p. One can then change from Δ'_p to Δ_n by canceling a certain number of elements in Δ'_p. Thus we can write

$$|\Delta'_p - \Delta_n| < \Pi'_p - \Pi_n.$$

However, since the product (17.28) is absolutely convergent, we will have

$$\lim \Pi'_p = \lim \Pi_n \quad (n, p = \infty).$$

Thus we also will have

$$\lim \Delta'_p = \lim \Delta_n.$$

Q.E.D.

Let us now imagine that the matrix (17.25) is indefinite in both directions so that the columns and rows are numbered from $-\infty$ to $+\infty$.

The term belonging simultaneously to the nth row and to the pth column will be denoted by a_{np}. Furthermore, n and p can take all positive or negative integral values, including the value of zero.

We will denote by Δ_n the determinant formed by taking the $2n+1$ rows numbered $-n, -n+1, -n+2, \ldots, -1, 0, 1, 2, \ldots, n-1, n$ and the $2n+1$ columns bearing the same numerals. The determinant of infinite order will converge if Δ_n tends to a finite and determined limit.

We will still assume that the terms of the main diagonal are equal to one, i.e., that $a_{nn} = 1$.

Then, using the same method of reasoning as above, we will find that the determinant converges absolutely provided that the series

$$\sum |a_{np}| \quad (n \gtrless p;\ n, p \text{ varying from } -\infty + \infty)$$

is convergent.

Let us now assume that, in our matrix, i.e., according to the above definition in our determinant of infinite order, all elements of a certain row are replaced by a sequence of quantities

$$\ldots,\quad x_{-n},\quad \ldots,\quad x_{-1},\quad x_0,\quad x_1,\quad x_2,\quad \ldots,\quad x_n,\quad \ldots,$$

which all are smaller in absolute value than a certain positive number k. We state that the determinant will remain convergent if the series

$$\sum |a_{np}| \quad (n \gtrless p)$$

converges.

In fact, as mentioned above, let us take $2n+1$ rows and $2n+1$ columns of the matrix, so as to form the determinant Δ_n. Let us assume that the sum of the absolute values of the elements in each row be formed, except for the row whose elements had been replaced by the quantities x. Let us then form the product Π_n of the $2n$ sums obtained in this manner. Any term of the determinant Δ_n will be a term of the product Π_n, multiplied by one of the quantities x or by this quantity with a changed sign. Consequently, according to the hypothesis

$$|x_i| < k,$$

we must have

$$|\Delta_n| < k\Pi_n.$$

On canceling some of the elements of Δ_n, this determinant will become Δ_n' and the product Π_n will become Π_n'. Some of the terms of the product Π_n will vanish and the corresponding terms of Δ_n will also vanish. Thus we have

$$|\Delta_n' - \Delta_n| < k(\Pi_n - \Pi_n').$$

Let us note now that, for passing from the determinant Δ_{n+p} to the determinant Δ_n, it is sufficient to cancel there certain elements. We will find

$$|\Delta_{n+p} - \Delta_n| < k(\Pi_{n+p}\Pi_n)$$

and, as above, we will derive from this that Δ_n tends to a finite and determined limit provided that this is also so for Π_n, namely, precisely when the series

$$\sum |a_{np}| \quad (n \gtrless p)$$

converges.

186. Let us apply these principles to the particular case treated by Hill in his paper on the motion of the perigee of the moon.[14]

Let us return to Eqs. (17.19) of no. 184:

$$A_n[q^2 - (h+2n)^2] = \frac{q_1}{2}(A_{n-1} + A_{n+1})$$

We have an infinity of linear equations with an infinity of unknowns. To justify their treatment by the conventional rules of calculus and to calculate their determinant, we will see to it first that the main diagonal has all its elements equal to one; consequently, we will write this equation in the form

$$A_n - \frac{q_1}{2[q^2 - (h+2n)^2]}(A_{n-1} + A_{n+1}) = 0. \qquad (17.19a)$$

Again denoting by a_{np} the element of the determinant belonging to the nth row and to the pth column, we will have

$$a_{nn} = 1; \quad a_{n\,n-1} = a_{n\,n+1} = -\frac{q_1}{2[q^2 - (h+2n)^2]};$$

$$a_{np} = 0 \quad \begin{pmatrix} p < n-1 \\ \text{or } p > n+1 \end{pmatrix}.$$

To have this determinant converge, it is thus sufficient that the series

$$\sum_{n=-\infty}^{n=+\infty} \left| \frac{q_1}{q^2 - (h+2n)^2} \right|$$

converges, a condition that is obviously fulfilled.

Evidently, this determinant is a function of h which, in agreement with Hill, I will call $\square(h)$.

Then, h will be determined by the equation

$$\square(h) = 0, \qquad (17.29)$$

to which we will return later.

Let us next suppose that we replace the elements of the zeroth row in this determinant by the unknowns x, so that we replace

$$\ldots, \quad a_{0\,-p} = 0, \quad \ldots, \quad a_{0\,-1} = \frac{-q_1}{2(q^2 - h^2)}, \quad a_{00} = 1,$$

$$a_{01} = \frac{-q_1}{2(q^2 - h^2)}, \quad \ldots, \quad a_{0p} = 0, \quad \ldots,$$

respectively, by

$$\ldots, \quad x_{-p}, \quad \ldots, \quad x_{-1}, \quad x_0, \quad x_1, \quad \ldots, \quad x_p, \quad \ldots .$$

According to the above statements, the determinant Δ obtained in this manner will still converge, provided that the quantities $|x|$ are smaller than a given number k. This will be a linear function of x and can be written in the form

$$\Delta = \cdots + A_{-p}x_{-p} + \cdots + A_0 x_0 + \cdots + A_p x_p + \cdots .$$

In addition, A_p is obviously obtained by assigning the value 1 to x_p and the value 0 to the other unknowns x.

We state that the quantities A_n, defined in this manner, satisfy Eqs. (17.19). In fact, if we assign a value of a_{np} to x_p, i.e., a value of

$$0, \quad \frac{-q_1}{2[q^2 - (h + 2n)^2]} \quad \text{or} \quad 1,$$

depending on whether

$$|n - p| > 1, \quad |n - p| = 1 \quad \text{or} \quad n = p,$$

then our determinant will become

$$A_n - \frac{q_1}{2[q^2 - (h + 2n)^2]}(A_n + A_{n+1})$$

which must be zero since it has two identical rows. Consequently, Eq. (17.19) will be satisfied.

There exists an exception for $n = 0$, since the determinant then no longer has two identical rows but still is zero because of the fact that it reduces to $\square(h)$ which is zero in virtue of Eq. (17.29).

Finally, the series

$$\sum A_p e^{(2p+h)it}$$

converges since this is obtained by setting, in Δ,

$$x_p = e^{(2p+h)it},$$

The absolute value of x_p is then bounded, which, as demonstrated above, is a sufficient condition for convergence of Δ.

Application of the Hadamard Theorem

187. This leaves Eq. (17.29) to be studied:

$$\square(h)=0$$

For this, we first have to define the determinant, Hill has called $\nabla(h)$.

Let us return to our determinant $\square(h)$ and let us multiply the zeroth row by

$$q^2-h^2,$$

and the $n(n\gtrless 0)$th row by

$$\frac{q^2-(h+2n)^2}{4n^2}.$$

We state that the resulting determinant $\nabla(h)$ will still be convergent. In fact, recalling the above definition of the limit of a determinant indefinite in both directions, we will find that

$$\nabla(h)=\square(h)(q^2-h^2)\Pi,$$

where Π is the limit toward which tends the product of the $2m$ factors:

$$\frac{q^2-(h+2n)^2}{4n^2},$$

where $n=\pm 1, \pm 2, \ldots, \pm m$ when m increases indefinitely. Consequently, Π is the limit of the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{h^2-q^2}{2n^2}-\frac{h^2}{n^2}+\frac{(h^2-q^2)^2}{16n^4}\right),$$

which obviously is convergent. Thus $\nabla(h)$ will converge.

Let us denote by b_{np} the element of this determinant that belongs to the nth row and to the pth column. This yields

$$b_{00}=q^2-h^2,\quad b_{nn}=\frac{q^2-(h+2n)^2}{4n^2}\quad (n\gtrless 0),$$

$$b_{01}=b_{0\,-1}=-\frac{q_1}{2},\quad b_{n\,n+1}=b_{n\,n-1}=-\frac{q_1}{8n^2}\quad (n\gtrless 0),$$

$$b_{np}=0\quad (|n-p|>1).$$

We will then replace, in $\nabla(h)$, the quantity h by x and study the properties of the function $\nabla(x)$ defined in this manner.

We state primarily that this is an integral function.

In fact, we obviously will have, on replacing h by x,

$$|b_{00}|<q^2+x^2,\quad |b_{nn}|<\frac{q^2+(x+2n)^2}{4n^2}.$$

Consequently, according to the inequality (17.23) of no. 185, we will have

$$\nabla(x) < (q^2 + x^2 + |q_1|) \prod \left(\frac{q^2 + |q_1| + (x + 2n)^2}{4n^2} \right). \quad (17.30)$$

However, putting, for abbreviation, $q^2 + |q_1| = \lambda$ and combining all product factors corresponding to values of n equal but of opposite sign, we can write this infinite product as

$$\prod \left(1 + \frac{x^2 - \lambda^2}{2n^2} - \frac{x^2}{n^2} + \frac{(x^2 - \lambda^2)^2}{16n^4} \right),$$

which evidently is convergent and still finite. Consquently, this must be the case also for $\nabla(x)$.

In this demonstration, we supposed x to be real. However, if x were imaginary, no essential changes would be required here and it would be sufficient to write

$$|q^2| + |q_1| + |x + 2n|^2,$$

instead of

$$q^2 + |q_1| + (x + 2n)^2.$$

Thus $\nabla(x)$ is still finite irrespective of the imaginary value of x; consequently, this represents an integral function.

If one would wish to demonstrate in detail that $\nabla(x)$ also has the other characters of an entire function, i.e., be continuous and have a derivative, it would be sufficient to note that the determinant whose limit is $\nabla(x)$ converges uniformly.

Let us call $\nabla_n(x)$ the determinant formed by taking, in $\nabla(x)$, the $2n + 1$ rows and the $2n + 1$ columns labeled from $-n$ to $+n$. We then have

$$\nabla(x) = \lim \nabla_n(x).$$

In the x-plane let there be some closed contour C. Let z be a point on this contour and let x be a point interior to this contour. Since $\nabla_n(x)$ is a polynomial, we obviously will have

$$2i\pi\nabla_n(x) = \int \frac{\nabla_n(z)dz}{z - x},$$

where the integral is, of course, taken along the contour C. The function

$$2i\pi\varphi(x) = \int \frac{\nabla(z)dz}{z - x}$$

obviously will be a holomorphic function of x; we state that $\varphi(x)$ is equal to $\nabla(x)$.

In fact, since it follows from the above demonstration that the convergence of $\nabla(x)$ is uniform, we can take n sufficiently large that

$$|\nabla(x) - \nabla_n(x)| < \epsilon, \quad |\nabla(z) - \nabla_n(z)| < \epsilon$$

at the point x and over the entire contour C. Consequently,

$$2i\pi|\varphi(x) - \nabla_n(x)| < \epsilon l,$$

where l is the length of the contour C divided by the minimum of $|z - x|$.

Thus the differences $|\varphi(x) - \nabla_n(x)|$ and $|\nabla(x) - \nabla_n(x)|$ can be made as small as desired, which can take place only if $\nabla(x) = \varphi(x)$.

Consequently, $\nabla(x)$ is holomorphic. Q.E.D.

We now state that $\nabla(x)$ is periodic.

Let us designate by $E_n(x)$ the finite determinant obtained by taking in $\nabla(x)$ the $2n+1$ rows and the $2n+1$ columns labeled $-n+1$ to $n+1$ and let us denote by $E'_n(x)$ the determinant obtained by taking the corresponding rows and columns in $\Box(x)$.

Proof of the convergence of a determinant, indefinite in both directions, has been given in no. 185 when the principal diagonal has all its elements equal to unity. This proof does not assume that one is restricted to taking as many rows whose numbers are negative as rows whose numbers are positive. Thus we have

$$\lim E'_n(x) = \Box(x) \quad \text{for} \quad n = \infty.$$

Moreover, it is clear that

$$E_n(x) = E'_n(x)(q^2 - x^2)\Pi',$$

where Π' is the product of the factors

$$q^2 - \frac{(x+2m)^2}{4m^2}, \tag{17.31}$$

where m takes the values $\pm 1, \pm 2, \ldots, \pm(n-1), n$, and $n+1$.

In addition, we also have

$$\nabla_n(x+2) = \left(\frac{n+1}{n}\right)^2 E_n(x),$$

which is immediately obvious when comparing the determinants.

We will let n tend to infinity so that the left-hand side will tend to $\nabla(x+2)$. As to the right-hand side, it will tend to

$$\Box(x)(q^2 - x^2)\lim \Pi'.$$

We found above that

$$\nabla(x) = \Box(x)(q^2 - x^2)\lim \Pi,$$

where Π is the product of the factors (17.31) in which m is assigned the values $\pm 1, \pm 2, \ldots, \pm n$.

We thus have

$$\frac{\Pi'}{\Pi} = \frac{q^2 - \dfrac{(x+2n+2)^2}{4(n+1)^2}}{q^2 - \dfrac{(x-2n)^2}{4n^2}},$$

whence

$$\lim \frac{\Pi'}{\Pi} = 1,$$

from which we finally obtain

$$\nabla(x+2) = \nabla(x). \qquad \text{Q.E.D.}$$

Furthermore, we have

$$\nabla_n(-x) = \nabla_n(x)$$

and, consequently,

$$\nabla(x) = \nabla(-x).$$

Continuing our investigation of the entire function $\nabla(x)$, we propose to prove that it is of genus zero when considering it as a function of x. It is known that an entire function is designated as being of genus zero when it can be expanded in an infinite product of the form

$$A\left(1 - \frac{x}{b_1}\right)\left(1 - \frac{x}{b_2}\right)\cdots\left(1 - \frac{x}{b_n}\right)\cdots.$$

More generally, it is said that an integral function is of genus p when it is expandable in a product of an infinite number of primary factors of the form

$$A\left(1 - \frac{x}{b}\right)e^{P},$$

where P is a polynomial of the order p in x.

To prove this essential point, we must use certain inequalities, which I will first establish.

Let us look for an upper limit of

$$|\nabla(y+ix)|.$$

Since the function ∇ is periodic, with period 2, we can always assume that y lies between -1 and $+1$. We then will have

$$|q^2 + q_1 + (y + ix - 2n)^2| < q^2 + |q_1| + x^2 + (y-2n)^2$$

so that, formulating as above,

$$\lambda = q^2 + |q_1|,$$

which yields, making use of our fundamental inequality

$$|\nabla(y+ix)| < (\lambda + y^2 + x^2) \prod \left(\frac{\lambda + x^2 + (y-2n)^2}{4n^2}\right). \qquad (17.32)$$

The second member of this inequality is a function of x^2 which will be denoted by $F(x^2)$.

Let us put, for the moment, $x^2 = t^3$ and let us consider the function $F(t^3)$. It is easy to see that this function is of genus one.

In fact, the function $F(x)$ is of genus zero and can be brought to the form

$$F(x) = A \prod_n \left(1 - \frac{x}{b_n^3}\right).$$

We represent by b_n^3 the roots of the equation $F(x) = 0$. From this, it follows that

$$F(t^3) = A \prod_n \left(1 - \frac{t}{b_n}\right)\left(1 - \frac{\alpha t}{b_n}\right)\left(1 - \frac{\alpha^2 t}{b_n}\right)$$

or

$$F(t^3) = A \prod_n \left(1 - \frac{t}{b_n}\right) e^{t/b_n} \prod_n \left(1 - \frac{\alpha t}{b_n}\right) e^{\alpha t/b_n} \prod_n \left(1 - \frac{\alpha^2 t}{b_n}\right) e^{\alpha^2 t/b_n}.$$

It is easy to verify that the three products on the right-hand side are absolutely convergent.

I demonstrated earlier[15] that, if a function $\varphi(x)$ is of genus 1, one will have

$$\lim \varphi(x) e^{-\alpha x^2} = 0,$$

if x tends to infinity with a fixed argument in such a manner that $e^{-\alpha x^2}$ tends to zero.

Thus if α and t are real and positive, we will have

$$\lim F(t^3) e^{-\alpha t^2} = 0.$$

As y varies from -1 to $+1$, the left-hand side will tend to its limit in a uniform manner, from which this consequence is obtained: Two positive numbers α and K can be found, such that

$$|\nabla(y+ix)| < K \exp(\alpha|x^{4/3}|).$$

Setting $y + ix = z$ and noting that

$$|x| < |z|,$$

we find

$$|\nabla(z)| < K\exp(\alpha|z^{4/3}|).$$

Let us now consider the expansion of $\nabla(z)$

$$\nabla(z) = \sum C_n z^n,$$

from which we obtain

$$2i\pi C_n = \int \frac{\nabla(z)dz}{z^{n+1}},$$

where the integral is taken along a circle of some radius, having the origin as center.

From this it is concluded that

$$|C_n| < \frac{K\exp(\alpha|z^{4/3}|)}{|z^n|},$$

no matter what $|z|$ might be. However, the minimum of

$$\exp(\alpha z^{4/3})z^{-n}$$

is

$$\exp(3n/4)\left(\frac{3n}{4\alpha}\right)^{-3n/4},$$

whence

$$|C_n| < \frac{K\exp(3n/4)}{\left(\frac{3n}{4\alpha}\right)^{3n/4}}.$$

It will be noted that, since the function $\nabla(z)$ is even, the coefficients C_{2n+1} are zero.

We propose to demonstrate that ∇, considered as being a function of z^2, is of genus 0. In view of a theorem developed by Hadamard (Ref. 16)[R15] it is sufficient for this to establish that

$$|C_{2n}| < K'\Gamma(n+1)^{-\mu},$$

where μ is larger than one.

Now, we have

$$|C_{2n}|\Gamma(n+1)^{+\mu} < K\exp(3n/2)\left(\frac{3n}{2\alpha}\right)^{3n/2}\Gamma(n+1)^{+\mu}.$$

Replacing $\Gamma(n+1)$ by its approximate value, the right-hand side becomes

$$K\,e^{3n/2-n\mu}\left(\frac{3}{2\alpha}\right)^{3n/2} n^{n\mu-3n/2}(2\pi n)^{\mu/2}.$$

It is now a question of demonstrating, for a value of $\mu > 1$, that this expression remains bounded. Now, if it

$$\mu < \tfrac{3}{2},$$

it will tend to zero when n increases indefinitely.

Therefore, it is sufficient to take

$$1 < \mu < \tfrac{3}{2}.$$

From this it results that the function $\nabla(z)$ can be expanded in a product of the form

$$\nabla(z) = A\Pi\left(1 - \frac{z^2}{b_n}\right). \tag{17.33}$$

All that remains is to define the zeros of the function $\nabla(x)$; according to the very nature of the problem, these zeros will be

$$x = \pm (h + 2n),$$

where n is an integer. In fact, it is for these values and only for these values that Eqs. (17.19) and (17.19a) of no. 186 and thus also Eq. (17.20) of no. 186 can be satisfied.

Consequently, the zeros of $\nabla(x)$ will be the same as those of

$$\cos \pi x - \cos \pi h.$$

Since these two functions are both expandable in infinite products of the form (17.33) and since the factors of these two products, except for the constant A, are the same, the two functions can differ only by a constant factor, so that we will have

$$\nabla(x) = A(\cos \pi x - \cos \pi h). \tag{17.34}$$

However, $\nabla(x)$ is not only a function of x but also an integral function of q^2 and of q_1; it is possible to demonstrate in exactly the same manner that this is a function of degree 0 with respect to q^2 as well as with respect to q_1.

However, there is more to this. For example, let

$$\nabla(x) = \sum D_n q_1^n,$$

$$A = \sum D'_n q_1^n,$$

$$A \cos \pi h = \sum D''_n q_1^n,$$

one finds

$$D_n = D'_n \cos \pi x - D''_n.$$

It can be demonstrated, in exactly the same manner as above, that

$$|D_n| < K'\Gamma(n)^{-\mu},$$

if μ is taken between 1 and $\frac{3}{2}$. Since this inequality must hold irrespective of the value of x, the quantity D'_n of necessity must satisfy an inequality of the same form, so that A will be a function of genus 0 with respect to q_1; in the same manner, it can be demonstrated that it is also a function of genus 0 with respect to q^2.

Now, A can never vanish since $\nabla(x)$ never vanishes identically (no matter what x might be). However, a function of genus 0 which does not vanish is reduced to a constant.

Consequently, A is independent of both q^2 and q_1.

Let us write the equality (17.34) to show the value of q_1, in the form of

$$\nabla(x,q_1) = A(\cos \pi x - \cos \pi h).$$

For $q_1 = 0$, h is equal to q. This yields

$$\nabla(x,0) = A(\cos \pi x - \cos \pi q),$$

whence, on dividing and setting $x = 0$,

$$\frac{1 - \cos \pi h}{1 - \cos \pi q} = \frac{\nabla(0,q_1)}{\nabla(0,0)} = \frac{\Box(0,q)}{\Box(0,0)}$$

or, finally,

$$1 - \cos \pi h = \Box(0,q_1)(1 - \cos \pi q), \tag{17.35}$$

since

$$\Box(0,0) = 1.$$

It is from the equality (17.35) that Hill derived the value of h.

In view of the above considerations, the legitimacy of his method can be considered as rigorously established.

Miscellaneous Remarks

188. In the particular case treated here, some of these results could be obtained also without using Hadamard's theorem.

In fact, let us note first that, although q^2 and q_1 are imaginary, the fundamental equality (17.32) still holds provided that

$$\lambda = |q^2| + |q_1|.$$

If we then recall the known expansion

$$\sin \pi x = \pi x \prod_n \left(1 - \frac{x}{n}\right),$$

we can derive from this

$$\cos \pi x - \cos \pi y = \frac{\pi^2}{2}(y^2 - x^2) \prod_n \left(\frac{(y-2n)^2 - x^2}{4n^2} \right),$$

whence

$$F(x^2) = \frac{2}{\pi^2}(\cos \pi\sqrt{-x^2-\lambda} - \cos \pi y).$$

The inequality (17.32) is valid irrespective of the value of q^2 and q_1, provided that x and y are real.

It is known that the ratio

$$\frac{\cos ix}{e^{|x|}}$$

tends to $\frac{1}{2}$ when x increases indefinitely through real values. It follows from this that a numerical constant B can be found such that

$$F(x^2) < Be^{\pi\sqrt{x^2+\lambda}};$$

from which we deduce

$$|\nabla(y + ix)| < Be^{\pi\sqrt{x^2+L}},$$

whence

$$|\nabla(z)| < Be^{\pi\sqrt{|z^2|+\lambda}} < Be^{\pi\sqrt{\lambda}} e^{\pi|z|}.$$

Let us now consider the ratio

$$\frac{\nabla(z)}{\cos \pi z - \cos \pi h}.$$

The numerator vanishes each time the denominator vanishes, from which it follows that this ratio is an entire function in z as well as in q^2 and in q_1

Since this ratio is a periodic function of z, we can always assume that the real part of z remains between -1 and $+1$. Thus let the imaginary part tend to infinity and let us define the behavior of our ratio

$$\frac{V(z)}{\cos \pi z - \cos \pi h} = \frac{\nabla(z)}{e^{\pi|z|}} : \frac{\cos \pi z - \cos \pi h}{e^{|\pi z|}}.$$

The first factor on the right-hand side, in absolute value, remains below $Be^{\pi\sqrt{\lambda}}$. The second factor tends to $\frac{1}{2}$. Thus, our ratio remains finite. Consequently, this is an entire function of z remaining constantly below a certain limit. This function, according to a known theorem, must reduce to a constant independent of z.

One still has to return to Hadamard's theorem for demonstrating that this fuction is also independent of q^2 and of q_1.

Extension of the Preceding Results

189. All the above methods, except that of Gyldén, can be applied to any equation of the form

$$\frac{d^2x}{dt^2} + x\varphi(t) = 0, \tag{17.36}$$

where $\varphi(t)$ is a periodic function of t, which consequently can be expanded in a trigonometric series.

There are only minor changes to be made, which the reader can readily do himself if he wishes to treat an equation of the form of Eq. (17.36) in this manner. Most of the results still hold, except for a few that are valid only if the function φ is even.

In the attempt to apply his method to Eq. (17.36), Gyldén conceived an ingenious trial and error method which need not be discussed here since it is used only in extremely rare cases.

Let us now assume that the function $\varphi(t)$ is not periodic but has the form

$$\varphi(t) = 1 + \mu\psi(t),$$

where μ is a very small numerical coefficient and $\psi(t)$ is the sum of n terms of the form

$$A_i \sin(\alpha_i t + \beta_i),$$

so that

$$\psi(t) = \sum_{i=1}^{i=n} A_i \sin(\alpha_i t + \beta_i).$$

Here, A_i, α_i, and β_i are constants; however, the terms α_i are not mutually commensurable unless the function φ is periodic.

In this case, the above procedures are still applicable, but the resultant series, which can be arranged in powers of μ, are no longer convergent, so that these methods no longer have any value other than that possessed by ordinary methods of formal calculation, according to Chap. 8.

Thus Eq. (17.36) can be formally satisfied, by setting

$$x = \sum B_m \cos(h + \gamma_m)t + \sum C_m \sin(h + \gamma_m)t. \tag{17.37}$$

In this formula, h, B_m, and C_m are series arranged in powers of μ and whose coefficients are constants. The terms γ_m are linear combinations of α_i with integral coefficients of, such that

$$\gamma_m = N_1\alpha_1 + N_2\alpha_2 + \cdots + N_n a_n$$

and the summation must be extended over all combinations of integral values of $N_1, N_2, \ldots, N_n$.

The divergence of series (17.37) may cause surprise. Let us assume that the terms α_i have the form

$$\alpha_i = m_i\lambda + m_i',$$

where m_i and m_i' are integers and λ is a constant which is the same for all the α_i.

Let us vary λ, while keeping constant the values of μ, A_i, β_i, m_i, and m_i'.

For all commensurable values of λ, the terms α_i are mutually commensurable and the function φ will be periodic. From no. 29, we then know that Eq. (17.36) admits of a solution of the form (17.37) and that, in addition, this solution is not purely formal and the series are convergent.

Since an infinity of numbers exists in any interval, it is surprising that the resultant series, when λ varies in as small an interval as one pleases, can be an infinite number of times convergent and an infinite number of times divergent.

This paradoxical fact can be better understood if one studies the simple example given below.

Consider the equation of the first order

$$\frac{dx}{dt} = x\varphi . \tag{17.38}$$

We will assume that φ is a series of the form

$$\varphi = \sum A_{m\,n}\mu^{|m| + |n|} \cos(m\lambda - n)t .$$

Here, m and n take all possible integral values, λ is a constant, the terms $A_{m\,n}$ are constant coefficients, and μ is a very small parameter in whose powers we will expand.

An integration then yields

$$\log x = A_{00}t + \sum \frac{A_{m\,n}\mu^{|m| + |n|}}{m\lambda - n} \sin(m\lambda - n)t , \tag{17.39}$$

This solution must be modified when λ is rational. Let $\lambda = p/q$, in which p and q are relatively prime; then, $m\lambda - n$ will be zero when we have

$$m = hq, \quad n = hp ,$$

where h is an integer.

This yields

$$\log x = Bt + \sum_{\infty} \frac{A_{m\,n}\mu^{|m| + |n|}}{m\lambda - n} \sin(m\lambda - n)t , \tag{17.40}$$

where, under the summation sign Σ, only values are assigned to m and n that do not cause $m\lambda - n$ to vanish and where

$$B = \sum_{h=-\infty}^{h=+\infty} A_{hq\,hp}\mu^{|hq|+|hp|} .$$

If, in Eqs. (17.38) and (17.39), we change from logarithms to numbers, it will be found in each of the cases that

$$x = e^{Ct}\psi(t) ,$$

where $\psi(t)$ is a series in powers of μ whose coefficients are formed from a finite number of terms in

$$\sin(m\lambda - n)t$$

or

$$\cos(m\lambda - n)t.$$

However, two differences exist between the two cases:

(i) If λ is rational, the series $\psi(t)$ is convergent; if, conversely, λ is irrational, the series $\psi(t)$ can be divergent and the solution will become purely formal.

(ii) The value of the exponent C is not the same in the two cases. If λ is incommensurable, C will be equal to $A_{0.0}$ while, if λ is rational, C will be equal to B. Thus C is not a continuous function of λ.

This must be the same for h in the case of Eq. (17.36); this yields an explanation for the fact that, for this type of question, one cannot reason by continuity.

CHAPTER 18

Case of Nonlinear Equations

Inhomogeneous Equations

190. We have seen in no. 177 that Eq. (16.6c) of no. 169, by a suitable change of variables, could be reduced to the form

$$\frac{d^2x}{dt^2} + x(q^2 - q_1 \cos 2t) = \varphi(t) . \tag{18.1}$$

In this expression, $\varphi(t)$ is a known function of t and the expression itself is a sum of terms of the form

$$\beta \cos \lambda t \quad \text{or} \quad \beta \sin \lambda t .$$

In the preceding chapter, we have shown how to integrate the equation without a right-hand side, i.e., Eq. (18.1) where we set $\varphi(t) = 0$; we know, on the other hand, that the integration of a linear inhomogeneous equation can always be reduced to that of a homogeneous equation.

This settles the question. In no. 184, we even considered Eq. (18.1) by setting there

$$\varphi(t) = \beta \cos \lambda t ,$$

and we showed how to satisfy this equation by setting

$$x = \sum B_n \cos(\lambda + 2n)t , \tag{18.2}$$

and we also proved that the terms B_n were defined by relations (17.21) and (17.21a) of no. 184.

Similarly, if we set

$$\varphi(t) = \beta \sin \lambda t ,$$

Eq. (18.1) will be satisfied by setting

$$x = \sum B_n \sin(\lambda + 2n)t , \tag{18.2a}$$

provided that the quantities B_n are still defined by the relations (17.21) and (17.21a).

It is obvious that, if $\varphi(t)$ is a sum of terms of the form $\beta \cos \lambda t$ and

$\beta \sin \lambda t$, a particular solution of Eq. (18.1) will be obtained which again will be a sum of terms of the form

$$B_n \cos(\lambda + 2n)t \quad \text{or} \quad B_n \sin(\lambda + 2n)t\,,$$

which will yield the general solution by adding, to this particular solution, the general solution of the equation without second member.

An exception is the case in which one of the coefficients B_n, defined by Eqs. (17.21) and (17.21a) of no. 184, would be infinite. This is what happens, as is easy to see, if λ is equal to $h + 2n$ where n is an integer.

In this case, it is still possible to integrate Eq. (18.1) but the time t is removed from under the sine and cosine signs so that the solution no longer retains its purely trigonometric form.

For example, assuming that

$$\varphi(t) = \beta \cos ht\,,$$

the general solution will have the form

$$x = t \sum A_n \sin(h + 2n)t + \sum (B_n + C_1 A_n)\cos(h + 2n)t$$

$$+ C_2 \sum A_n \sin(h + 2n)t\,.$$

Thus the necessary and sufficient condition for having the solution retain its trigonometric form is that none of the λ corresponding to the various terms of $\varphi(t)$ be equal to $h + 2n$.

If now λ were very close to $h + 2n$, without being rigorously equal to $h + 2n$, then one of the coefficients B_n, without being infinite, would become very large.

This would constitute no drawback if Eq. (18.1), i.e., Eq. (16.6c) of no. 169, were rigorously exact. However, this is not the case: The equation is only approximate, as demonstrated in Chap. 16; to have it become sufficiently approximate, it would be necessary that ρ—denoted here by x—always remain very small.

Thus if one of the coefficients B_n were very large, x would not remain very small. The neglected terms could become so large as to render the method of approximation illusory.

Therefore, in the sequence of approximations, one must take care that no term whose argument λt differs very little from $(h + 2n)t$ ever appears on the right-hand side of Eq. (18.1).

In a more general manner, let us consider the equation

$$\frac{d^2x}{dt^2} + xf(t) = \varphi(t)\,, \tag{18.3}$$

where $f(t)$ and $\varphi(t)$ are functions of t expandable in trigonometric series.

Let $\beta \cos \lambda t$ or $\beta \sin \lambda t$ be a term of $\varphi(t)$ and let $\alpha \cos \mu t$ or $\alpha \sin \mu t$ be a term of $f(t)$.

Let us then consider the homogeneous equation:

$$\frac{d^2x}{dt^2} + xf(t) = 0 .$$

Let x_1 and x_2 be two independent solutions of this equation and let x_1' and x_2' be their derivatives with respect to t; we then have

$$x_1x_2' - x_2x_1' = C ,$$

where C is a constant which still can be assumed as equal to one.

The general solution of the inhomogeneous equation will then read:

$$x = -x_1 \int x_2\varphi(t)dt + x_2 \int x_1\varphi(t)dt . \qquad (18.4)$$

According to no. 188, x_1 and x_2 are a sum of terms of the form

$$A \begin{matrix} \sin \\ \cos \end{matrix} (h + \gamma)t ,$$

where h is a constant which is the same for all terms, while γ is a linear combination with integral coefficients of the coefficients μ.

What now is the condition for having the expression (18.4) retain its trigonometric form? It is sufficient that, with $x_1\varphi(t)$ and $x_2\varphi(t)$ being assumed as expanded in trigonometric series, no wholly known term be present; or else that the expansion of x_1 or of x_2 contain no term having the same argument as one of the terms of $\varphi(t)$; or finally, since λt is any one of the arguments of the terms $\varphi(t)$, that $\lambda - h$ be no linear combination with integral coefficients of the μ.

If, specifically, the function $f(t)$ is periodic such that

$$\mu = n\alpha ,$$

where n is an integer, then the ratio $(\lambda - h)/\alpha$ should not be an integer.

If $f(t)$ is a periodic function of two arguments αt and βt, in such a manner that

$$\mu = m\alpha + n\beta ,$$

where m and n are integers, then no relations of the form

$$\lambda - h = m\alpha + n\beta$$

should exist.

These conditions are sufficient but not necessary. If, in fact, a term of x_1 and a term of $\varphi(t)$ have the same argument, their product will give a wholly known term in the expansion of $x_1\varphi(t)$. Thus we will obtain as many wholly known terms in the product $x_1\varphi(t)$ as there are pairs of terms of the same argument contained in the two factors. However, it might happen that these terms mutually cancel.

The necessary and sufficient condition thus is that the completely known term of $x_1\varphi(t)$ and that of $x_2\varphi(t)$ be zero.

Equation of the Evection

191. Let us apply the above considerations to the integration by successive approximations of the equation

$$\frac{d^2x}{dt^2}+x(q^2-q_1\cos 2t)=\alpha\varphi(x,t)\ . \tag{18.5}$$

Here, α is a very small coefficient and $\varphi(x,t)$ is a known function of x and t, whose terms all have the form

$$Ax^p\cos\lambda t+\mu\ ,$$

where p is an integer while A, λ, and μ are arbitrary constants.

We will write this equation in the form

$$\frac{d^2x}{dt^2}+x[q^2+\beta+(-q_1+\gamma)\cos 2t\,]=\beta x+\gamma x\cos 2t+\alpha\varphi\ , \tag{18.6}$$

where β and γ are very small constants whose values will be determined below, by modifying them at each approximation.

As first approximation, we will set

$$\beta=\gamma=0\ ,\quad \varphi=\varphi(0,t)\ .$$

This yields an equation of the same form as Eq. (18.1) furnishing a first approximate value of x which will be denoted by ξ_1. The corresponding value of the number h will be denoted by h_1.

The function ξ_1 will retain its trigonometric form and will contain no secular term since, in general, none of the differences $(\lambda-h_1)/2$ will be an integer.

For the second approximation, it is necessary to write

$$\varphi=\varphi(\xi_1,t)\ .$$

However, if a value of zero is retained for β and γ, then the expansions of $x_1\varphi$ and $x_2\varphi$ will contain wholly known terms and the time, according ot our above statements, will be removed from under the trigonometric signs.

Therefore, it is necessary to assign new values β_2 and γ_2 to β and to γ, which are selected such that the general integral of the equation

$$\frac{d^2x}{dt^2} + x[q^2 + (- q_1 + \ )\cos 2t\ ]$$
$$= \beta_2\xi_1 + \gamma_2\xi_1 \cos 2t + \alpha\varphi(\xi_1,t) = \psi_1 \qquad (18.6a)$$

contains no secular terms. We know the necessary and sufficient condition for having this be the case. Let $x_1^{(1)}$ and $x_2^{(1)}$ be two independent integrals of the equation

$$\frac{d^2x}{dt^2} + x(q^2 - q_1 \cos 2t) = 0\ .$$

It is necessary that the expansions of $x_1^{(1)}\ \psi_1$ and of $x_2^{(1)}\ \psi_1$ contain no wholly known term. It is obvious that one can always choose β_2 and γ_2 to have this be the case.

After this, let us consider the equation

$$\frac{d^2x}{dt^2} + x[q^2 + \beta_2 + (- q_1 + \gamma_2)\cos 2t\ ] = 0\ .$$

Let $x_1^{(2)}$ and $x_2^{(2)}$ be two integrals of this equation and let h_2 be the corresponding value of the number h; then, $x_1^{(2)}$ and $x_2^{(2)}$ can be expanded in cosines and sines of $(h_2 + 2n)t$, where n is an integer.

Let us note now that ξ_1 contains terms of two kinds. Those of the first kind depend on the sines and cosines of

$$(h_1 + 2n)t\ ,$$

while those of the second kind depend on the sines and cosines of

$$(\lambda + 2n)t\ ,$$

where λt is one of the arguments on which φ depends.

Then, let ξ_1' be what becomes of ξ_1 on replacing there h_1 by h_2 in the terms of the first kind. Let ψ_1' be what becomes of ψ_1 on replacing there ξ_1 by ξ_1'.

Instead of the equation

$$\frac{d^2x}{dt^2} + x[q^2 + \beta_2 + (- q_1 + \gamma_2)\cos 2t\ ] = \psi_1\ ,$$

which it would be logical to use since it is obtained by setting, on both sides of Eq. (18.6),

$$\beta = \beta_2, \quad \gamma = \gamma_2$$

and, on the right-hand side,

$$x = \xi_1\ ,$$

we will consider the equation

$$\frac{d^2x}{dt^2} + x[q^2 + \beta_2 + (- q_1 + \gamma_2)\cos 2t\ ] = \psi_1'\ . \qquad (18.7)$$

In fact, h_2 differs very little from h_1 so that the difference $\psi_1 - \psi_1'$ is actually of the order of the terms neglected here. Let us consider some solution of this equation (18.7). Since $x_1^{(2)}$ and $x_2^{(2)}$ differ little from $x_1^{(1)}$ and $x_2^{(1)}$ and since ψ_1' differs little from ψ_1, the wholly known terms of

$$x_1^{(2)}\psi_1' \quad \text{and} \quad x_2^{(2)}\psi_1'$$

will also differ little from those of

$$x_1^{(1)}\psi_1 \quad \text{and} \quad x_2^{(2)}\psi_1 ,$$

which are zero; consequently, they will be very small. Therefore, in the considered solution of Eq. (18.7), the secular terms will be very small and can be neglected. I will then call ξ_2 not the solution of Eq. (18.7) itself, but rather what becomes of this solution when these secular terms are eliminated.

Let then

$$\psi_2 = \beta_3\xi_2 + \gamma_3\xi_2 \cos 2t + \alpha\varphi(\xi_2,t) .$$

We will determine β_3 and γ_3 in such a manner that the wholly known terms of

$$x_1^{(2)}\psi_2 \quad \text{and} \quad x_2^{(2)}\psi_2$$

will be zero.

Let us now form the equation

$$\frac{d^2x}{dt^2} + x[q^2 + \beta_3 + (-q_1 + \gamma_3)\cos + 2t] = 0 .$$

Let $x_1^{(3)}$ and $x_2^{(3)}$ be two solutions of this equation and let h_3 be the corresponding value of h.

Let ξ_2' be what becomes of ξ_2 on replacing there h_2 by h_3, which means that ξ_2' is derived from ξ_2 as ξ_1' is derived from ξ_1. Let ψ_2' be what becomes of ψ_2 on replacing there ξ_2 by ξ_2'.

Let us then consider the equation

$$\frac{d^2x}{dt^2} + x[q^2 + \beta_3 + (-q_1 + \gamma_3)\cos 2t] = \psi_2' , \qquad (18.8)$$

and let ξ_3 be what becomes of one of the solutions of this equation on deducting there the secular terms, in such a manner that ξ_3 is deduced from a solution of Eq. (18.8) by the same rule by which ξ_2 is deduced from a solution of Eq. (18.7).

It is easy to see that these secular terms are of the same order as those neglected in this third approximation.

Having thus defined ξ_3, one can proceed to the next approximation, applying the same rule.

A few remarks should be made here:

To form Eq. (18.7), we replaced the coefficient h_1 in ξ_1 and in ψ_1 by h_2, meaning that we replaced ξ_1 and ψ_1 by ξ'_1 and ψ'_1, which is done also in the subsequent approximations.

If we had not done this, we would have introduced a much larger number of arguments than necessary, which would have constituted a serious drawback.

In return, it seems at first glance that we could have completely avoided any secular terms; in fact,then ψ_1 would contain terms of the argument $(h_1 + 2n)t$ while $x_1^{(2)}$ and $x_2^{(2)}$ would contain terms of the argument $(h_2 + 2n)t$ such that the products $x_1^{(2)}\psi_1$, $x_2^{(2)}\psi_1$ would contain no wholly known terms but only terms in

$$\cos(h_1 - h_2)t, \quad \sin(h_1 - h_2)t .$$

However, this would be an illusion since, because of the fact that the difference $h_1 - h_2$ is very small, these terms would have a very long period. An integration would introduce very small divisors and the convergence of the approximations would become illusory.

On the other hand, it seems that the success of the method is contingent on the following circumstance: At each approximation, we must satisfy two conditions, since we must cancel the wholly known terms of

$$x_1^{(i+1)}\psi'_i, \quad x_2^{(i+1)}\psi'_i$$

and we must fix precisely the values of β_{i+1} and γ_{i+1}.

One could be tempted to believe that this was the reason for the fact that Gyldén transposed the term

$$q_1 x \cos 2t ,$$

to the left-hand side despite the smallness of the coefficient q_1, and that he merely wished to retain two terms on the left-hand side so as to have two indeterminate coefficients available.

However, this again would be in error.

The principles of Chap. 9 actually show that, even if q_1 and γ were zero, the approximations could be continued without introducing secular terms. It is true that we would then have two conditions to satisfy; however, after one would have disposed of the only remaining arbitrary coefficient such as to satisfy the first of these conditions, then the second condition—as demonstrated in no. 127—would be satisfied identically.

This will be better understood later, after we will have modified the method of successive approximations of this number, so as to give it the form shown in no. 192.

192. Let ξ_i be the value of x obtained in the ith approximation by the method used in the preceding number. This will be a sum of terms depending on the sine or the cosine of angles such as

$$\varphi = (m_1 h_i + 2m_2 + m_3\lambda_3 + m_4\lambda_4 + \cdots + m_n\lambda_n)t ,$$

where $m_1, m_2, \ldots, m_n$ are integers while h_i is the ith approximate value of the number h, while $\lambda_3 t, \lambda_4 t, \lambda_n t$ are the arguments of the various terms of $\varphi(x,t)$.

Let us put $h_i t = w_i$, so that

$$\varphi = m_1 w_i + (2m_2 + m_3\lambda_3 + \cdots + m_n\lambda_n)t .$$

Then, ξ_i can be considered as a function of two variables w_i and t. In addition, this function can be expanded in powers of the small parameter α which enters the right-hand side of Eq. (18.5). Similarly, h_i can be expanded in powers of α.

Thus the problem discussed in the preceding number can be formulated as follows: We have attempted to formally satisfy Eq. (18.5) on replacing there x by a series expandable in powers of α and in sines and cosines of multiples of

$$w, \quad 2t, \quad \lambda_3 t, \quad \lambda_4 t, \quad \ldots, \quad \lambda_n t .$$

The auxiliary variable w must itself be equal to ht, since the number h can be expanded in powers of α.

The solution of this problem can be given in a form more satisfying to the mind, by arranging the approximations as done below.

If we prove the fact that x depends on t in two manners, first directly and secondly on the basis that x is also a function of w and w is a function of t, then Eq. (18.5) can be written as

$$h^2 \frac{d^2x}{dw^2} + 2h \frac{d^2x}{dw\,dt} + \frac{d^2x}{dt^2} + x(q^2 - q_1 \cos 2t) = \alpha\varphi(x,t) . \qquad (18.9)$$

Since x must be expanded in powers of α, we will write

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots \qquad (18.10)$$

and, similarly for h,

$$h = h_0 + \alpha h_1 + \alpha^2 h_2 + \cdots . \qquad (18.11)$$

(Thus h_i no longer has the same meaning as in the preceding number.)

Let us substitute the expansions (18.10) and (18.11) into the partial differential Eq. (18.9). The two sides of this equation are then expanded in powers of α. Let us equate, on both sides of Eq. (18.9), first all terms independent of α, then all coefficients of α, and finally those of $\alpha^2, \ldots,$ which will yield a sequence of equations which we will denote by E_0, E_1,

$E_2, \ldots$, such that the equation E_i is obtained on equating the coefficients of α^i.

The equation E_0 should be used for determining h_0 and x_0, the equation E_1 for determining h_1 and $x_1, \ldots$, and finally the equation E_i for determining h_i and x_i.

To facilitate writing our equations, we will agree—as in Chap. 15—to represent any known function by Φ.

Then, E_0 can be written as

$$h_0^2 \frac{d^2x_0}{dw^2} + 2h_0 \frac{d^2x_0}{dw\,at} + \frac{d^2x_0}{dt^2} + x_0(q^2 - q_1 \cos 2t) = 0\,.$$

Similarly, E_1 (recalling that h_0 and x_0 are assumed to have been already determined by means of E_0) is written in the form

$$2h_0h_1 \frac{d^2x_0}{dw^2} + 2h_1 \frac{d^2x_0}{dw\,dt} + h_0^2 \frac{d^2x_1}{dw^2} + 2h_0 \frac{d^2x_1}{dw\,dt} + \frac{d^2x_1}{dt^2} + x_1(q^2 - q_1 \cos 2t) = \Phi\,,$$

and, in general, E_i will be written as

$$2h_0h_i \frac{d^2x_0}{dw^2} + 2h_i \frac{d^2x_0}{dw\,dt} + h_0^2 \frac{d^2x_i}{dw^2} + 2h_0 \frac{d^2x_i}{dw\,dt} + \frac{d^2x_i}{dt^2} + x_i(q^2 - q_1 \cos 2t) = \Phi\,.$$

The equation E_0 is easy to integrate; in fact, it reduces to Eq. (18.1) of no. 190 which has been the topic of Chap. 17. We will obtain an integral by setting

$$x_0 = \sum A_n \cos(w + 2nt)\,,$$

where the coefficients A_n are the same as in no. 178 while h_0 is equal to the number denoted by h in Chap. 17.

We will have a further equation of this type by setting

$$x_0 = \sum A_n \sin(w + 2nt)\,.$$

Thus if we put

$$\xi = \sum A_n \cos(w + 2nt), \quad \eta = \sum A_n \sin(w + 2nt)$$

and if β and γ are arbitrary constants, we will still have an integral by setting

$$x_0 = \beta\xi + \gamma\eta\,.$$

This, incidentally, is the only integral that is periodic in w and t.

Let us then pass to the equation E_1. If h_1 were known, one could write

$$h_0^2 \frac{d^2x_1}{dw^2} + 2h_0 \frac{d^2x_1}{dw\,dt} + \frac{d^2x_1}{dt^2} + x_1(q^2 - q_1 \cos 2t) = \Phi . \qquad (18.12)$$

How would we then integrate Eq. (18.12)?

Let us put

$$\xi' = h_0 \frac{d\xi}{dw} + \frac{d\xi}{dt} = -\sum A_n (h_0 + 2n)\sin(w + 2nt) ,$$

$$\eta' = h_0 \frac{d\eta}{dw} + \frac{d\eta}{dt} = +\sum A_n (h_0 + 2n)\cos(w + 2nt) .$$

The determinant $\xi\eta' - \xi'\eta$ will be a constant that can still be assumed as equal to one since only the ratios of the coefficients A_n are determined and since A_0 can be arbitrarily chosen.

Let us now apply the variational method of constants. If, by β and γ, we no longer denote two constants but two functions of w and of t, we can define these two functions by the equations

$$x_1 = \beta\xi + \gamma\eta ,$$

$$h_0 \frac{dx_1}{dw} + \frac{dx_1}{dt} = \beta\xi' + \gamma\eta' .$$

If, for abbreviation, we put

$$\beta' = h_0 \frac{d\beta}{dw} + \frac{d\beta}{dt} ,$$

$$\gamma' = h_0 \frac{d\gamma}{dw} + \frac{d\gamma}{dt} ,$$

then Eq. (18.12) can be replaced by

$$\beta'\xi + \gamma'\eta = 0 ,$$

$$\beta'\xi' + \gamma'\eta' = \Phi ;$$

whence

$$\beta' = -\Phi\eta , \qquad\qquad (18.13)$$
$$\gamma' = \Phi\xi .$$

These equations (18.13) are easy to integrate.

For example, let us take the second of Eq. (18.13); here $\Phi\xi$ can be expanded in a series of the form

$$\Phi\xi = B_0 + \sum B \cos(mw + \mu t + k) ; \qquad (18.14)$$

where B and k are constants and m is an integer; μ is a linear combination with integral coefficients of 2 and of λ_i. The wholly known term B_0 has thus been derived.

The equation

$$h_0 \frac{d\gamma}{dw} + \frac{d\gamma}{dt} = \Phi\xi$$

will then yield

$$\gamma = B_0 t + \sum \frac{B \sin(mw + \mu t + k)}{mh_0 + \mu} + \psi(w - h_0 t) ,$$

where ψ is an arbitrary function of $w - h_0 t$.

If we wish γ to be expandable in a trigonometric series of the same form as the series (18.14), the following is necessary:
(i) This function ψ must be zero (since we do not assume here that a relation of the form $mh_0 + \mu = 0$ is in existence). Therefore, we will use $\psi = 0$.
(ii) The term B must be zero.

If we are to be able to solve the problem postulated here, two conditions must be satisfied:

The wholly known term of $\Phi\xi$, like that of $\Phi\eta$, must be zero.

We will choose h_1 such as to satisfy one of these conditions while the other must be satisfied identically, unless the formulated problem is not feasible.

In a similar manner, the equation E_i is used for determining x_i and h_i. So as to have x_i retain the trigonometric form, two conditions are necessary. One is satisfied by properly selecting h_i and the other one must be satisfied identically.

Thus either the proposed problem is impossible; or else our conditions must be identities.

193. To demonstrate that these conditions are actually satisfied identically, it remains to establish the feasibility of the problem. In the same way the method of no. 127 would not have been legitimate if we had not previously demonstrated in no. 125 the feasibility of expansion.

Let us consider a system of canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1,2, \ldots ,n) . \qquad (18.15)$$

We assume that F can be expanded in powers of a parameter μ, in the form of

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots ,$$

but we no longer assume, as in no. 125, that F_0 is independent of y_i.

We suppose that F is periodic, with period 2π with respect to y_i.

Finally, we assume that we were able to integrate the equations

$$\frac{dx_i}{dt}=\frac{dF_0}{dy_i}, \quad \frac{dy_i}{dt}=-\frac{dF_0}{dx_i} \tag{18.16}$$

and that the solution satisfies the following conditions:
(i) The variables x_i and y_i will be functions of n constants of integration

$$x_1', \quad x_2', \quad \ldots, \quad x_n',$$

and of n arguments

$$y_1', \quad y_2', \quad \ldots, \quad y_n'.$$

(ii) These n arguments themselves will be functions of time such that

$$y_i'=\lambda_i t+\overline{\omega}_i.$$

Here, λ_i will be constants that depend on the n first constants of integration x_i', while $\overline{\omega}_i$ will be n new integration constants.
(iii) The terms x_i and y_i-y_i' will be periodic functions of y_i', with period 2π.
(iv) The expression

$$\sum x_i dy_i-\sum x_i' dy_i'$$

will be an exact differential.

We obviously then have

$$F_0(x_i,y_i)=\text{const.}, \tag{18.17}$$

meaning that F_0 will depend only on the constants of integration x_i'.

We recall here the theorem of no. 4 which can also be formulated in this manner.

If a change of variables is performed by passing from one system of conjugate variables (x_i,y_i) to another system of conjugate variables (x_i',y_i'), the condition for keeping the canonical form unaltered is that the expression

$$\sum x_i' dy_i'-\sum x_i dy_i$$

be an exact differential.

From this it follows that if, in the case under consideration, we use x_i' and y_i' as new variables, Eqs. (18.15) will retain their canonical form and will become

$$\frac{dx_i'}{dt}=\frac{dF}{dy_i'}, \quad \frac{dy_i'}{dt}=-\frac{dF}{dx_i'}. \tag{18.18}$$

It is obvious that
(i) F will be periodic with respect to y_i';
(ii) F_0 will depend only on x_i' because of Eq. (18.17).

This means that Eqs. (18.18) thus satisfy the conditions of nos. 125 and 127 from which it follows that they can be formally satisfied in the manner given below:

The terms x_i' and y_i' can be expanded in powers of μ, in the form

$$x_i' = x_i'^0 + \mu x_i'^1 + \mu^2 x_i'^2 + \cdots,$$
$$y_i' = y_i'^0 + \mu y_i'^1 + \mu^2 y_i'^2 + \cdots.$$

The terms $x_i'^k$ and $y_i'^k$ will be functions of n integration constants and of n arguments

$$w_i = n_i t + \overline{\omega}_i',$$

where n_i are constants that can be expanded in powers of μ while $\overline{\omega}_i'$ are arbitrary constants.

The terms $x_i'^k$ and $y_i'^k$ will be periodic with respect to w_i, except for $y_i'^0$ which reduces to w_i. We should mention that $x_i'^0$ is a constant.

Now, we only have to substitute these values of x_i' and y_i' into the equations that yield the old variables as a function of these new variables x_i' and y_i'; this shows that Eqs. (18.18) can be formally satisfied in the following manner:

The terms x_i and y_i can be expanded in powers of μ, in the form

$$x_i = x_i^0 + \mu x_i^1 + \cdots,$$
$$y_i = y_i^0 + \mu y_i^1 + \cdots.$$

The quantities x_i^k and y_i^k will be periodic with respect to w_i, except for y_i^0. However, $y_i^0 - w_i$ will be periodic. Nevertheless, it will not happen that x_i^0 reduces to a constant and y_i^0 to w_i.

194. Let us apply these principles to Eq. (18.5) of no. 191 which, giving it another number, I can write as

$$\frac{d^2x}{dt^2} + x(q^2 - q_1 \cos 2t) = \alpha\varphi(x,t). \qquad (18.19)$$

Let us attempt to reduce this equation to the canonical form.

Let ψ be a function of x and of t such that

$$\varphi(x,t) = \frac{d\psi}{dx}.$$

Here, φ just as ψ can be expanded in powers of x and in the sines and cosines of multiples of

$$2t, \quad \lambda_3 t, \quad \lambda_4 t, \quad \ldots, \quad \lambda_n t.$$

For greater symmetry, let us put

$$2 = \lambda_2 .$$

Next, let us put

$$y = \frac{dx}{dt}, \quad y_i = \lambda_i t \quad (i = 2,3, \ldots ,n) ,$$

and let us assume that, in φ, ψ, and in the term $q_1 \cos 2t$, the quantities $\lambda_i t$ had everywhere been replaced by y_i in such a manner that both sides of Eq. (18.19) become functions of x, of d^2x/dt^2, and of y_i which are periodic with period 2π with respect to y_i.

Let us introduce $n - 1$ auxiliary variables

$$x_2, \quad x_3, \quad \ldots, \quad x_n ,$$

and let us put

$$F = \frac{y^2}{2} - \alpha\psi + \frac{x^2}{2} (q^2 - q \cos y_2) - \sum \lambda_i x_i .$$

We can replace Eq. (18.19) by the system of canonical equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{dF}{dy}, \quad \frac{dy}{dt} = -\frac{dF}{dx}, \\ \frac{dx_i}{dt} &= \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 2,3, \ldots ,n) . \end{aligned} \tag{18.20}$$

If we next put (see no. 181)

$$x = \frac{1}{q} \sqrt{2x_1} \cos y_1, \quad y = q\sqrt{2x_1} \sin y_1 ,$$

then the expression

$$x\,dy - x_1 dy_1$$

will be an exact differential. Consequently, the canonical form of the equations will not be altered if we use x_i, y_i $(i = 1, 2, \ldots , n)$ as variables.

In addition, F will be periodic with respect to $y_1, y_2, \ldots , y_n$, so that

$$q^2x^2 + y^2 = 2qx_1 ,$$

The small parameter α here plays the role of μ, and one can see that F is expanded in powers of α.

If we set $\alpha = 0$, F will reduce to

$$F_0 = qx_1 - \frac{q_1}{q_2} x_1 \cos^2 y_1 \cos y_2 - \sum \lambda_i x_i .$$

We can find a function S dependent on n arbitrary constants $x_1', x_2', \ldots ,$ x_n' and satisfying the equation

$$\frac{dS}{dy_1}\left(q - \frac{q_1}{q_2}\cos^2 y_1 \cos y_2\right) - \sum \lambda_i \frac{dS}{dy_i} = \text{const.} \tag{18.21}$$

With slightly different notations, this constitutes the equation of no. 181. In no. 181 we had shown that, considering q_1 a very small coefficient analogous to the parameter μ of no. 125, the methods of no. 125 can be applied to this equation. The function $S - x_1'y_1 - x_2'y_2 - \cdots - x_n'y_n$ is a function of x_1', y_1, and y_2 which is periodic only in y_1 and y_2 (see no 181). To convince oneself of this, it is merely necessary to apply the method of no. 125 to Eq. (18.21), making q_1 play the role of μ.

It follows from this that the equations

$$\frac{dx_i}{dt} = \frac{dF_0}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF_0}{dx_i}, \tag{18.22}$$

can be satisfied by setting, as in no. 3,

$$x_i' = \frac{dS}{dy_i}, \quad y_i = \frac{dS}{dx_i'},$$

and, furthermore,

$$y_i' = \lambda_i t + \overline{\omega}_i .$$

Here, λ_i and $\overline{\omega}_i$ are constants of which the second one is arbitrary.

We will simply have

$$x_i = x_i' \quad \text{and} \quad y_i = y_i' ,$$

for $i > 2$.

We also will have $y_2 = y_2'$.

So far as y_1' is concerned, it will be equal to

$$-ht + \overline{\omega}_1 ,$$

such that the coefficient λ_1 will be nothing else but the number h with changed sign.

It is easy to find the function S or else the expression of x_i and y_i as a function of x_i', y_i'. These are obtained readily when the number h and the coefficients A_n, determined in the preceding chapter, are known.

We should note here that, according to the definition of the new variables x_i' and y_i', the expression

$$dS = \sum x_i dy_i + \sum y_i' dx_i'$$

and, consequently, also the expression

$$\sum (x_i dy_i - x_i' dy_i')$$

will be exact differentials.

Moreover, the terms x_i' and $y_i - y_i'$ will be periodic functions of y_i'.

Finally, we obtain

$$F_0 = +hx_1' - \lambda^2 x_2' - \lambda_2 x_2' - \cdots - \lambda_n x_n' .$$

Thus if we take x_i' and y_i' as new variables, then the canonical form of Eqs. (18.20) will not be altered and these can be written as

$$\frac{dx_i'}{dt} = \frac{dF}{dy_i'}, \quad \frac{dy_i'}{dt} = -\frac{dF}{dx_i'} . \tag{18.23}$$

In addition, F will be periodic with respect to y_i' and, for $\alpha = 0$, $F = F_0$ will depend only on x_i'.

This returns us to the conditions of nos. 125 and 127 so that we can conclude that the terms x_i' and y_i' and thus also x_i and y_i can be formally expressed as a function of α, of n arbitrary constants, and of n variables w_k, in such a manner that the functions $x_i', y_i' - w_i, x_i, y_i - w_i$ can be expanded in powers of α and will be periodic with respect to w_k. They will then have the form

$$w_k = n_k t + \overline{\omega}_k ,$$

where the quantities $\overline{\omega}_k$ are new integration constants while n_k will be constants expandable in powers of α.

It is also easy to demonstrate that, in the particular case in question here, we have the following expression for $k > 1$:

$$y_k = y_k' = w_k, \quad n_k = \lambda_k .$$

To satisfy not only eqs. (18.20) but also Eq. (18.19) from which these have been deduced, it is necessary to take

$$\overline{\omega}_k = 0, \quad w_k = \lambda_k t .$$

It results from all this that the problem proposed in the preceding number is feasible and, consequently, that the conditions of which we spoke at the end of this number must be satisfied identically.

195. Since this must take place irrespective of the value of q_1 and even at $q_1 = 0$ and since this fact could not have escaped Gyldén's attention, it cannot have been to avoid these secular terms that he transposed the term in $q_1 x \cos 2t$ to the left-hand side despite the fact that the coefficient q_1 is very small: He did this for an entirely different reason which we will attempt to explain below.

If we refer to the preceding chapter, we will see that the coefficients A_n become infinite when the number h is an integer. Thus these coefficients are very large when the number h is close to an integer or else, since h differs little from q, when the number q is close to an integer.

Thus if, writing the equation of the preceding chapter in the form

$$\frac{d^2x}{dt^2} + q^2x = + q_1 x \cos 2t ,$$

we had applied the procedures of no. 127 and made q_1 play the role of μ, then the convergence would have been very slow in the case in which q is close to a whole number.

Let us now consider the equation

$$\frac{d^2x}{dt^2} + q^2x = \alpha\varphi(x,t) . \tag{18.24}$$

Let

$$Ax^m \cos \lambda t \quad \text{or} \quad Ax^m \sin \lambda t$$

be any term of $\varphi(x,t)$ and let m be a positive integer or zero. If $m = 0$, this term will be independent of x and, without causing inconvenience, could stay on the right-hand side. If $m > 1$, the term will contain a factor x^2 which generally will be very small and cannot have a significant influence.

This leaves the case in which $m = 1$.

According to what we have just seen, the procedures of no. 127 can be applied to the equation

$$\frac{d^2x}{dt^2} + q^2x = \alpha Ax \cos \lambda t , \tag{18.25}$$

and, if one makes α play the role of μ, the convergence is slow or rapid depending on whether $2q/\lambda$ is or is not be close to an integer. The convergence will be slow specifically when $2q/\lambda$ is close to unity; in fact, according to what we have seen in no. 179, the expression of $F_1(t)$ contains $q^2 - 1$ in the denominator.

It results from this that the function $F(t)$, expanded as in no. 179 in powers of q_1, contains terms in

$$\frac{q_1}{q^2 - 1} .$$

The function $F(t)$ which satisfies the equation

$$\frac{d^2x}{dt^2} + q^2x = q_1 x \cos 2t \tag{18.26}$$

is thus very large if q is close to one. However, Eq. (18.25) is reduced to Eq. (18.26) by changing there t into $2t/\lambda$, q into $\lambda q/2$, and αA into $\lambda^2 q_1/4$.

The integral of Eq. (18.25) can thus become large and its convergence will be slow if $2q/\lambda$ is close to one, as I have just said.

Consequently, if the right-hand side of Eq. (18.24) contains a term such that x enters as a factor at the first power and if its argument λt is such that $2q/\lambda$ is close to one, then one can greatly increase the speed of convergence by bringing this term over to the left-hand side.

Let us see whether this case occurs, when applying Gyldén's method to the three-body problem.

Let us return to Eq. (16.6a) of no. 168:

$$\frac{d^2\rho}{dv_0^2} + \rho = B .$$

The terms of B are of the order of magnitude of the perturbing forces; they depend on $v_0 + \chi$, $v_0' + \chi'$, ρ, and ρ'. We can assume that we have caused χ, χ', and ρ' to vanish there by applying the procedures of nos. 170–172 or that, by using similar procedures, we had replaced v_0' as a function of v_0.

In that case, B will no longer depend on ρ and on v_0, and its terms will have the form

$$A\rho^m \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix} \lambda v_0.$$

As to λ, this will be equal to

$$m + \mu n ,$$

where m and n are integers while μ is the ratio of the mean motions of the two planets.

In B, let us distinguish the two following terms:

$$\alpha\rho \quad \text{and} \quad \beta\rho \cos 2v_0 ,$$

and let us put

$$B = \alpha\rho + \beta\rho \cos 2v_0 + B' .$$

It is possible to transpose $\alpha\rho$ to the left-hand side and to write

$$\frac{d^2\rho}{dv_0^2} + \rho(1 - \alpha) = B' + \beta\rho \cos 2v_0 .$$

This equation has the same form as Eq. (18.24). To know whether it is convenient to bring the term $\beta\rho \cos 2v_0$ to the left-hand side, it is necessary to define whether the quantity, corresponding to $2q/\lambda$, is close to one. However, this quantity is equal to

$$\sqrt{1 - \alpha} ,$$

and α is of the order of the perturbing function. Thus the rapidity of convergence will be increased greatly by bringing this term to the left-hand side, while the same reasons do not exist for transposing the other terms of B'.

Let us now study this question a little more closely. The difficulty here is due to the fact that the coefficient of ρ is close to one; or else to the fact that this coefficient of ρ reduces to one as soon as the perturbing masses become zero.

When the perturbing masses are zero, the motion will become Keplerian and the equations of motion will reduce to

$$v = v_0, \quad \frac{d^2u}{dv_0^2} + u = 0 .$$

If, with the perturbing masses remaining zero, the two planets were attracted by a central star but according to a law differing completely from the Newtonian law, then these equations would become

$$v = v_0, \quad \frac{d^2u}{dv_0^2} + \varphi(u) = 0 ,$$

where $\varphi(u)$ is a function of u depending on the law of attraction.

Next, as in no. 169, let us put

$$u = u_1 + \rho ,$$

where u_1 is a known function of v_0 which differs little from u; neglecting all powers higher than ρ, the equation will become

$$\frac{d^2\varphi}{dv_0^2} + \varphi'(u_1)\rho = A ,$$

where φ' is the derivative of φ while A is a known function of v_0 as well as of $\varphi'(u_1)$.

For example, if u_1 were a constant or if $\varphi(u)$ were a linear function, then $\varphi'(u_1)$ would be a constant generally differing from one so that the difficulty encountered earlier would not occur.

Thus the difficulty which made us transpose the term in q_1 to the left-hand side exists with no other law but that of Newton.

This is due to the fact that, if Newton's law is adopted and if the perturbing masses are still assumed as zero, then the perihelions are fixed which is no longer true for any other law of gravitation.

This is exactly what we emphasized at the beginning of Chap. 11.

Thus the difficulty which was overcome by Gyldén by transposing the term in q_1 to the left-hand side is exactly the same as that overcome by us with the procedures given in Chap. 11.

Variational Equation

196. Equation (16.5b) of no. 169, known as the variational equation, is written as

$$\frac{d^2\chi}{dv_0^2} - C\sin(mv_0 + n\mu v_0 + m\chi + k) = A\,, \qquad (18.27)$$

where C is a constant and A is a sequence of very small terms, which we will assume depend only on χ.

Let us then set

$$mv_0 + n\mu v_0 + m\chi + k = V\,,$$

the equation will become

$$\frac{d^2V}{dv_0^2} - \frac{C}{m}\sin V = \frac{A}{m}\,,$$

where A is a very small function of V and of v_0. Since A is very small, we can write

$$A = \alpha m\varphi(V,v_0)\,,$$

where α is a very small coefficient; after this, we can expand the expression in ascending powers of α.

We thus have

$$\frac{d^2V}{dv_0^2} - \frac{C}{m}\sin V = \alpha\varphi(V,v_0)\,.$$

In this form, it can be seen that Eq. (18.27) appears as a particular case of the following equation:

$$\frac{d^2x}{dt^2} + f(x) = \alpha\varphi(x,t)\,, \qquad (18.28)$$

where f and φ are arbitrary functions while α is a very small coefficient.

This is also the case for Eq. (16.6d) of no. 169 which can be written in the form

$$\frac{d^2\rho}{dv_2^0} + \rho(1+\alpha) - C\rho^3 = B\,,$$

where B is a sum of very small terms that can be transformed by the methods of nos. 170–172 in such a manner that one can assume them to contain only ρ and v_0.

Thus equation (18.28) will be the one to be studied below.

Before going further, a remark is required here.

Let us consider Eq. (18.5) of no. 191. We have made an effort to expand the solution of this equation in powers of α. In Chap. 16 we did not formulate the problem in exactly the same manner; we stated there that, on the right-hand side of this equation, the term x first had to be replaced by zero and then by its first approximate value, and so on.

However, it is easy to demonstrate that these two modes of approximation come to the same thing. If, in fact, we set $\alpha = 0$ in this equation, it will reduce to

$$\frac{d^2x}{dt^2} + x(q^2 - q_1 \cos 2t) = 0$$

and will then admit $x = 0$ as a particular solution, which is exactly the value of x which we had supposed in first approximation. The same holds for Eq. (16.6d) of no. 169, which can now be written as

$$\frac{d^2\rho}{dv_0^2} + A\rho - C\rho^3 = \alpha f(\rho, v_0) .$$

If one sets $\alpha = 0$, the equation will admit $\rho = 0$ as particular solution. However, in Chap. 16, we have exactly supposed $\rho = 0$ as first approximation.

The two methods of approximation thus are still equivalent.

This is no longer entirely so with respect to Eq. (18.27) of the present number, which we had written as

$$\frac{d^2V}{dv_0^2} + \frac{C}{m} \sin V = \alpha\varphi(V, v_0) .$$

Setting $\alpha = 0$, this equation reduces to

$$\frac{d^2V}{dv_0^2} - \frac{C}{m} \sin V = 0 , \tag{18.29}$$

and obviously admits of $V = 0$ as particular solution. However, what we supposed in Chap. 16 as first approximation, was not

$$V = 0 ,$$

but rather

$$\chi = 0 ,$$

whence

$$V = mv_0 + n\mu v_0 + k ,$$

which obviously is not a solution of Eq. (18.29).

The two methods of approximation are not absolutely equivalent. However, because of the smallness of the coefficient C, it is possible to take, as first approximation, a solution of Eq. (18.29) instead of setting $\chi = 0$ without much slowing the rapidity of convergence. This is exactly the manner in which Gyldén operated.

Thus let us return to the Eq. (18.28):

$$\frac{d^2x}{dt^2} + f(x) = \alpha\varphi(x, t)$$

As in no. 191, we will assume that $\varphi(x,t)$ is a periodic function, with period 2π with respect to the arguments

$$\lambda_2 t, \quad \lambda_3 t, \quad \ldots, \quad \lambda_n t,$$

and we will put

$$y = \frac{dx}{dt}, \quad y_i = \lambda_i t, \quad \varphi = \frac{d\psi}{dx}.$$

Similarly, we will put

$$f = \frac{d\theta}{dx}.$$

If, next, we put

$$F = \frac{y^2}{2} + \theta - \alpha\psi - \sum \lambda_i x_i,$$

then Eq. (18.28) can be replaced by the canonical equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{dF}{dy}, & \frac{dy}{dt} &= -\frac{dF}{dx}, \\ \frac{dx_i}{dt} &= \frac{dF}{dy_i}, & \frac{dy_i}{dt} &= -\frac{dF}{dx_i}. \end{aligned} \tag{18.30}$$

We propose to integrate these equations formally as follows: Our variables must be expanded in powers of α and the coefficients will be periodic functions, with period 2π, of n parameters

$$w, \quad w_2, \quad w_3, \quad \ldots, \quad w_n,$$

with

$$w = ht + \overline{\omega}_1, \quad w_i = h_i t + \overline{\omega}_i.$$

Obviously, as in no. 194, it is necessary to set

$$h_i = \lambda_i, \quad \overline{\omega}_i = 0, \quad y_i = w_i.$$

As to the number h, this can be expanded in powers of α.

The results of no. 193 can be recapitulated as follows: If a similar problem is feasible for $\alpha = 0$, it will still be feasible if α is no longer assumed as zero.

Now, if we set $\alpha = 0$, our equation reduces to

$$\frac{d^2x}{dt^2} + f(x) = 0. \tag{18.31}$$

This equation is readily integrated by quadrature, yielding

$$x = \omega(w), \quad y = h\omega'(w), \quad w = ht + \overline{\omega}_1,$$

$$y_i = w_i = \lambda_i t.$$

Here, ω and ω' are functions of w and of an integration constant β; they are periodic, with period 2π with respect to w. The number h is a function of β, and ϖ_1 is a new constant of integration.

Since the problem postulated above is feasible for $\alpha = 0$, it will still be feasible for $\alpha \lesseqgtr 0$.

It now remains to solve this problem effectively.

For this, we rewrite Eq. (18.28) to show explicitly that x depends on t first directly and then through the intermediary of w. Thus we follow a method wholly similar to that of no. 192.

This will yield

$$h^2 \frac{d^2x}{dw^2} + 2h \frac{d^2x}{dw\,dt} + \frac{d^2x}{dt^2} + f(x) = \alpha\varphi(x,t) . \qquad (18.32)$$

We will replace the quantities x and h by their expansions in powers α

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots ,$$
$$h = h_0 + \alpha h_1 + \alpha^2 h^2 + \cdots ,$$

and will then equate the coefficients of like powers of α. This yields the equations

$$h_0^2 \frac{d^2x_0}{dw^2} + 2h_0 \frac{d^2x_0}{dw\,dt} + \frac{d^2x_0}{dt^2} + f(x_0) = 0 , \qquad (18.33)$$

$$2h_0h_1 \frac{d^2x_0}{dw^2} + 2h_1 \frac{d^2x_0}{dw\,dt} + h_0^2 \frac{d^2x_1}{dw_2}$$
$$+ 2h_0 \frac{d^2x_1}{dw\,dt} + \frac{d^2x_1}{dt^2} + f'(x_0)x_1 = \Phi , \qquad (18.34)$$

$$2h_0h_2 \frac{d^2x_0}{dw_2} + 2h_2 \frac{d^2x_0}{dw\,dt} + h_0^2 \frac{d^2x_2}{dw^2}$$
$$+ 2h_0 \frac{d^2x_2}{dw\,dt} + \frac{d^2x_2}{dt^2} + f'(x_0)x_2 = \Phi , \qquad (18.35)$$

$$\vdots$$

We will use Φ to denote any known function of t and of w. The right-hand side Eq. (18.34) is known since h_0 and x_0 had been determined by means of Eq. (18.33). The right-hand side of Eq. (18.35) is known since h_0, h_1, x_0, x_1 had been determined by means of Eqs. (18.33) and (18.34), and so on.

Equation (18.33) reduces to Eq. (18.31), so that we have

$$x_0 = \omega(w,\beta) ,$$

where ω is a function of w and of the constant β and is periodic with respect to w.

Let us now consider Eq. (18.34); if h_1 were known, this equation would be written as

$$h_0^2 \frac{d^2 x_1}{dw^2} + 2h_0 \frac{d^2 x_1}{dw\, dt} + \frac{d^2 x_1}{dt^2} + f'(x_0) x_1 = \Phi \,. \qquad (18.34a)$$

This is an equation linear on the right-hand side. Consequently, we have to consider the homogeneous equation:

$$h_0^2 \frac{d^2 z}{dw^2} + 2h_0 \frac{d^2 z}{dw\, dt} + \frac{d^2 z}{dt^2} + f'(x_0) z = 0 \,.$$

Obviously, this equation admits of the following particular solution:

$$z = z_1 = \frac{d\omega}{d\beta}, \quad z = z_2 = \frac{d\omega}{dw}.$$

As in no. 192, let us now put

$$z_1' = h_0 \frac{dz_1}{dw} + \frac{dz_1}{dt}, \quad z_2' = h_0 \frac{dz_2}{dw} + \frac{dz_2}{dt}.$$

The determinant $z_1 z_2' - z_2 z_1'$ will be a constant which we will denote by k. Let us note, in passing, that we wrote the equations as though x_0, z_1, z_2, z_1', z_2' depended simultaneously on w and on t whereas these functions, in reality, depend only on w so that many of the terms of these equations are actually zero.

Then, let γ and δ be two quantities defined by the equations

$$x_1 = \gamma z_1 + \delta z_2 \,,$$

$$h_0 \frac{dx_1}{dw} + \frac{dx_1}{dt} = \gamma z_1' + \delta z_2' \,.$$

For abbreviation, let us put

$$\gamma' = h_0 \frac{d\gamma}{dw} + \frac{d\gamma}{dt},$$

$$\delta' = h_0 \frac{d\delta}{dw} + \frac{d\delta}{dt}.$$

Equation (18.34a) can then be replaced by the two following expressions:

$$\gamma' z_1 + \delta' z_2 = 0 \,,$$

$$\gamma' z_1' + \delta' z_2' = \Phi \,,$$

whence

$$\gamma' = -\Phi z_2 \,,$$

$$\delta' = \Phi z_1 \,.$$

These equations can be integrated by the same procedure as used for the analogous equations of no. 192, and no difficulty will be encountered here provided that the mean values of Φz_1 and Φz_2 are zero.

One can then choose h_1 such that one of these mean values will vanish while the other will vanish identically since we already know that the problem is feasible.

Equation (18.35) and all subsequent equations can be treated in the same manner.

In certain specific cases, the integration of Eq. (18.31) reduces to elliptic functions. This happens, for example, when $f(x)$ is a polynomial of the third degree in x or when $f(x)$ reduces to a constant factor multiplied by $\sin x$, i.e., in the case of Eqs. (16.6d) and (16.5b) of no. 169.

Summary

197. Above, we have made an attempt to interpret the essence of the Gyldén methods rather than follow scrupulously his mode of presentation. It remains to state what, in our opinion, one should think of these methods.

Each time that the ratio of the mean motions is not very close to being rational, the methods by Newcomb discussed in Chaps. 9–15 will appear simpler—specifically with the improvements made by us—and more satisfactory than those by Gyldén.

Nevertheless, a study of Gyldén's methods retains its full usefulness. In fact, numerous cases exist in which the ratio of the mean motions is too close to being rational for having the methods of Chaps. 9–15 still remain applicable. To treat these cases, Gyldén applied procedures analogous to those which he used successfully in simpler cases and, again, has obtained the same success.

Therefore, it seems worthwhile to penetrate further into the spirit of these methods, whether one either wishes to apply them directly or merely to use them as a means for developing new theories that might be more satisfactory for some reason.

The essence of these methods can be expressed in a single statement: If any term becomes very large and slows down the convergence, one takes account of it from the first approximation on.

Generalization of Periodic Solutions

198. The theory of equations studied in this chapter is connected with a proposition which Gyldén used to some extent without specifically formulating it. We cannot just disregard it here.

Let us consider the equation

$$\frac{d^2x}{dt_2} - \alpha x = \mu f(x,t,\mu) , \qquad (18.35)$$

where α is some constant and μ is a very small parameter; f is a function of x, t, and μ which can be expanded in powers of x and μ as well as in sines and cosines of multiples of n arguments

$$\lambda_1 t, \quad \lambda_2 t, \quad \ldots, \quad \lambda_n t .$$

If there were only a single argument $\lambda_1 t$, the function f would be a periodic function of t with period $2\pi/\lambda_1$. In that case, Eq. (18.35) would admit of a periodic solution with the same period. In fact, for $\mu = 0$, this equation, no matter what the constant α might be, obviously will admit of a periodic solution which will be

$$x = 0 .$$

Thus in view of the principles exposed in Chap. 3, the equation will also admit of a solution for small values of μ.

Can this result be generalized for the case in which f contains n different arguments

$$\lambda_1 t, \quad \lambda_2 t, \quad \ldots, \quad \lambda_n t ?$$

Does Eq. (18.35) then admit of a solution of the form

$$x = x_1 \mu + x_2 \mu^2 + x_3 \mu^3 + \cdots , \qquad (18.36)$$

where $x_1, x_2, x_3, \ldots$ can be expanded in sines and cosines of multiples of $\lambda_i t$?

To check on this point, we will use a method resembling that given in no. 45 and which, although more general, will also be simpler since, by assuming $\alpha = 0$, we had intentionally introduced in no. 45 a difficulty which does not occur in the general case.

Let us assume the problem as solved and let us substitute for x in f the series (18.36). After this substitution, f can be expanded in powers of μ, first because this function had already been expandable in powers of this variable before the substitution and secondly because the value of x given by Eq. (18.36) can itself be expanded in powers of μ. We thus will have

$$f = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \cdots$$

where φ_0 will depend only on t; φ_1 on t and x_1; φ_2 on t, x_1, and x_2; and so forth.

Then, Eq. (18.35), upon equating the coefficients of the various powers of μ, will yield

$$\begin{aligned} \frac{d^2x_1}{dt^2} - \alpha x_1 &= \varphi_0 , \\ \frac{d^2x_2}{dt^2} - \alpha x_2 &= \varphi_1 , \\ \frac{d^2x_3}{dt^2} - \alpha x_3 &= \varphi_2 , \\ &\vdots \end{aligned} \tag{18.37}$$

which will permit us to determine by recurrence the various functions x_1, $x_2, x_3, \ldots$.

Equations (18.37) have the form

$$\frac{d^2x_i}{dt^2} - \alpha x_i = \varphi_{i-1} .$$

If φ_{i-1} can be expanded in sines and cosines of multiples of λt and can be written in the form

$$\varphi_{i-1} = \sum A \cos[(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)t + k] ,$$

where m_i are integers while k and A are any constants, then we can take

$$x_i = -\sum \frac{A \cos[(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)t + k]}{\alpha + (m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)^2} \tag{18.38}$$

and x_i will be of the wanted form.

This leaves us to determine whether series (18.36) is convergent. This will be the case whenever α is positive.

Let us now assume that α is positive; in that case, we will have

$$\frac{1}{\alpha} > \frac{1}{\alpha + (m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)^2} .$$

Let us return to the notations of Chap. 2 and introduce a new function of t of the same form as φ_{i-1}, which we will designate as φ'_{i-1}. Let us assume that this function is such that

$$\varphi_{i-1} \ll \varphi_{i-1} \quad (\arg e^{\pm i\lambda_1 t}, \ldots, e^{\pm \lambda_n t}) .$$

Let us then define x'_i by the equation

$$\alpha x'_i = \varphi_{i-1}$$

and x_i by Eq. (18.38), obviously yielding

$$x_i \ll x'_i .$$

Let there then be a function $f'(x,t,\mu)$ of the same form as $f(x,t,\mu)$ such that

$$f \ll f' \quad (\arg x, \mu, e^{\pm i\lambda_1 t}, e^{\pm i\lambda_2 t}, \ldots, e^{\pm i\lambda_n t}) .$$

Let us consider Eq. (18.39) which will define a new function x'

$$\alpha x' = \mu f'(x',t,\mu) . \tag{18.39}$$

From this equation, x' can be derived as a convergent series expanded in powers of μ:

$$x' = x_1'\mu + x_2'\mu^2 + x_3'\mu^3 + \cdots ;$$

whose coefficients are arranged in sines and cosines of multiples of $\lambda_i t$.

On substituting this series for x' in f', we obtain

$$f' = \varphi_0' + \mu\varphi_1' + \mu^2\varphi_2' + \cdots ,$$

where φ_0' depends only on t; φ_1' on t and on x_1'; φ_2' on t, on x_1', and on x_2'; . . . .

We will also have

$$\varphi_k(x_1,x_2, \ldots ,x_k,t) \ll \varphi_k'(x_1,x_2, \ldots ,x_k,t) ,$$
$$\arg(x_1,x_2 \ldots ,x_k,e^{\pm i\lambda t}) .$$

For abbreviation, we will write $e^{\pm i\lambda t}$ for the n arguments $e^{\pm i\lambda_1 t}$, $e^{\pm i\lambda_2 t}$, $\ldots$, $e^{\pm i\lambda_n t}$.

Equation (18.39) will give

$$\alpha x_1' = \varphi_0' , \quad \alpha x_2' = \varphi_1', \quad \ldots ,$$

from which we find successively

$$\varphi_0 \ll \varphi_0' \quad (\arg e^{\pm i\lambda t}) ,$$
$$x_1 \ll x_1' \quad (\arg e^{\pm i\lambda t}) ,$$
$$\varphi_1(x_1,t) \ll \varphi_1'(x_1,t) \quad (\arg x_1,e^{\pm i\lambda t}) .$$
$$\varphi_1(x_1,t) \ll \varphi_1'(x_1',t) \quad (\arg e^{\pm i\lambda t}) ,$$
$$x_2 \ll x_2' \quad (\arg e^{\pm i\lambda t}) ,$$
$$\varphi_2(x_1,x_2,t) \ll \varphi_1'(x_1,x_2,t) \quad (\arg x_1,x_2,e^{\pm i\lambda t}) ,$$
$$\varphi_2(x_1,x_2,t) \ll \varphi_2'(x_1',x_2',t) \quad (\arg e^{\pm i\lambda t}) ,$$
$$x_3 \ll x_3' \quad (\arg e^{\pm i\lambda t}) ,$$
$$\vdots$$

and finally,

$$x \ll x' \quad (\arg \mu, e^{\pm i\lambda t}) ,$$

which proves that series (18.36) converges well.

Thus this series converges in two cases:
(i) irrespective of the value of α, when only a single argument $\lambda_1 t$ exists;
(ii) irrespective of the number of the arguments when α is positive.

CHAPTER 19

Bohlin Methods

Delaunay Method

199. Let us return to the hypotheses and notations of no. 125. We have shown that in the application of the method of no. 125 divisors of the form

$$n_1^0 m_1 + n_2^0 m_2 + \cdots + n_n^0 m_n,$$

are introduced, where m_i are integers.

It results from this that the method becomes illusory as soon as one of these divisors becomes very small.

Among the methods conceived for overcoming this difficulty, that by Delaunay was chronologically the first; its presentation will facilitate understanding all others.

Let us first consider a system of canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{19.1}$$

and assume that F is exclusively a function of $x_1, x_2, \ldots, x_n$ and of

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n,$$

and that it is periodic, with period 2π with respect to this latter quantity. We also assume that m_i are integers.

Integration of system (19.1) then reduces to that of the partial differential equation

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, m_1 y_1 + m_2 y_2 + \cdots + m_n y_n\right) = C,$$

where C is an arbitrary constant. This integration is easy to perform.

In fact, let us put

$$S = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n + \varphi(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n),$$

so that the equation becomes

$$F(x_1^0 + m_1\varphi', x_2^0 + m_2\varphi', \ldots, x_n^0 + m_n\varphi', m_1 y_1 + \cdots + m_n y_n) = C.$$

Let us solve this equation for φ', so that

$$\varphi' = \text{function of } \sum m_i y_i, \text{ of } x_1^0, x_2^0, \ldots, x_n^0, \text{ and of } C.$$

This expression is then integrated with respect to $\Sigma\, m_i\, y_i$ by considering C and x_k^0 as constants; then, φ and thus also S will be obtained as a function of $\Sigma m_i\, y_i$, of x_k^0, and of C.

It is necessary to go into more details here; for this, we will consider a simple particular case by setting

$$m_1 = 1, \quad m_2 = m_3 = \cdots = m_n = 0,$$
$$F = x_1^2 + \mu \cos y_1,$$

where μ is very small.

Our equation then becomes

$$\left(\frac{dS}{dy_1}\right)^2 + \mu \cos y_1 = C,$$

whence

$$\frac{dS}{dy_1} = \sqrt{C - \mu \cos y_1}.$$

Several cases must be considered here:

(i) We have

$$C > |\mu|.$$

In this case, the radical $\sqrt{C - \mu \cos y_1}$ is always real and never vanishes. Two values are possible for it: one that is positive for all values of y_1 and another that is negative for all values of y_1. For example, let us take the first value; it can be expanded in cosines of multiples of y_1, in such a manner as to yield

$$\frac{dS}{dy_1} = x_1^0 + \sum B_n \cos ny_1.$$

I have separated out the wholly known term and denoted it by x_1^0. Clearly, x_1^0 is a function of C so that also C will be a function of x_1^0. Moreover, the quantities B_n will be functions of C and thus also of x_1^0.

We then have

$$S = x_1^0\, y_1 + \sum \frac{B_n}{n} \sin ny_1,$$

which gives S as a function of y_1 and of the arbitrary constant x_1^0.

(ii) We have

$$-|\mu| < C < |\mu|.$$

In this case, the quantity under the radical sign

$$C - \mu \cos y_1$$

is not always positive so that one cannot assign all possible values to y_1 but only those for which the root is real.

It then becomes possible to introduce an auxiliary variable ϵ by putting, for example,

$$\mu \cos y_1 = C \cos \epsilon,$$

from which it follows that

$$\frac{dS}{dy_1} = \sqrt{2C}\, \sin(\epsilon/2),$$

or

$$\frac{dS}{d\epsilon} = \sqrt{\frac{\mu^2 - C^2 \cos^2 \epsilon}{C}}\,.$$

Since C^2 is smaller than μ^2, the root on the right-hand side will always be real and can be expanded in a trigonometric series of the form

$$\frac{dS}{d\epsilon} = B_0 + \sum B_n \cos n\epsilon,$$

whence

$$S = B_0\epsilon + \sum \frac{B_n}{n} \sin n\epsilon,$$

which yields S as a function of the auxiliary variable ϵ and of the constant C.

(iii) We have

$$C = |\mu|.$$

Let, for example,

$$\mu > 0, \quad C = \mu.$$

We then obtain

$$\frac{dS}{dy_1} = \sqrt{\mu}\, \sqrt{1 - \cos y_1} = \sqrt{2\mu}\, \sin \frac{y_1}{2}$$

or

$$S = -\sqrt{2\mu}\, \cos \frac{y_1}{2}\,.$$

Here, S is expressed as a function of y_1 and again will be a periodic function of y_1, except that the period no longer will be 2π but 4π.

We should add to this that, if $C < |-\mu|$, then the root will always be imaginary and, if $C = |-\mu|$, it will cease being imaginary only for $y_1 = 0$. The above statements can be interpreted in two ways:

(a) First, by consideration of elliptic functions.

It is obvious that

$$S = \int \sqrt{C - \mu \cos y_1}\, dy_1$$

is an elliptic integral and that, if we put

$$u = \int \frac{dy_1}{\sqrt{C - \mu \cos y_1}},$$

then the expressions

$$\sin y_1, \quad \cos y_1, \quad \sqrt{C - \mu \cos y_1}$$

will be doubly periodic functions of u.

The various cases investigated above then correspond to various hypotheses that can be made with respect to the discriminant of elliptic functions.

(b) By geometry.

In fact, we can plot curves by adopting the polar coordinates and by taking $A + dS/dy_1$ as radius vector, where A is any constant for a polar angle y_1. This yields a pattern as given in Fig. 7.

The solid lines correspond to the hypotheses $C > |\mu|$ while the broken line represents the hypotheses

$$-|\mu| < C < |\mu|,$$

while the dot-dash curve, which has a double point in B, corresponds to the case of $C = |\mu|$; finally, the curve corresponding to $C = -|\mu|$ refers to a single point A.

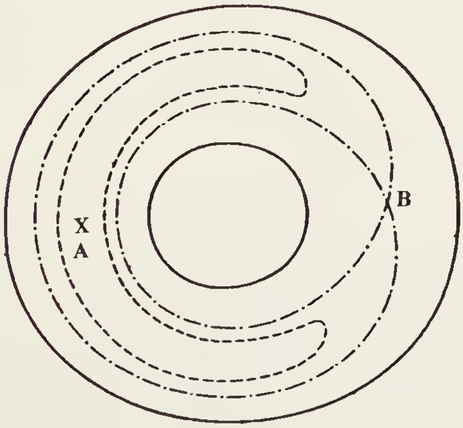

Figure 7

If the methods of no. 125 were applied to the above problem, one would have arrived at an expansion of S in powers of μ. In fact, the radical

$$\sqrt{C - \mu \cos y_1}$$

is actually expandable in powers of μ, which thus also is the case for S. However, the series is convergent only if

$$C < |\mu|.$$

If this condition is not satisfied, the methods of no. 125 become illusory and it is then necessary to apply Delaunay's method, i.e., the method just discussed. This method can also be advantageously used as soon as C is of the same order of magnitude as μ, since the convergence of the series given in no. 125 will then be very slow.

Let us note that the expansion of the radical has the form

$$\sum \sqrt{C} \left(\frac{\mu}{C}\right)^n \varphi_n(y_1)$$

from which it can be seen that, if C is small, the convergence becomes very slow and may even stop entirely.

If we set $C = C_1 \mu$, the expansion becomes

$$\sum \sqrt{C_1} \sqrt{\mu} \, \varphi_n(y_1) \frac{1}{C_1^n},$$

and all its terms will be of the same degree in μ. It is also obvious that

$$\sqrt{C - \mu \cos y_1} = \sqrt{\mu} \sqrt{C_1 - \cos y_1}.$$

200. Let us now pass to a somewhat more general case and let us assume that F is exclusively a function of $x_1 = dS/dy_1$ and of y_1, periodic in y_1.

The partial differential equation becomes

$$F\left(\frac{dS}{dy_1}, y_1\right) = C$$

and must first be solved with respect to C.

Let us assume that we have

$$F = F_0 + F_1 \mu + F_2 \mu^2 + \cdots$$

and that F_0 depends only on $dS/dy_1 = x_1$.

Then, several cases are in question.

Let us assume that F, which already can be expanded in powers of μ, will be also holomorphous in x_1, which incidentally will happen in all applications.

In that case, according to the methods of nos. 30 *et seq.*, the equation

$$F(x_1, y_1) = C \tag{19.2}$$

can be solved for x_1.

For $\mu = 0$, the equation is written as

$$F_0(x_1) = C. \tag{19.3}$$

Let x_1^0 be a value satisfying this Eq. (19.3). If the derivative of F_0 is denoted by F_0' and if

$$F_0'(x_1^0) \gtrless 0,$$

then x_1 can be deduced from Eq. (19.2) in the form of a series arranged in powers of μ, where the coefficients are functions of y_1.

If, conversely,

$$F_0'(x_1^0) = 0, \quad F_0''(x_1^0) \gtrless 0,$$

we still will have x_1 in the form of a series but now this series is expandable in powers of $\sqrt{\mu}$ rather than in powers of μ.

Let us examine these two cases in succession.

First, let $F_0'(x_1^0) \gtrless 0$.

We then put, since x_1 and thus also S can be expanded in powers of μ,

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \cdots$$

and we assume, in addition, that dS_0/dy_1 reduces to the constant x_1^0. Next, the other functions $S_1, S_2, \ldots$ are calculated by recursion, with the computational procedure being exactly the same as that of no. 125.

Let us now pass to the second hypothesis where $F_0'(x_1^0) = 0$.

In that case, S can be expanded in powers of $\sqrt{\mu}$ and we can write

$$S = S_0 + \sqrt{\mu}\, S_1 + \mu S_2 + \cdots.$$

We still assume that

$$\frac{dS_0}{dy_1} = x_1^0, \quad S_0 = x_1^0\, y_1.$$

We then have

$$F_0\left(\frac{dS}{dy_1}\right) = F_0 + \frac{F_0''}{2}\left(\sqrt{\mu}\,\frac{dS_1}{dy_1} + \mu\,\frac{dS_2}{dy_1} + \cdots\right)^2$$

$$+ \frac{F_0'''}{6}\left(\sqrt{\mu}\,\frac{dS_1}{dy_1} + \cdots\right)^3 + \cdots.$$

On the right-hand side, we assume that x_1 in $F_0, F_0'', F_0''', \ldots$, has been replaced by x_1^0.

Similarly, let us put

$$C = C_0 + C_1\sqrt{\mu} + C_2\mu + \cdots,$$

thus demonstrating that the constant on the right-hand side can depend on μ.

Consequently, equating on both sides of

$$F = C$$

all coefficients of like powers of μ, we obtain

$$\begin{aligned}
F_0 &= C_0,\\
0 &= C_1,\\
\tfrac{1}{2}F_0''\left(\frac{dS_1}{dy_1}\right)^2 &= -F_1(x_1^0, y_1) + C_2,\\
F_0''\frac{dS_1}{dy_1}\frac{dS_2}{dy_1} &= \Phi + C_3,\\
F_0''\frac{dS_1}{dy_1}\frac{dS_3}{dy_1} &= \Phi + C_4,\\
&\vdots
\end{aligned} \tag{19.4}$$

In the third equation of system (19.4), we assume S_0 as known; in the fourth equation, we assume S_1 as known; in the fifth equation, we assume S_0, S_1, S_2 as known, and so on.

I continue to use Φ to designate any known function.

The third equation of system (19.4) will permit calculating dS_1/dy_1; since F_0'' is a constant, we obtain

$$\frac{dS_1}{dy_1} = \sqrt{\frac{2}{F_0''}[C_2 - F_1(x_1^0, y_1)]}\ .$$

Various circumstances may occur that correspond to the diverse cases treated in the simpler example discussed above.

It may happen that C_2 remains larger than F_1 irrespective of the value attributed to y_1. Then, dS_1/dy_1 is a periodic function of y_1 whose period is 2π.

Or else it can happen that the condition

$$C_2 > F_1$$

is satisfied only for certain values of y_1. Then, the function S_1 also will be real only for certain values of y_1.

Once S_1 is determined, the fourth equation of system (19.4) will yield S_2, the fifth will yield S_3, and so on.

The solution is entirely satisfactory in the first case, namely, when S_1 is always real. However, in the opposite case, there is one point to be emphasized:

The values of y_1, for which the various functions S_1, S_2, S_3, ... pass from the real to the imaginary are given by the equation

$$C_2 = F_1(x_1^0, y_1).$$

One could believe that it is for these very values that S passes from the real to the imaginary. However, this is not so; the values for which S passes from the real to the imaginary are given by the equations

$$F = C_0 + C_2\mu + C_3\mu\sqrt{\mu} + \cdots, \quad \frac{dF}{dx_1} = 0.$$

It is true that these are very close to the former values if μ is very small but they are not identical with them.

To overcome this difficulty, various means exist. For example, since C_2, C_3, ... are arbitrary, it is possible to set $C_3 = 0$ and do this also for all other C with odd subscript.

Next, we will successively calculate

$$S_1, \quad S_2, \quad S_3, \quad \ldots,$$

which will yield

$$\frac{dS}{dy_1} = \frac{dS_0}{dy_1} + \frac{dS_1}{dy_1}\sqrt{\mu} + \frac{dS_2}{dy_1}\mu + \frac{dS_3}{dy_1}\mu\sqrt{\mu} + \cdots.$$

Since nothing distinguishes $\sqrt{\mu}$ from $-\sqrt{\mu}$, we again have a solution by setting

$$\frac{dS}{dy_1} = \frac{dS_0}{dy_1} - \frac{dS_1}{dy_1}\sqrt{\mu} + \frac{dS_2}{dy_1}\mu - \frac{dS_3}{dy_1}\mu\sqrt{\mu} + \cdots.$$

These two solutions are either both real or both conjugate imaginary. It follows from this that

$$S_0, \quad S_2, \quad S_4, \quad \ldots$$

are always real.

In addition, the expression

$$\frac{dS_1}{dy_1} + \mu\frac{dS_3}{dy_1} + \mu^2\frac{dS_5}{dy_1} + \cdots \tag{19.5}$$

is always real or purely imaginary from which it follows that, to obtain the equation yielding the values of y_1 for which S passes from the real to the imaginary, it is sufficient to equate the expression (19.5) to zero.

How does the passage from the case in which S is always real to the case in which S is sometimes real and sometimes imaginary take place?

This can be better explained by constructing a pattern similar to that given in Fig. 7.

As radius vector, we will use dS/dy_1 and, as polar angle, y_1, after which we plot the curves

$$F\left(\frac{dS}{dy_1}, y_1\right) = C,$$

or at least those among them for which dS/dy_1 differs little from x_1^0.

These curves will differ very little from those where the radius vector is equal to

$$x_1^0 + \sqrt{\mu}\,\frac{dS_1}{dy_1},$$

and where dS_1/dy_1 is given by the formula

$$\sqrt{\frac{2}{F_0''}(C_2 - F_1)}.$$

For plotting these curves, a hypothesis is necessary as to the manner in which the function F_1 varies when y_1 varies from zero to 2π. For example, let us assume that F_1 passes through a maximum, then through a minimum, then through another maximum larger than the first, and then through a minimum smaller than the first. This will yield a pattern of the type shown in Fig. 8.

One can see that as C_2 diminishes one obtains successively:
if C_2 is larger than the largest maximum, two concentric curves, shown in Fig. 8 as dashed lines (- - -);
if C_2 is equal to the large maximum, a curve with a double point, shown as a solid line (—);
if C_2 falls between the two maxima, a curve analogous to that shown as a dot-dash line (–·–·–);
if C_2 is equal to the small maximum, a curve with a double point shown as a dotted line ($\cdots$).

If C_2 becomes smaller than the smallest maximum, the curve is decomposed into two other curves which are shown by the line (+ + +). One of these curves reduces to a point and vanishes when C_2 becomes equal to the largest maximum. The other curve reduces to a point and then vanishes as soon as C_2 becomes equal to the smallest minimum.

It is obvious that passing from one case to the other takes place via a curve with a double point, which necessitates a closer study of these curves and, specifically, of the first curve which is represented by the solid line.

If we assume a moving body traveling along this curve in a continuous motion, this body will—for example—start from the double point, circulate around one of the loops of the curve, return to the double point, move

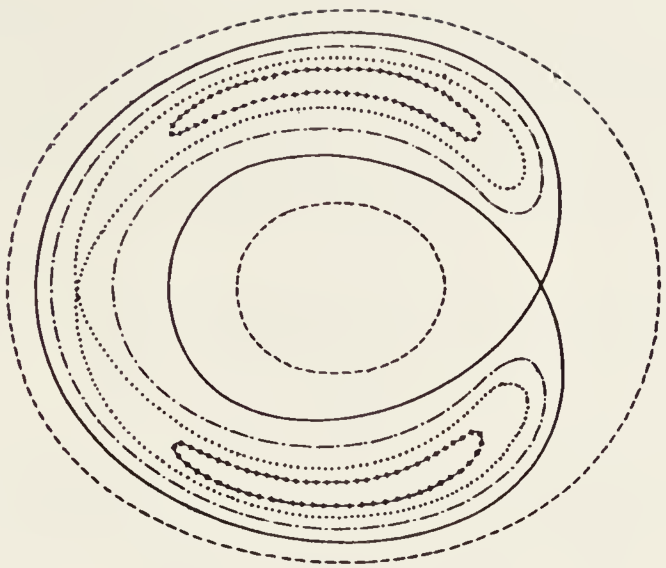

Figure 8

along the second loop, and finally return to its starting point. It is obvious that the motion will still be periodic but that the period is doubled, such that dS/dy_1 is a periodic function of y_1 but with a period which now is 4π instead of 2π as before.

Let us now return to Eqs. (19.4).

It will be found that, if the value corresponding to the maximum of F_1 is assigned to C_2, then the radical

$$\sqrt{\frac{2}{F_0''}(C_2 - F_1)}\ ,$$

which is equal to dS_1/dy_1, will be a periodic function of y_1, with period 4π, and thus will be expandable in sines and cosines of multiples of $y_1/2$.

As soon as y_1 increases by 2π, the root will change sign so that the series will contain only odd multiples of $y_1/2$. The function vanishes twice.

If, in fact, y_1^0 is the value of y_1 that corresponds to the maximum of F_1, then the function dS_1/dy_1 will vanish for $y_1 = y_1^0$ and for $y_1 = y_1^0 + 2\pi$. In that case, irrespective of the value of the constants $C_3, C_4, \ldots$, Eqs. (19.4) show that

$$\frac{dS_2}{dy_1},\quad \frac{dS_3}{dy_1},\quad \ldots$$

are periodic functions of y_1, with period 2π; however, these functions can become infinite for

$$y_1 = y_1^0 \quad \text{or} \quad y_1 = y_1^0 + 2\pi.$$

Nevertheless, we know that we can select the constants $C_3, \ldots$ in such a manner that this case will not occur. The existence of the solid-line curve in Fig. 8 proves this sufficiently. Let us see how to make this choice.

If we assume that the constants with odd subscripts

$$C_3, \quad C_5, \quad \ldots,$$

are zero, then Eqs. (19.4) will not change on changing $\sqrt{\mu}$ into $-\sqrt{\mu}$.

It follows from this that, if the function

$$\frac{dS_0}{dy_1} + \sqrt{\mu}\,\frac{dS_1}{dy_1} + \mu\,\frac{dS_2}{dy_1} + \mu\sqrt{\mu}\,\frac{dS_3}{dy_1} + \cdots$$

satisfies our equation, this will be the same for the function

$$\frac{dS_0}{dy_1} - \sqrt{\mu}\,\frac{dS_1}{dy_1} + \mu\,\frac{dS_2}{dy_1} - \mu\sqrt{\mu}\,\frac{dS_3}{dy_1} + \cdots.$$

These are the two solutions of Eqs. (19.4), and it is obvious that passage from one to the other is done by changing $\sqrt{\mu}$ into $-\sqrt{\mu}$. However, Eqs. (19.4) will also remain unaltered on changing y_1 into $y_1 + 2\pi$. Thus it is also possible to pass from one solution to the other by changing y_1 into $y_1 + 2\pi$.

This leads to the following consequence:

On changing y_1 into $y_1 + 2\pi$, the functions with even subscripts dS_{2n}/dy_1 will not change and the functions with odd subscripts dS_{2n+1}/dy_2 will change sign.

Nevertheless, since dS_1/dy_1 vanishes for $y_1 = y_1^0$ and for

$$y_1 = y_1^0 + 2\pi$$

and since this derivative enters as a factor on the left-hand side of Eqs. (19.4), it might happen that

$$\frac{dS_2}{dy_1}, \quad \frac{dS_3}{dy_1}, \quad \ldots$$

becomes infinite for $y_1 = y_1^0 + 2k\pi$, which actually takes place if the constants $C_4, \ldots$ are not properly chosen.

However, it is possible to make this choice in such a manner that the functions dS_p/dy_1 still remain finite.

To demonstrate this, let us consider the equation

$$F\left(\frac{dS}{dy_1}, y_1\right) = C,$$

which can be written in the form

$$F(x_1, y_1) = C.$$

If we consider x_1 and y_1 as the coordinates of a point, this equation represents a curve. Let us postulate that this curve has a double point; this results in

$$\frac{dF}{dx_1}=\frac{dF}{dy_1}=0,$$

which can also be written as

$$\begin{aligned}\frac{dF_0}{dx_1}+\mu\frac{dF_1}{dx_1}+\mu^2\frac{dF_2}{dx_1}+\cdots=0,\\ \frac{dF_1}{dy_1}+\mu\frac{dF_2}{dy_1}+\mu^2\frac{dF_3}{dy_1}+\cdots=0,\end{aligned} \tag{19.5a}$$

since F_0 does not depend on y_1.

Let us solve Eqs. (19.5a) for x_1 and y_1. At $\mu=0$, we find

$$x_1=x_1^0, \quad y_1=y_1^0.$$

The functional determinant of Eqs. (19.5a), for $\mu=0$, $x_1=x_1^0$, $y_1=y_1^0$, can be written as

$$\frac{d^2F_0}{dx_1^2}\,\frac{d^2F_1}{dy_1^2}$$

and, in general, is not zero. Therefore, one can solve Eqs. (19.5a) and one will find that x_1 and y_1 can be expanded in powers of μ. Let, then,

$$x_1=\alpha, \quad y_1=\beta,$$

be the series obtained in this fashion. The expression

$$F(\alpha,\beta)$$

obviously can be expanded in powers of μ. Let then

$$F(\alpha,\beta)=C_0+C_2\mu+C_4\mu^2+\cdots \tag{19.6}$$

be this expansion. We state that, if the constants C_{2p} in Eqs. (19.4) are assigned the values derived from series (19.6), the functions dS_p/dy_1 will remain finite.

To demonstrate this, let us put

$$x_1=\alpha+x', \quad y_1=\beta+y.$$

$$F(x_1,y_1)=F(\alpha,\beta)+F'$$

and consider the equation

$$F'(x',y')=0,$$

which has the same form as Eq. (19.2). Consequently, we can treat it in the same manner, meaning that we can put

$$x' = \frac{dS'_0}{dy'} + \sqrt{\mu}\,\frac{dS'_1}{dy'} + \cdots$$

and determine the functions dS'_p/dy' by Eqs. (19.4') analogous to Eqs. (19.4) which will differ from the latter only by the fact that the symbols are primed. However, the constants C_p will all be zero and, for

$$x' = y' = 0$$

we will have

$$F' = \frac{dF'}{dx'} = \frac{dF'}{dy'} = 0.$$

Thus if F' is considered as being expandable in powers of x' and y', the series will start with terms of the second degree in x' and y', no matter what μ might be. The expansion of $F'_0, F'_1, \ldots$ will thus also start with second-degree terms. It follows from this that, if one considers the functions Φ on the right-hand side of Eqs. (19.4a) as being expandable in the vicinity of $y' = 0$ in powers of y' and of dS'_p/dy', then the series will always start with second-degree terms.

One sees immediately that dS'_1/dy' vanishes for $y' = 0$. Thus it could be feared that dS'_p/dy' might become infinite for $y' = 0$. Far from this, however, we can actually state that for this value of y' the function dS'_p/dy' is zero.

In fact, let us assume that this holds for

$$\frac{dS'_1}{dy'}, \quad \frac{dS'_2}{dy'}, \quad \ldots, \quad \frac{dS'_{p-1}}{dy'},$$

I now say that this will be valid also for dS'_p/dy'.

Let us consider the equation

$$F''_0 \frac{dS'_1}{dy'} \frac{dS'_p}{dy'} = \Phi,$$

where F''_0 denotes the second derivative of F'_0. Here, Φ can be expanded in powers of y' and of $dS'_1/dy', \ldots, dS'_{p-1}/dy'$. Since $y' = 0$ is a single zero for these various quantities and since the expansion of Φ starts with second-degree terms, $y' = 0$ will be a double zero for Φ.

This will be a single zero for dS'_1/dp'.

Thus it will be a single zero for dS'_p/dy'.

Q.E.D.

We thus find

$$x' = \frac{dS'}{dy'},$$

$$S' = S'_0 + \sqrt{\mu}\, S'_1 + \mu S'_2 + \cdots, \quad S'_0 = 0.$$

We deduce from this

$$x_1 = \alpha + \frac{dS'(y_1 - \beta)}{dy_1}. \tag{19.7}$$

Here, $S'(y_1 - \beta)$ represents the function S' where the argument y' has been replaced by the argument $y_1 - \beta$. Let

$$\alpha = \alpha_0 + \mu\alpha_1 + \mu_2\alpha_2 + \cdots,$$
$$\beta = \beta_0 + \mu\beta_1 + \mu^2\beta^2 + \cdots,$$
$$x_1 = \frac{dS_0}{dy_1} + \sqrt{\mu}\,\frac{dS_1}{dy_1} + \cdots$$

be the expansions of α, β, and x_1, so that, on equating the two sides of Eq. (19.7), we obtain

$$\frac{dS_0}{dy_1} = \alpha_0, \quad \frac{dS_1}{dy_1} = \frac{dS'_1(y_1 - \beta_0)}{dy_1}, \quad \frac{dS_2}{dy_1} = \alpha_1 + \frac{dS'_2}{dy_1},$$
$$\frac{dS_3}{dy_1} = \frac{dS'_3}{dy_1} - \beta_1\frac{d^2S'_1}{dy_1^2}, \quad \frac{dS_4}{dy_1} = \alpha_2 + \frac{dS'_4}{dy_1} - \beta_1\frac{d^2S'_2}{dy_1^2}, \quad \ldots, \tag{19.8}$$

In the derivatives of S'_p, the term y' must be replaced by the argument $y'_1 - \beta^0$.

It is thus obvious that the function dS_p/dy_1 remain finite.

Once we have demonstrated the feasibility of determining the constants C_p in such a manner as to prevent the functions dS_p/dy_1 from becoming infinite, this determination can actually be performed without having to search first for the expansions of α and of β.

It is sufficient to make use of Eqs. (19.4) here.

Let us consider one of these equations

$$F''_0 \frac{dS_1}{dy_1}\frac{dS_{p-1}}{dy_1} = \Phi + C_p.$$

If p is even, we will take

$$C_p = -\Phi(y_1^0),$$

and, since Φ is a periodic function with period 2π, we will also have

$$C_p = -\Phi(y_1^0 + 2\pi),$$

such that dS_{p-1}/dy_1 will become infinite neither for $y_1 = y_1^0$ nor for $y_1 = y_1^0 + 2\pi$.

If p is odd, it is necessary to set $C_p = 0$, and the condition

$$\Phi(y_1^0) = 0$$

which entrains the condition

$$\Phi(y_1^0 + 2\pi) = 0$$

since Φ changes sign when y_1 increases by 2π] will be satisfied identically.

From this it also follows that dS_{p-1}/dy_1 never becomes infinite.

Finally, it results from this that dS_p/dy_1 can be expanded in sines and cosines of multiples of y_1 if p is even and in sines and cosines of odd multiples of $y_1/2$ if p is odd.

We have dwelled at some length on several points that are rather self-evident since we wish to discuss a similar but much more difficult problem later in the text, of which we attempted to demonstrate the analogies here.

201. Let us see now how to pass from the first case, namely, from the case in which

$$F_0'(x_1^0) \gtrless 0$$

and in which the methods of no. 125 are applicable, to the second case in which

$$F_0'(x_1^0) = 0$$

and which will be studied here in some detail.

Let us note first that $F_0'(x_1^0)$ is the same term that we had denoted by $-n_1^0$ in no. 125 and in other portions of this book. Now, putting

$$\frac{dS}{dy_1} = \frac{dS_0}{dy_1} + \mu \frac{dS_1}{dy_1} + \mu^2 \frac{dS_2}{dy_1} + \cdots,$$

we find a series of equations of the form

$$n_1^0 \frac{dS_p}{dy_1} = \Phi + C_p. \tag{19.9}$$

As already explained in no. 125, C_p can be determined arbitrarily. We assume that this is done in such a manner that the mean value of $\Phi + C_p$ is zero, and consequently that S_p will be a periodic function of y_1.

It is obvious that, in the expansion of dS_p/dy_1, various powers of n_1^0 enter the denominator such that, if n_1^0 is small, certain terms of dS_p/dy_1 may become perceptible. We must first of all consider the maximum exponent that n_1^0 might have in the denominator of the various terms of dS_p/dy_1.

I say that this maximum exponent is equal to $2p - 1$.

In fact, S is a function of y_1 on the one hand and of the parameter μ and of the integration constant x_1^0 on the other hand. We are disregarding the constants C_p, which are wholly determined by the conditions

$$F_0(x_1^0) = C_0,$$

which is the mean value of

$$(\Phi + C_p) = 0.$$

Instead of x_1^0, we can also take n_1^0 as constant of integration. In that case, S will be a function of y_1, of μ, and of n_1^0. Let us expand this in powers of μ and n_1^0. The series will contain negative powers of n_1^0.

The equation

$$n_1^0 \frac{dS_1}{dy_1} = \Phi + C_1$$

demonstrates that the expansion of dS_1/dy_1 in ascending powers of n_1^0 will start with a term in $1/n_1^0$.

Let us now pass to the equation

$$n_1^0 \frac{dS_2}{dy_1} = \Phi + C_2.$$

Here, Φ will depend on dS_1/dy_1. However, Φ is obtained by replacing the variable $x_1 = dS/dy_1$ in F by the expansion

$$\frac{dS_0}{dy_1} + \mu \frac{dS_1}{dy_1} + \cdots$$

and retaining in the expansion all terms in μ^2, one thus finds that Φ can contain dS_1/dy_1 to, at most, the second power since the cube of dS_1/dy_1 would have to be accompanied by the factor μ^3 and thus could not yield a term in μ^2.

Consequently, the expansion of Φ and thus also of C_2 will start with a term in

$$\left(\frac{1}{n_1^0}\right)^2$$

and, finally, that of dS_2/dy_1, with a term in

$$\left(\frac{1}{n_1^0}\right)^3.$$

The rule is clear: The expansion of dS_p/dy_1 starts with a term in

$$\left(\frac{1}{n_1^0}\right)^{2p-1}.$$

In fact, let us assume that this holds for

$$\frac{dS_1}{dy_1}, \quad \frac{dS_2}{dy_1}, \quad \ldots, \quad \frac{dS_{p-1}}{dy_1};$$

I say that it then also holds for dS_p/dy_1.

Let us now consider the equation

$$n_1^0 \frac{dS_p}{dy_1} = \Phi + C_p.$$

Here, Φ is a polynomial in

$$\frac{dS_1}{dy_1}, \quad \frac{dS_2}{dy_1}, \quad \ldots, \quad \frac{dS_{p-1}}{dy_1}.$$

Let us consider any term u of this polynomial and let us attempt to evaluate the sum of the indices q of various factors of the form dS_q/dy_1 that enter into u.

Since this term u is derived from a term in μ^2 in the expansion of

$$F\left(\frac{dS_0}{dy_1} + \mu \frac{dS_1}{dy_1} + \cdots\right),$$

this sum can at most be equal to p. In addition, if this sum is equal to p and since none of the indices q is equal to p, the considered term u will contain at least two factors.

The expansion of u in powers of n_1^0 will start with a term in

$$\left(\frac{1}{n_1^0}\right)^{\sum (2q-1)}.$$

However,

$$\sum q \leqslant p.$$

If

$$\sum q < p,$$

$$\sum (2q-1) \leqslant 2p-2;$$

and if

$$\sum q = p,$$

we will also have

$$\sum (2q-1) \leqslant 2p-2,$$

since at least two factors are present here.

Thus, the expansion of Φ and consequently also that of C_p will start with a term in

$$\left(\frac{1}{n_1^0}\right)^{2p-2}$$

while the expansion of dS_p/dy_1 will start with a term in

$$\left(\frac{1}{n_1^0}\right)^{2p-1}. \qquad \text{Q.E.D.}$$

However, since n_1^0 is an arbitrary constant, let us replace it by some expansion

$$n_1^0 = \alpha_0 + \mu\alpha_1 + \mu^2\alpha_2 + \cdots.$$

Then, S will be expanded in positive powers of μ, as well as in negative and positive powers of

$$\alpha_0 + \mu\alpha_1 + \mu^2\alpha_2 + \cdots.$$

If α_0 is not zero, these positive and negative powers can themselves be expanded in positive powers of μ such that, finally, S will have been expanded in positive powers of μ.

These expansions, according to the statements made in no. 125, are the same as those one obtains by starting from Eqs. (19.1) but assigning, to the constants C_p values differing from those given them above.

Now, instead of this, let us assume n_1^0 to be very small and let us replace n_1^0 by a series of the form

$$n_1^0 = \alpha_1\sqrt{\mu} + \alpha_2\,\mu + \alpha_3\,\mu\sqrt{\mu} + \cdots. \qquad (19.10)$$

This time, the negative powers of

$$\alpha_1\sqrt{\mu} + \alpha_2\,\mu + \cdots$$

can no longer be expanded in positive powers of $\sqrt{\mu}$, whereas

$$\frac{\mu^p}{(n_1^0)^{2p-1}}$$

can be expanded in positive powers of $\sqrt{\mu}$, and the series will start with a term in $\sqrt{\mu}$.

If, according to our above discussions, we now note that S can be expanded in powers of

$$\frac{\mu}{n_1^0}, \quad \frac{\mu^2}{(n_1^0)^3}, \quad \frac{\mu^p}{(n_1^0)^{2p-1}}, \quad \mu, \quad n_1^0,$$

it can be concluded that S is expandable in positive powers of $\sqrt{\mu}$.

The series obtained in this manner do not differ from those obtained in the preceding number by means of Eqs. (19.4) and by assigning diverse values to the constants C_p.

To avoid confusion, we will use

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \cdots \tag{19.11}$$

for representing the series obtained when starting from Eqs. (19.9), in which the constants C_p—as mentioned above—are determined in such a manner that the mean value of $\Phi + C_p$ becomes zero.

For the moment, let us represent by

$$S = T_0 + \sqrt{\mu}\, T_1 + \mu T_2 + \mu\sqrt{\mu}\, T_3 + \cdots \tag{19.12}$$

what one obtains on replacing n_1^0 in Eq. (19.11) by its expansion (19.10) and by arranging the series in powers of $\sqrt{\mu}$.

What do the terms T_0, T_1, etc. then represent?

The term T_0 will be obtained on replacing the constant n_1^0 in S_0 by zero.

The term T_1 will be obtained in the following manner: To show explicitly that S_0 depends on n_1^0 we write $S_0(n_1^0)$. We have found

$$S_0(n_1^0) = x_1^0\, y_1,$$
$$S_0(0) = T_0\,.$$

This yields

$$S_0(n_1^0) = S_0(\alpha_1\sqrt{\mu} + \alpha_2\,\mu + \cdots) = T_0 + \frac{dS_0}{dn_1^0}\sqrt{\mu}\,\alpha_1 + \cdots$$

or

$$S_0(n_1^0) = T_0 + \alpha_1 \frac{dx_1^0}{dn_1^0} y_1\sqrt{\mu} + \cdots = T_0 - \frac{\alpha_1\sqrt{\mu}}{F_0''} y_1 + \cdots,$$

where F_0'' has the same significance as in Eqs. (19.4) of the preceding number.

On the other hand, in T_1 we will have terms arising from $S_1, S_2, \ldots$; these are obtained in the following manner:

In series (19.11), we will take all terms in

$$\frac{\mu^p}{(n_1^0)^{2p-1}}.$$

Let

$$S_1' \frac{\mu}{n_1^0} + S_2' \frac{\mu^2}{(n_1^0)^3} + S_3' \frac{\mu^3}{(n_1^0)^5} + \cdots, \tag{19.13}$$

be the collection of these terms.

We will then have

$$T_1 = -\frac{\alpha_1 y_1}{F_0''} + \frac{S_1'}{\alpha_1} + \frac{S_2'}{\alpha_1^3} + \frac{S_3'}{\alpha_1^5} + \cdots$$

From this it follows that, if one groups all terms in $\mu^p/(n_1^0)^{2p-1}$ in series (19.11), i.e., all those terms belonging to expansion (19.13), and if one forms the square of

$$\frac{dT}{dy_1} = -\frac{\alpha_1}{F_0''} + \frac{1}{\alpha_1}\frac{dS_1'}{dy_1} + \frac{1}{\alpha_1^3}\frac{dS_2'}{dy_1} + \cdots,$$

then this square will reduce to two terms

$$\left(\frac{dT_1}{dy_1}\right)^2 = \frac{\alpha_1^2}{F_0''^2} - \frac{2}{F_0''}\frac{dS_1'}{dy_1}.$$

This result is even more remarkable in that it can be extended, as will be demonstrated below, to all equations of dynamics.

To obtain T_2, it is necessary to consider not only S_0 and all terms in

$$\frac{\mu^p}{(n_1^0)^{2p-1}},$$

but also terms in

$$\frac{\mu^p}{(n_1^0)^{2p-2}}, \quad \ldots .$$

To summarize, in passing from the case where the methods of no. 125 are applicable to the case where they no longer are applicable one proceeds as follows: When n_1^0 is very small, the order of magnitude of a given term no longer depends only on the exponent of μ but also on that of n_1^0. If one supposes that n_1^0 is of the same order as $\sqrt{\mu}$, one combines all terms which thus become of the same order and sums them.

202. All these results can be extrapolated directly to the more general case which we had considered at the beginning of no. 199.

Let us first assume that F depends on $x_1, x_2, \ldots, x_n$, and y_1. We next will have to consider the equation

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, y_1\right) = \text{const.} \tag{19.14}$$

For integrating this equation, we will assign to

$$\frac{dS}{dy_2}, \quad \ldots, \quad \frac{dS}{dy_n}$$

any constant values

$$x_2^0, \quad \ldots, \quad x_n^0,$$

yielding an equation

$$F\left(\frac{dS}{dy_1}, x_2^0, \ldots, x_n^0, y_1\right) = C,$$

of the same form as that discussed in the two preceding numbers.

However, the solution S, instead of containing only one arbitrary constant x_1^0, will contain n such constants, which will be $x_1^0, x_2^0, \ldots, x_n^0$.

If now the fundamental equation is written as

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, m_1 y_1 + m_2 y_2 + \cdots + m_n y_n\right) = C, \quad (19.15)$$

it will be easy to restore it to the form of Eq. (19.14). In fact, let us put

$$\begin{aligned} m_1 y_1 + m_2 y_2 + \cdots + m_n y_n &= y_1', \\ m_1^2 y_1 + m_2^2 y_2 + \cdots + m_n^2 y_n &= y_2', \\ &\vdots \\ m_1^n y_1 + m_2^n y_2 + \cdots + m_n^n y_n &= y_n', \end{aligned} \quad (19.16)$$

where m_i^k are integers chosen in such a manner that the determinant of the coefficients of Eqs. (19.16) becomes equal to 1. This is always possible provided that $m_1, m_2, \ldots, m_n$ are relatively prime, which it is always permissible to assume.

The partial differential equation (19.15) will then become

$$F\left(\frac{dS}{dy_1'}, \frac{dS}{dy_2'}, \ldots, \frac{dS}{dy_n'}, y_1'\right) = C,$$

and thus is reduced to the form of Eq. (19.14).

All statements made with respect to equations of the form (19.14) can be applied also to equations of the form (19.15).

We can find solutions of Eq. (19.15) which are expandable, like those of Eq. (19.14), at times in powers of μ and at times in powers of $\sqrt{\mu}$.

For $\mu = 0$, S reduces to

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n.$$

The complete solution of the partial differential equation (19.15) must contain n arbitrary constants. As arbitrary constants, we could take x_1^0, $x_2^0, \ldots, x_n^0$ or else $n_1^0, n_2^0, \ldots, n_n^0$ by putting

$$n_i^0 = -\frac{dF_0}{dx_i^0}.$$

However, it is more convenient to introduce an infinite number of arbitrary constants among which only n distinct constants will exist. These constants will be

$$n_1^0, \quad n_2^0, \quad \ldots, \quad n_n^0, \quad C_0, \quad C_1, \quad C_2, \quad \ldots, \quad C_p, \quad \ldots,$$

by equating the right-hand side of Eq. (19.15) to

$$C = C_0 + C_1 \mu + C_2 \mu^2 + \cdots.$$

If

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0 \gtrless 0,$$

then S can be expanded in powers of μ; if, conversely,

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0 = 0,$$

then S can be expanded in powers of $\sqrt{\mu}$.

Specifically, let us assume that, assigning any values to $n_1^0, n_2^0, \ldots, n_n^0$, one chooses the constants C_p such that

$$S - n_1^0\, y_1 - n_2^0\, y_2 - \cdots - n_n^0\, y_n$$

becomes a periodic function of y. This returns us to an expansion that corresponds to that of the beginning of the preceding numbers, derived from Eqs. (19.14) of the present number.

In this expansion, various powers of

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0,$$

will enter the denominator.

Let us next replace the constants of integration n_i^0 by various expansions in powers of $\sqrt{\mu}$.

For example, let

$$n_i^0 = \alpha_i^0 + \sqrt{\mu}\, \alpha_i^1 + \mu \alpha_i^2 + \cdots.$$

We assume that

$$m_1 \alpha_1^0 + m_2 \alpha_2^0 + \cdots + m_n \alpha_n^0 = 0.$$

From this it results that the expansion of

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

will start with a term in $\sqrt{\mu}$.

If the terms of S are then arranged in positive and ascending powers of $\sqrt{\mu}$, we obtain various series analogous to those studied in detail in no. 201.

203. It is now easy to understand the basic principle of the Delaunay method.

Let us return to the general case of the equations of dynamics. Consequently, let us assume that our function

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$$

no longer depends only on $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$ but also on n arguments $y_1, y_2, \ldots, y_n$ and that it is periodic with respect to these arguments.

If none of the linear combinations with integer coefficients

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

is very small, the methods of no. 125 can be applied without difficulty. However, if one of these combinations is very small, one separates in F the terms depending on the argument

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n .$$

Here, F is assumed as expanded in a trigonometric series, i.e., in a sequence of terms each of which is the product of

$$\cos(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n)$$

or

$$\sin(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n)$$

(p being integers) and coefficients that are functions of $x_1, x_2, \ldots, x_n$.

Let us consider the terms that are such that

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \cdots = \frac{p_n}{m_n},$$

and let

$$F' = F_0' + \mu F_1' + \cdots$$

be the ensemble of these terms.

These terms will specifically comprise all terms of F that are independent of $y_1, y_2, \ldots, y_n$ and, for example, all terms of F_0, such that

$$F_0' = F_0.$$

Let us now consider the equation

$$F'\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, m_1 y_1 + m_2 y_2 + \cdots + m_n y_n\right) = C.$$

This can be readily integrated with the aid of the procedures discussed in the first portions of this chapter.

Let

$$S = x_1' y_1 + x_2' y_2 + \cdots + x_n' y_n + S'$$

be one of the solutions of this equation. The coefficients $x_1', x_2', \ldots, x_n'$ are the constants of integration which had been denoted until now by x_1^0 but which will here be denoted by x_1' since they will later be used as new independent variables.

So far as S' is concerned, this is a periodic function of

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n ,$$

which, in addition, depends on $x_1', x_2', \ldots, x_n'$, in such a manner that the mean-square value of dS/dy_1 is nothing else but x_1' and that the expression of S considered here does not differ from that obtained from Eqs. (19.14) of no. 202.

Let us now put

$$x_i = \frac{dS}{dy_i}, \quad y_i' = \frac{dS}{dx_i'}.$$

Let us take x_i' and y_i' as new variables. The canonical form of the equations will not be altered. The function F, expressed as a function of x_i' and of y_i' will retain its form; only the coefficients of the terms in

$$m_1 y_1' + m_2 y_2' + \cdots + m_n y_n'$$

will be much smaller than those of the corresponding terms in

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n.$$

The long-period inequalities will have vanished since, in all, they have been taken into consideration ever since the first approximation.

Bohlin Method

204. The drawback of the Delaunay method is the fact that it requires numerous changes of variables. This inconvenience can be avoided by a process devised by Bohlin, which I had also suggested, but a few days after Bohlin.

Let us return to our general equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{19.17}$$

and let us assume that the expression

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

is very small.

It is now a question of integrating the equation

$$F\left(\frac{dS}{dy_1}, \frac{dS}{dy_2}, \ldots, \frac{dS}{dy_n}, y_1, y_2, \ldots, y_n\right) = C. \tag{19.18}$$

Let us put

$$S = S_0 + S_1\sqrt{\mu} + S_2\,\mu + S_3\,\mu\sqrt{\mu} + \cdots,$$
$$C = C_0 + C_2\,\mu + C_4\,\mu^2 + \cdots.$$

Let us substitute these values into Eq. (19.18), arrange it in powers of $\sqrt{\mu}$, and equate the coefficients of all similar powers of $\sqrt{\mu}$. From this we obtain

$$
\begin{gathered}
F_0\left(\frac{dS_0}{dy_1}, \frac{dS_0}{dy_2}, \ldots, \frac{dS_0}{dy_n}\right) = C_0\,; \\
\sum \frac{dF_0}{dx_i}\frac{dS_1}{dy_i} = 0, \\
\sum \frac{dF_0}{dx_i}\frac{dS_2}{dy_i} + \frac{1}{2}\sum \frac{d^2F_0}{dx_i dx_k}\frac{dS_1}{dy_i}\frac{dS_1}{dy_k} = \Phi + C_2, \\
\sum \frac{dF_0}{dx_i}\frac{dS_3}{dy_i} + \frac{1}{2}\sum \frac{d^2F_0}{dx_i dx_k}\frac{dS_1}{dy_i}\frac{dS_2}{dy_k} = \Phi, \\
\sum \frac{dF_0}{dx_i}\frac{dS_4}{dy_i} + \frac{1}{2}\sum \frac{d^2F_0}{dx_i dx_k}\frac{dS_1}{dy_i}\frac{dS_3}{dy_k} = \Phi + C_4, \\
\vdots
\end{gathered}
\tag{19.19}
$$

The meaning of these equations is as follows:

I still denote by Φ any known function, and I assume: in the third equation of system (19.19) that S_0 is known; in the fourth equation of system (19.19) that S_0 and S_1 are known; in the fifth equation of system (19.19) that S_0, S_1, and S_2 are known.

The right-hand side contains sometimes Φ and sometimes $\Phi + C_{2p}$ since we had assumed that the constants C with odd subscripts, i.e., the coefficients of odd powers of $\sqrt{\mu}$ in the expansion of C, are zero.

It is then necessary to define the meaning of the sign Σ in the second term on the left-hand side of the various equations of system (19.19). This sign extends over both indices i and k. It must be agreed that, in the third equation of system (19.19), the combination (i,k) appears twice if $i \gtrless k$ and once if $i = k$ and that, in the other equations of system (19.19), this combination appears twice in all cases.

As above, we will assume

$$\frac{dS_0}{dy_i} = x_i^0,$$

where x_i^0 are constants. In the derivatives of F_0 appearing in the equations of the system (19.19), it has been assumed that x_i have been replaced by x_i^0 in such a manner that

$$\frac{dF_0}{dx_i} = -n_i^0.$$

We will assume in addition that x_i^0 can be chosen such that

$$\sum m_i n_i^0 = 0 \tag{19.20}$$

and that no other linear combination with integral coefficients exists among the n_i^0.

Let us try to determine S in such a manner that

$$\frac{dS_p}{dy_i}$$

will be periodic functions of y_i.

The first equation of system (19.19) simply determines C_0; the second equation is written as

$$\sum n_i^0 \frac{dS_1}{dy_i} = 0, \tag{19.21}$$

and can be satisfied only if the dS_1/dy_i are functions of $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$ alone. If S_1, for example, were to contain a term

$$A \cos(p_1 y_1 + \cdots + p_n y_n),$$

then the left-hand side of Eq. (19.21) would contain a term

$$-A(p_1 n_1^0 + \cdots + p_n n_n^0)\sin(p_1 y_1 + \cdots + p_n y_n)$$

which could not vanish unless

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \cdots = \frac{p_n}{m_n}.$$

Consequently, we will have

$$S_1 = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + f(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n),$$

where the derivative of f is periodic.

Let us now pass to the third equation of system (19.19) and, let us equate all terms on the two sides of this equation that depend on the sines and cosines of multiples of

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n.$$

The first term on the left-hand side, which can be written as

$$-\sum n_i^0 \frac{dS_2}{dy_i},$$

will not contain such terms; if S_2 contained a term

$$A \cos(p_1 y_1 + \cdots + p_n y_n),$$

where

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \cdots = \frac{p_n}{m_n},$$

then the corresponding term of the expression

$$-\sum n_i^0 \frac{dS_2}{dy_i}$$

would be written as

$$\sum A(p_1 n_1^0 + \cdots + p_n n_n^0)\sin(p_1 y_1 + \cdots + p_n y_n)$$

and would vanish because of relation (19.21).

The second term on the left-hand side, conversely, depends only on S_1 and is a function of $m_1 y_1 + \cdots + m_n y_n$ alone. Thus all these terms contain only sines or cosines of multiples of

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n.$$

Let us now introduce a new notation:

Let U be some function whose derivatives dU/dy_i are periodic functions of U. This can be expanded in a series all of whose terms will have one of the following forms:

$$\alpha_i y_i, \quad \alpha \cos(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n),$$
$$\alpha \sin(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n).$$

Let us eliminate, in this series, all trigonometric terms except those for which

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \cdots = \frac{p_n}{m_n}.$$

The remaining ensemble of terms could be denoted by $[U]$ and could be called the mean-square value of U.

Then, we will have

$$\left[\frac{dU}{dy_i}\right] = \frac{d[U]}{dy_i}; \quad \sum n_i^0 \frac{d[U]}{dy_i} = \text{const.}$$

and, if V is some periodic function,

$$\left[[V]\frac{dU}{dy_i}\right] = [V]\left[\frac{dU}{dy_i}\right].$$

Thus we obtain

$$-\left[\sum n_i^0 \frac{dS_2}{dy_i}\right] = \text{const.},$$
$$\left[\frac{dS_1}{dy_i}\frac{dS_1}{dy_k}\right] = \frac{dS_1}{dy_i}\frac{dS_1}{dy_k}, \tag{19.22}$$
$$[\Phi] = -[F_1].$$

In F_1, we assume that the terms x_i have been replaced by x_i^0. The function Φ entering the third equation of system (19.22) is that of the third equation of system (19.19).

The constant on the right-hand side of the first equation of system (19.22) can be denoted by $C_2 - C_2'$.

Then, equating the mean values on both sides of the third equation of system (19.19), we will find

$$\tfrac{1}{2}\sum \frac{d^2F_0}{dx_i dx_k}\frac{dS_1}{dy_i}\frac{dS_1}{dy_k} = C_2' - [F_1]. \tag{19.23}$$

This equation has the same form as those studied in nos. 199–202 and, specifically, the same form as the second equation of system (19.4) in no. 200.

Therefore, as in the case of this second equation of system (19.4), we will obtain three different cases.

Let us recall that S_1 is of the form

$$S_1 = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + f(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n),$$

whence

$$\frac{dS_1}{dy_i} = \alpha_i + m_i f'.$$

Let us substitute this value of dS_1/dy_i into Eq. (19.23). This equation then becomes an equation of the second degree with respect to f', so that we can write

$$Af'^2 = 2Bf' + D = C_2' - [F_1], \tag{19.24}$$

where A, B, and D are constants depending on the constants α_i. These latter constants α can be arbitrarily chosen.

So that f' and thus also dS_1/dy may be periodic functions of $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$, it is necessary and sufficient that Eq. (19.24) always have its roots real, i.e., that the inequality

$$B^2 - AD + AC_2' - A[F_1] > 0,$$

be satisfied for all values of

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n.$$

Since the constants α_i are arbitrary, we will take

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0. \tag{19.25}$$

This does not restrict the generality, as will be shown below.

Incidentally, we could obtain exactly the same result by assuming

$$\frac{\alpha_1}{m_1} = \frac{\alpha_2}{m_2} = \cdots = \frac{\alpha_n}{m_n},$$

since, if this condition is satisfied, the expression

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$$

becomes a function of $m_1 y_1 + \cdots + m_n y_n$ alone, which can be included in f.

However this might be, if the conditions (19.25) are assumed as satisfied, Eq. (19.24) is simplified and can be written as

$$Af'^2 = C_2 - [F_1]. \qquad (19.24a)$$

Let us then assume that one constructs curves for various values of the constant C_2' by taking as radius vector f' plus any constant and as polar angle

$$m_1 y_1 + m_2 y_2 + \cdots + m_n y_n ;$$

one obtains a pattern completely similar to that in Fig. 3.

To be specific let us assume that A is positive. Then, if f' is to be periodic, it must it remain real, meaning that C_2' must be larger than the maximum of $[F_1]$.

In this case, f' and thus also dS_1/dy_i is a periodic function of $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$ which never vanishes.

Having thus determined S_1 it is now a question of determining S_2. This function must be of the form

$$\alpha_1^2 y_1 + \alpha_2^2 y_2 + \cdots + \alpha_n^2 y_n + \varphi,$$

where φ is periodic; in general, S_p must be of the form

$$\alpha_1^p y_1 + \alpha_2^p y_2 + \cdots + \alpha_n^p y_n + \varphi,$$

where φ is periodic. For simplification, we will assume

$$\frac{\alpha_1^p}{m_1} = \frac{\alpha_2^p}{m_2} = \cdots = \frac{\alpha_n^p}{m_n}, \qquad (19.26)$$

which, as demonstrated below, will not restrict the generality.

We have

$$-\left[\sum n_i^0 \frac{dS_p}{dy_i}\right] = C_p' - C_p, \qquad (19.27)$$

which is an equation analogous to the first equation of system (19.22). If conditions (19.26) are satisfied, we will have $C_p' = C_p$ and, specifically, $C_2' = C_2$.

After this, let us return to the third equation of system (19.19) which, now that we have C_2 and that S_1 is wholly determined, can be written in the form

$$\sum n_i^0 \frac{dS_2}{dy_i} = \Phi.$$

The known function Φ is periodic in $y_1, y_2, \ldots, y_n$.

Thus let

$$\Phi = \sum A \cos(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n + \beta),$$

so that Eq. (19.27) will yield

$$S_2 = \sum \frac{A \sin(p_1 y_1 + p_2 y_2 + \cdots + p_n y_n + \beta)}{p_1 n_1^0 + p_2 n_2^0 + \cdots + p_n n_n^0} + \psi(m_1 y_1 + m_2 y_2 + \cdots + m_n y_n),$$

where ψ is an arbitrary function of $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$. This solution will become illusory if, for any term of Φ, we had

$$p_1 n_1^0 + p_2 n_2^0 + \cdots + p_n n_n^0 = 0,$$

i.e.,

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \cdots = \frac{p_n}{m_n}.$$

However, this cannot happen since

$$[\Phi] = 0.$$

In fact, we have exactly determined S_1 in such a manner that the mean values of both sides of the third equation of system (19.19) are equal. This must thus be the same for both sides of Eq. (19.27) which differs from the third equation of system (19.19) only by the fact that certain terms have been transposed from one side to the other.

However,

$$\left[\sum n_i^0 \frac{dS_2}{dy_i}\right] = 0,$$

since $C_2' = C_2$.

Consequently,

$$[\Phi] = 0. \qquad \text{Q.E.D.}$$

To find the value of S_2, it is necessary to determine the arbitrary function

$$\psi = [S_2].$$

For this purpose, let us equate the mean values on both sides of the fourth equation of system (19.19). Because of relations (19.26), we obtain

$$\left[\sum n_i^0 \frac{dS_3}{dy_i}\right] = 0,$$

and, in addition,

$$\left[\frac{d^2F_0}{dx_i dx_k} \frac{dS_1}{dy_i} \frac{dS_2}{dy_k}\right] = \frac{d^2F_0}{dx_i dx_k} \frac{dS_1}{dy_i} \frac{d\,[S_2]}{dy_k},$$

since S_1 depends only on $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$. Thus we have

$$\tfrac{1}{2} \sum \frac{d^2 F_0}{dx_i dx_k} \frac{dS_1}{dy_i} \frac{d[S_2]}{dy_k} = [\Phi].$$

If, by $[S_2']$, we denote the derivative of $[S_2]$ with respect to

$$m_1 y_1 + \cdots + m_n y_n,$$

we obtain

$$\frac{dS_1}{dy_i} = m_i f', \quad \frac{d[S_2]}{dy_k} = m_k [S_2'],$$

and can write

$$\frac{1}{2} \sum \frac{d^2 F_0}{dx_i dx_k} m_i m_k [S_2'] = \frac{[\Phi]}{f'}.$$

Since f' does not vanish, $[S_2']$ is a periodic function of $m_1 y_1 + \cdots + m_n y_n$ which does not become infinite, while $[S_2]$ will have the form

$$a(m_1 y_1 + \cdots + m_n y_n) + \psi,$$

where a is a constant coefficient and ψ is a series developed in sines and cosines of multiples of $m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$.

Since S_2 is thus wholly determined, the fourth equation of system (19.19) can be written as

$$\sum n_i^0 \frac{dS_3}{dy_i} = \Phi;$$

it takes a form completely analogous to that of Eq. (19.27) and can be treated in exactly the same manner; and so on.

We stated above that the hypotheses (19.25) and (19.26) do not restrict the generality.

In fact, let us consider a solution of our fundamental equation conforming to these hypotheses (19.25) and (19.26). Let S be this solution and let

$$S = S_0 + \sqrt{\mu} S_1 + \mu S_2 + \cdots.$$

Moreover, let

$$S_0 = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n,$$

and

$$S_p = \alpha_1^p y_1 + \alpha_2^p y_2 + \cdots + \alpha_n^p y_n + \text{periodic function}.$$

By virtue of the hypotheses (19.25) and (19.26), the terms α satisfy the condition

$$\frac{\alpha_1^p}{m_1} = \frac{\alpha_2^p}{m_2} = \cdots = \frac{\alpha_n^p}{m_n}$$

and, in addition, are functions of the integration constants x_i^0 and C_p.

Since the x_i^0 are arbitrary constants, they can be replaced by any expansions

$$x_i^0 = \beta_i^0 + \sqrt{\mu}\,\beta_i^1 + \mu\,\beta_i^2 + \cdots,$$

where β_i^p are new arbitrary constants.

If, in S, we replace the x_i^0 by these expansions and then again arrange the series with respect to powers of $\sqrt{\mu}$, we obtain

$$S = S_0' + \sqrt{\mu} S_1' + \mu S_2' + \cdots,$$

where

$$S_p' = \alpha_1'^p y_1 + \alpha_2'^p y_2 + \cdots + \alpha_n'^p y_n + \text{periodic function},$$

which means that we will have been able to select β_i^p in such a manner that the constants $\alpha_i'^p$ are any constants.

Consequently, our hypotheses have produced no basic restriction of the generality. Q.E.D.

Case of Libration

205. What will happen now if C_2 is not larger than the maximum of $[F_1]$ and if, consequently, S_1 is not always real? In these cases, in which one says that there is libration, certain difficulties occur which can be overcome by an artifice analogous to that used for the elliptic functions in no. 199. To simplify the presentation somewhat, we will assume that

$$m_1 = 1, \quad m_2 = m_3 = \cdots = m_n = 0.$$

We are entitled to do so since, if this were not the case, a change of variable analogous to the change [Eq. (19.16)] in no. 202 could be made.

It is no longer possible to arrange matters such that dS_p/dy_i are periodic functions of $y_1, y_2, \ldots, y_n$. However, we can at least attempt to find a function S such that the terms dS_p/dy_i be periodic functions of

$$y_2, \quad y_3, \quad \ldots, \quad y_n.$$

Then, what we denoted by $[U]$ in the foregoing number is nothing else but the mean value of U, considered as a periodic function of $y_2, y_3, \ldots, y_n$.

This will yield

$$\left[\sum n_i^0 \frac{dS_p}{dy_i}\right] = \text{const}, \tag{19.28}$$

and, in fact,

$$\left[\frac{dS_p}{dy_2}\right], \quad \left[\frac{dS_p}{dy_3}\right], \quad \ldots, \quad \left[\frac{dS_p}{dy_n}\right]$$

will reduce to constants, while, on the other hand, the relation

$$\sum m_i n_i^0 = 0$$

reduces here to

$$n_0^1 = 0,$$

such that the left-hand side of Eq. (19.28) contains no term in $[dS_p/dy_1]$.

It can be assumed that not only the terms dS_p/dy_1 but also the S_p (at least for $p > 0$) are periodic funcitons of $y_2, y_3, \ldots, y_n$. This is a hypothesis identical to the hypotheses (19.25) and (19.26) of the preceding number which, as we have seen, do not restrict the generality. If this is admitted, the constant on the right-hand side of Eq. (19.28) will be zero.

After this, let us return to Eqs. (19.19) of the foregoing number. The second equation of this system indicates that S_1 depends only on y_1 while the first equation, on equating the mean values on both sides, yields

$$\frac{1}{2}\frac{d^2F_0}{dx_1^2}\left(\frac{dS_1}{dy_1}\right)^2 = C_2 - [F_1], \tag{19.29}$$

which determines S_1.

Taking Eq. (19.29) into consideration, the third equation of system (19.19) will become

$$-\sum n_i^0 \frac{dS_2}{dy_i} = [F_1] - F_1. \tag{19.30}$$

Since the right-hand side is a function of $y_1, y_3, \ldots, y_n$ whose mean value is zero, the application of an integration process, a procedure we have often made use of, will yield S_2 to within an arbitrary function of y_1, meaning that Eq. (19.30) will define

$$S_2 - [S_2].$$

For determining $[S_2]$, let us take the fourth equation of system (19.19) and let us equate the mean values on both sides, so that

$$\frac{d^2F_0}{dx_1^2}\frac{d[S_2]}{dy_1}\frac{dS_1}{dy_1} = [\Phi]. \tag{19.31}$$

From this, we will derive the value of $[S_2]$.

Knowing S_2 and taking Eq. (19.31) into consideration, we can then write the fourth equation of the system (19.19) in the form

$$-\sum n_i^0 \frac{dS_3}{dy_i} = \Phi.$$

Since the mean value of Φ is zero, this equation, which is of the same form as Eq. (19.30), can be treated in the same manner, yielding

$$S_3 - [S_3]$$

and so on.

It is obvious that the functions dS_p/dy_i determined in this manner are uniform functions of y_1 and of $\sqrt{C_2 - [F_1]}$.

206. To study our functions more fully, a change of variables must be made. For this, let us introduce an auxiliary function T, defined in the following manner: We will have

$$T = T_0 + T_1\sqrt{\mu} + T_2\,\mu$$

and

$$T_0 = x_1^0\, y_1 + x_2^0\, y_2 + \cdots + x_n^0\, y_n,$$

where x_i^0 will be constants that satisfy the conditions

$$F_0 = C_0, \quad n_1^0 = 0.$$

In other words, T_0 will be nothing else but what we had previously called S_0.

To define T_1, we will start from the same equation that had served for defining S_1, namely, Eq. (19.23) of no. 204 in which we will replace S_1 by T_1 and C_2' by C_2, which yields

$$\frac{1}{2}\sum \frac{d^2F_0}{dx_i^0 dx_k^0}\frac{dT_1}{dy_i}\frac{\alpha T_1}{dy_k} = C_2 - [F_1]. \qquad (19.23a)$$

To this, we will add the following equations (where x_1' are constants):

$$\frac{dT_1}{dy_i} = x_i' \quad (i = 2,3, \ldots ,n).$$

It should be noted here that, in postulating this latter hypothesis, T_1 is defined in the same manner as that used for defining S_1 above but deviating from the hypotheses (19.25) which stipulated that the constants x_i' must be zero.

Since the coefficients $d^2F_0/dx_i^0\, dx_k^0$ depend only on x_i^0, they must be constants. Thus, if we replace the terms dT_1/dy_i by x_i', then Eq. (19.23a) will become

$$A\left(\frac{dT_1}{dy_1}\right)^2 + 2B\frac{dT_1}{dy_1} + D = C_2 - [F_1], \tag{19.24b}$$

where A is a constant while B and D are two polynomials homogeneous with respect to x_i', with the first one being of the first degree and the second one of the second degree. From this, we derive

$$\frac{dT_1}{dy_1} = -\frac{B}{A} \pm \sqrt{\frac{B^2}{A^2} - \frac{D}{A} + \frac{C_2}{A} - \frac{[F_1]}{A}}.$$

We will then put

$$\frac{B^2}{A^2} - \frac{D}{A} + \frac{C_2}{A} = x_1',$$

and, for abbreviation,

$$[F_1] = A\psi,$$

whence

$$T_1 = x_2' y_2 + x_3' y_3 + \cdots + x_n' y_n - \frac{By_1}{A} + \int dy_1 \sqrt{x_1' - \psi}.$$

Next, we determine T_2 by the equation

$$\sum n_i^0 \frac{dT_2}{dy_i} = F_1 - [F_1],$$

analogous to Eq. (19.27) of no. 204.

This equation determines T_2, as indicated above, to within an arbitrary function of y_1. Without inconvenience, we could make any choice here. For example, let us assume

$$[T_2] = 0.$$

We will put

$$\sum n_i^0 x_i' = -C_1.$$

From this, the following will result:

(i) T_2 is a periodic function of y_i which does not depend on x_i', since this is the case for F_1 and for $[F_1]$ where we simply had assumed that the x_i were replaced by the constants x_i^0.

(ii) If, on the left-hand side of Eq. (19.18) of no. 204, the term S is replaced by T, this side will reduce to

$$C_0 + C_1\sqrt{\mu} + C_2\mu,$$

to within terms containing $\mu\sqrt{\mu}$ as factor; this is so since the functions T_0, T_1, and T_2 satisfy the first three equations of system (19.19) except that, in the second of these equations, the zero on the right-hand side must be replaced by C_1.

Let us now put

$$\begin{aligned} x_1 &= \frac{dT}{dy_1} = x_1^0 + \sqrt{\mu}\left(-\frac{B}{A} + \sqrt{x_1' - \psi}\right) + \mu\frac{dT_2}{dy_1}, \\ y_1' &= \frac{dT}{dx_1'} = \frac{\sqrt{\mu}}{2}\int\frac{dy_1}{\sqrt{x_1' - \psi}}, \\ x_i &= \frac{dT}{dy_i} = x_i^0 + x_i'\sqrt{\mu} + \mu\frac{dT_2}{dy_i}, \\ y_i' &= \frac{dT}{dx_i'} = y_i\sqrt{\mu} - \frac{y_1\sqrt{\mu}}{A}\frac{dB}{dx_i'} \quad (i = 2,3, \ldots ,n). \end{aligned} \tag{19.32}$$

If x_i' and y_i' as new variables are used instead of x_i and y_i, then the canonical form of the equations will not be altered.

Let us first study the third equation of system (19.32) in which y_1', y_1, and x_1' enter. If x_1' is considered as a constant and if we only vary y_1', I say that y_1 is a periodic function of y_1'.

It is here that the analogy with the use of the elliptic functions of no. 199 becomes evident. In the specific case treated there, we had

$$A = 1, \quad [F_1] = \cos y_1,$$

such that our third equation of system (19.32) becomes

$$y_1' = \frac{\sqrt{\mu}}{2}\int\frac{dy_1}{\sqrt{C_2 - \cos y_1}}.$$

The integral on the right-hand side is an elliptic integral so that $\cos y_1$ and $\sin y_1$ are doubly periodic functions of y_1'. However, two cases must be distinguished depending on whether

$$C_2 > 1 \quad \text{or} \quad C_2 < 1.$$

If $C_2 > 1$, the real period will be equal to

$$\frac{\sqrt{\mu}}{2}\int_0^{2\pi}\frac{dy_1}{\sqrt{C_2 - \cos y_1}},$$

and if

$$C_2 = \cos\alpha < 1$$

then the real period will be equal to

$$\sqrt{\mu}\int_{-\alpha}^{+\alpha}\frac{dy_1}{\sqrt{C_2 - \cos y_1}}.$$

In this particular case, y_1 is a uniform function of y_1' for the imaginary values as well as for the real values of y_1'. However, in the general case, y_1 is a uniform function of y_1' for only the real values, and, furthermore, $\cos y_1$ and $\sin y_1$ admit of a real period which is

$$\frac{\sqrt{\mu}}{2}\int_0^{2\pi}\frac{dy_1}{\sqrt{x_1'-\psi}}$$

if x_1' is above to the maximum of $[\psi]$; and

$$\sqrt{\mu}\int_\alpha^\beta\frac{dy_1}{\sqrt{x_1'-\psi}}$$

if x_1' is below this maximum and if $x_1'-\psi$ vanishes for $y_1=\alpha$ and for $y_1=\beta$, remaining positive for $\alpha<y_1<\beta$. We should add here that, in the first case, y_1 increases by 2π when y_1' increases by one period, whereas in the second case, i.e., in the case of libration, y_1 resumes its original value when y_1' increases by one period.

In the specific case of no. 199, not only are $\cos y_1$ and $\sin y_1$ doubly periodic functions of y_1' but also $\sqrt{C_2-\cos y_1}$. As for

$$S_1=\int\sqrt{C_2-\cos y_1}\,dy_1,$$

this expression increases by a constant quantity when y_1' increases by one period.

Similarly, in the general case

$$\sqrt{x_1'-\psi}\quad\left(\text{and, consequently, }\frac{dT_1}{dy_1}\right)$$

is a periodic function of y_1'. This function, just as y_1, also depends on x_1' which plays a role analogous to that of the modulus in the case of elliptic functions.

Before going further, let us note that the period of these various periodic functions of y_1 is proportional to $\sqrt{\mu}$.

It results from this that, in the case of libration, x_1, x_i, y_1, and $y_i'-y_i\sqrt{\mu}$ are periodic functions of y_1'. In addition, x_1 and x_i depend on y_i, but these are periodic functions, with period 2π, of these $n-1$ variables.

Thus, if we express the old variables x_i and y_i as a function of the new variables x_i' and y_i', it is evident that the terms x_i, $\cos y_i$, and $\sin y_i$ will be periodic functions of y_i'. The same will thus also hold for F, which is periodic, with period 2π with respect to y_i.

The period will be equal to

$$\sqrt{\mu}\int_{\alpha}^{\beta}\frac{dy_1}{\sqrt{x_1'-\psi}}$$

for y_1' and $2\pi\sqrt{\mu}$ for y_i'. For abbreviation, we will put the period relative to y_1' equal to $P\sqrt{\mu}$. It is obvious that P is a function of x_1', just as the period of the elliptic functions is a function of the modulus.

If we put

$$y_i' = z_i\sqrt{\mu},$$

whence

$$z_1 = \tfrac{1}{2}\int\frac{dy_1}{\sqrt{x_1'-\psi}}, \quad x_i = y_i - \frac{y_1}{A}\frac{dB}{dx_i'}, \tag{19.32a}$$

then F will be a periodic function of z_i; the period will be P for z_1 and 2π for the other z_i; in addition, F will be a function of x_i'. This function will be expandable in powers of $\sqrt{\mu}$. The first three terms of the series

$$C_0 + C_1\sqrt{\mu} + C_2\,\mu$$

will be independent of z_i and will be functions only of x_i'. The first term C_0 is an absolute constant; by definition, C_1 is a linear function of x_i', independent of x_1'. Finally, we have

$$C_2 = Ax_1' + D - \frac{B^2}{A},$$

from which it follows that C_2 is a polynomial of the first order with respect to the other x_i'.

Let us now put

$$F = C_0 + F^*\sqrt{\mu},$$

so that our equations become

$$\frac{dx_i'}{dt} = \frac{dF^*}{dz_i}, \quad \frac{dz_i}{dt} = -\frac{dF^*}{dx_i'}. \tag{19.33}$$

Like the function F in no. 125, the function F^* is periodic with respect to the variables of the second series which here are the z_i.

However, two obstacles prevent the methods of no. 125 from being directly applicable to Eqs. (19.33).

(a) The function F^* actually is periodic with respect to z_i except that, with respect to z_1, the period no longer is 2π but P.

To overcome this first difficulty, it is sufficient to make a minor change of variables. Setting

$$u_1 = \int \frac{P\,dx_1'}{2\pi}, \quad v_1 = \frac{2\pi z_1}{P} \tag{19.34}$$

the equations remain canonical and are written as

$$\begin{aligned} \frac{du_1}{dt} &= \frac{dF^*}{dv_1}, \quad \frac{dn_1}{dt} = -\frac{dF^*}{du_1}, \\ \frac{dx_i'}{dt} &= \frac{dF^*}{dz_i}, \quad \frac{dz_i}{dt} = -\frac{dF^*}{dx_i'} \quad (i = 2,3, \ldots, n), \end{aligned} \tag{19.35}$$

and this time F^* is periodic, with period 2π with respect to

$$v_1, \quad z_2, \quad z_3, \quad \ldots, \quad z_n.$$

(b) If we set $\mu = 0$, then F^* reduces to C_1, and C_1 does not depend on all variables of the first series but only on

$$x_2', \quad x_3', \quad \ldots, \quad x_n',$$

since n_1^0 is zero. Consquently, we do not have the conditions of no. 125 but rather with those of no. 134. We will demonstrate that the conclusions of that number are applicable.

Indeed, the function corresponding to the function denoted by R in no. 134 is here designated by C_2. It is easy to see that C_2 depends on x_1', and thus also on u_1, and depends only on variables of the first series.

The conditions to render the theorem of no. 134 applicable here are thus fulfilled so that we can conclude that n functions

$$u_1, \quad x_2', \quad x_3', \quad \ldots, \quad x_n'$$

exist which depend on n variables

$$v_1, \quad z_2, \quad z_3, \quad \ldots, \quad z_n$$

and on n arbitrary constants, and which satisfy the following conditions:
(i) On substituting these in F^*, the function reduces to a constant.
(ii) The expression

$$u_1 dv_1 + x_2' dz_2 + x_3' dz_3 + \cdots + x_n' dz_n = dV$$

is an exact differential
(iii) These n functions are periodic, with period 2π with respect to

$$v_1, \quad z_2, \quad z_3, \quad \ldots, \quad z_n.$$

Thus let us consider u_1 and x_i' as functions of v_1 and of z_i, resulting in n relations between these $2n$ variables; let us then return to the old variables x_i and y_i over the intermediary of Eqs. (19.32), (19.32a), and (19.34). In this manner, we will obtain n relations between x_i and y_i. Solving these relations for x_i will yield the terms x_i as a function of y_i, and it is clear that:

(A) If, in F, the terms x_i are replaced by their values as a function of y_i, then F will reduce to a constant.

(B) The expression

$$\sum x_i dy_i = dS \tag{19.36}$$

is an exact differential.

This is so since, according to the form of Eqs. (19.32), (19.32a), and (19.34), the difference

$$dS - \sqrt{\mu}\, dV$$

always is an exact differential.

(C) If the terms x_i are expressed as functions of v_1 and of z_i, then the x_i will be periodic functions of these variables. Similarly, if the x_i are expressed as functions of y_i, these functions will be periodic, with period 2π with respect to $y_2, y_3, \ldots, y_n$.

It results from this that the functions S, defined by Eq. (19.36), do not differ from those discussed in the preceding number since, in their definition, we only used Eq. (19.18) of no. 204 and only the condition that the quantities dS/dy_i be periodic with respect to $y_2, y_3, \ldots, y_n$.

Thus the two systems of equations

$$\begin{gathered} u_1 = \frac{dV}{dv_1}, \quad x_i' = \frac{dV}{dz_i}, \quad u_1 = \int \frac{P\,dx_1'}{2\pi}, \quad v_1 = \frac{2\pi z_1}{P}, \\ x_k = \frac{dT}{dy_k}, \quad z_k = \frac{dT_1}{dx_k'} \quad (i = 2,3,\ldots,n), \quad (k = 1,2,\ldots,n) \end{gathered} \tag{19.37}$$

and

$$x_k = \frac{dS}{dy_k} \tag{19.38}$$

are identical, provided that V satisfies the partial differential equation

$$F^*\left(\frac{dV}{dv_1}, \frac{dV}{dz_i}, v_1, z_i\right) = \text{const}, \tag{19.39}$$

and the condition that its derivatives are periodic with respect to v_1 and z_i and that S is defined as in the preceding number.

Here, V can be expanded in powers of $\sqrt{\mu}$ and can be written as

$$V = V_0 + \sqrt{\mu} V_1 + \mu V_2 + \cdots.$$

Each of the functions V_i can be written in the form

$$V_i = \beta_i^1 v_1 + \beta_i^2 z_2 + \cdots + \beta_i^n z_n + V_i',$$

where V_i' is periodic and where the n constants β_i^k, analogous to the constants $\alpha_{i\cdot k}$ of no. 125, can be arbitrarily selected. Similarly, we have

$$S = S_0 + \sqrt{\mu} S_1 + \cdots$$

and we have seen that S_p still depends on arbitrary constants, which we have previously called α_i^p.

So that the two systems (19.37) and (19.38) be identical, it is naturally necessary—if determinate values are assigned to the constants β_i^k—that corresponding values must be assigned to the constants α_i^p and vice versa.

Thus to each function V there corresponds a function S and vice versa.

However, in the previous numbers, we have subjected our constants α_i^p and thus also S to certain conditions, which are hypotheses (19.25) and (19.26). If one wishes to remain restricted to this, it is necessary that the constants β_i^k also satisfy certain conditions which are easy to formulate. We will state here only that the terms x_i' must vanish with $\sqrt{\mu}$.

Equations (19.37) and (19.38) permit us to express all our variables as functions of any n of these, so that we can assume that y_1 and x_k are expressed as functions of

$$v_1, \quad y_2, \quad y_3, \quad \ldots, \quad y_n.$$

Consequently, let

$$y_1 = \theta(v_1, y_2, y_3, \ldots, y_n),$$
$$x_k = \zeta_k(v_1, y_2, y_3, \ldots, y_n).$$

It is quite obvious that the functions θ and ζ_k are periodic, with period 2π with respect to each of the n variables on which they depend.

Considering, for the moment, $y_2, y_3, \ldots, y_n$ as constants and x_1 and y_1 as coordinates of a point in a plane, we can visualize the equations

$$y_1 = \theta(v_1), \quad x_1 = \frac{dS}{dy_1} = \zeta_1(v_1).$$

As we vary v_1, the point x_1, y_1 will describe a closed curve, since the functions θ and ζ_1 resume their original values when v_1 increases by 2π.

Thus, if $y_2, y_3, \ldots, y_n$ are regarded as constants, the equation

$$x_1 = \frac{dS}{dy_1}$$

will be that of a closed curve.

This is exactly the result at which we wished to arrive. However, it is of importance to define its significance. Indeed, we must not forget that all preceding theorems are valid, but only from the viewpoint of formal calculus.

The functions θ and ζ_k can be expanded in powers of $\sqrt{\mu}$, such that we can write

$$y_1 = \theta_0(v_1) + \sqrt{\mu}\theta_1(v_1) + \mu\theta_2(v_1) + \cdots,$$
$$x_1 = \zeta_1^0(v_1) + \sqrt{\mu}\zeta_1^1(v_1) + \mu\zeta_1^2(v_1) + \cdots, \tag{19.40}$$

and all functions $\theta_p(v_1)$ and $\zeta_1^p(v_1)$ are periodic, with period 2π.

The right-hand-side of Eqs. (19.40) are series developed in powers of $\sqrt{\mu}$ but which, in general, do not converge. Equations (19.40) thus are valid only from the viewpoint of formal calculus. Therefore, let us rewrite these equations but terminate the series at the terms in $\mu^{p/2}$. This yields

$$y_1 = \theta_0(v_1) + \mu^{1/2}\theta_1(v_1) + \cdots + \mu^{p/2}\theta_p(v_1),$$
$$x_1 = \zeta_1^0(v_1) + \mu^{1/2}\zeta_1^1(v_1) + \cdots + \mu^{p/2}\zeta_1^p(v_1). \tag{19.40a}$$

Obviously, Eqs. (19.40a) define a closed curve. Let us assume that, eliminating v_1 between these two equations, one solves them for x_1, so that

$$x_1 = P_0 + \mu^{1/2}P_1 + \cdots + \mu^{q/2}P_q + \cdots \tag{19.41}$$

where $P_0, P_1, \ldots, P_q, \ldots$ are functions of y_1. The right-hand-side of Eq. (19.41) is an indefinite but convergent series, and Eq. (19.41) is that of a closed curve.

In virtue of the principles of formal calculus, the value of x_1 obtained in this manner can differ from dS/dy_1 only by quantities of the order of $\mu^{(p+1)/2}$. Consequently, we will have

$$P_0 = \frac{dS_0}{dy_1}, \quad P_1 = \frac{dS_1}{dy_1}, \quad \ldots, \quad P_p = \frac{dS_p}{dy_1},$$

but we will not have

$$P_{p+1} = \frac{dS_{p+1}}{dy_1}.$$

Now, it is a question whether the curve

$$x_1 = \frac{dS_0}{dy_1} + \mu^{1/2}\frac{dS_1}{dy_1} + \cdots + \mu^{P/2}\frac{dS_p}{dy_1} \tag{19.42}$$

is a closed curve.

Let us return to Eq. (19.31). Since, in the case of libration, dS_1/dy_1 vanishes for two different values of y_1, it can be asked whether $d[S_2]/dy_1$ and thus also dS_2/dy_1 might not become infinite. This question must be answered in the negative since $[\Phi]$ vanishes at the same time as dS_1/dy_1. However, let us continue the approximation further.

For defining $d[S_3]/dy_1$, we will find an equation analogous to Eq. (19.31):

$$\frac{d^2F_0}{dx_1^2}\frac{d[S_3]}{dy_1}\frac{dS_1}{dy_1} = [\Phi] + C_4.$$

Will $d[S_3]/dy_1$ become infinite this time?

It is true that we can have a constant C_4 arranged such that dS_3/dy_1 does not become infinite for one of the values of y_1 that cancel dS_1/dy_1; however, in general, $[\Phi] + C_4$ will not vanish for the other value of y_1 that cancels dS_1/dy_1. Consequently, dS_3/dy_1 will become infinite no matter what the constant C_4 might be.

Thus Eq. (19.42) will not represent a closed curve since the right-hand side will become infinite.

Therefore, the above statement that the curve

$$x_1 = \frac{dS}{dy_1}$$

is closed can have no meaning by itself since the series S is divergent.

It has the following meaning:

The statement means that one can always find a function Φ_p of y_1 and of μ that can be expanded in powers of $\sqrt{\mu}$ such that the equation

$$x_1 = \frac{dS_0}{dy_1} + \mu^{1/2}\frac{dS_1}{dy_1} + \mu\frac{dS_2}{dy_1} + \cdots + \mu^{p/2}\frac{dS_p}{dy_1} + \mu^{(p+1)/2}\Phi_p$$

becomes that of a closed curve.

A simple example will illustrate the above point:

Let there be the curve

$$x = \sqrt{1 - y^2 + \mu y^2}\,.$$

This curve is an ellipse. Let us expand the right-hand side in powers of μ and let us terminate the series, for example, at the terms in μ^2, so that

$$x = \sqrt{1-y^2} + \frac{\mu}{2}\frac{y^2}{\sqrt{1-y^2}} - \frac{\mu^2}{8}\frac{y^2}{(1-y^2)^{3/2}},$$

which is not the equation of a closed curve since the right-hand side becomes infinite for $y = \pm 1$.

All these difficulties, which are purely artificial, can be avoided by the change of variables (19.32).

Limiting Case

207. Let us finally pass to the case in which C_2 is equal to the maximum of $[F_1]$, which is intermediate between the ordinary case and the case of libration.

Let us return to Eqs. (19.19) of no. 204 and to Eqs. (19.29), (19.30), and (19.31) of no. 205. We still are assuming $m_1 = 1$, $m_i = 0$ (for $i > 1$) and, consequently, $n_1^0 = 0$. In this case, the radical

$$\sqrt{C_2 - [F_1]}$$

and consequently (dS_1/dy_1), as shown in no. 200, is a periodic function of y_1 whose period, however, is no longer 2π but 4π. This function will change sign when y_1 is changed into $y_1 + 2\pi$. The function vanishes for a single value of y_1 which lies between zero and 2π and which is exactly the value at which the function $[F_1]$ reaches its maximum. Without restricting the generality, it can be assumed that this value is equal to zero. Then, we will have

$$\frac{dS_1}{dy_1} = 0$$

for

$$y_1 = 2k\pi$$

no matter what the integer k might be.

All this has been explained in detail in no. 200.

Let us now consider Eqs. (19.19) as well as the equations analogous to Eqs. (19.29) and (19.31), which are obtained by equating the mean values on both sides of Eqs. (19.19). These equations, as we have seen, allow us to determine the functions S_p by recurrence and, immediately show that the terms dS_p/dy_1 will be periodic functions of the y, where the period is 2π with respect to $y_2, y_3, \ldots, y_n$ and 4π with respect to y_1.

If the constants C_p are zero for an odd index p, which already had been assumed when writing Eqs. (19.19), then these equations (19.19) will not change on changing y_1 into $y_1 + 2\pi$ and also not on changing $\sqrt{\mu}$ into $-\sqrt{\mu}$.

From this, we can conclude, by a reasoning similar to that given in no. 200, that dS_p/dy_i will change into

$$(-1)^p \frac{dS_p}{dy_i},$$

when y_1 changes into $y_1 + 2\pi$.

Thus dS_p/dy_i is a periodic function with period 2π with respect to y_1, if p is even.

If p is odd, this function will change sign as soon as y_1 increases by 2π.

This raises the following question:

Are the functions dS_p/dy_i finite?

For determining $[S_2]$, we have Eq. (19.31):

$$\frac{d^2F_0}{dx_1^2}\frac{d[S_2]}{dy_1}\frac{dS_1}{dy_1} = [\Phi],$$

and, more generally, for determining $[S_p]$

$$\frac{d^2F_0}{dx_1^2}\frac{d[S_p]}{dy_1}\frac{dS_1}{dy_1}=[\Phi]+C_{p+1}, \tag{19.43}$$

where C_{p+1} is zero when $p+1$ is odd.

Since the function $[\Phi]$ on the right-hand side of Eq. (19.43) depends only on y_1, we will set it equal to $\varphi_{p+1}(y_1)$.

It is easy to demonstrate by recurrence that

$$\varphi_{p+1}(y_1+2\pi)=(-1)^{p+1}\varphi_{p+1}(y_1).$$

It could happen that $d[S_p]/dy_1$ becomes infinite, since dS_1/dy_1 might vanish for $y_1=2k\pi$ and since it could occur that, for this value of y_1, the right-hand side of Eq. (19.43) does not vanish.

Therefore, if we want the terms $d[S_p]/dy_1$ to remain finite, it is necessary that

$$\varphi_{p+1}(0)+C_{p+1}=\varphi_{p+1}(2\pi)+C_{p+1}=0. \tag{19.44}$$

If the conditions (19.44) are satisfied by all values of p, then $d[S_p]/dy_1$ and thus also dS_p/dy_i will remain finite.

If $p+1$ is even, it is easy to satisfy Eqs. (19.44). In fact, the constant C_{p+1} is arbitrary and it is sufficient to take it as equal to

$$-\varphi_{p+1}(0)=-\varphi_{p+1}(2\pi).$$

However, if $p+1$ is odd, then C_{p+1} is zero and the following condition must be satisfied:

$$\varphi_{p+1}(0)=0, \tag{19.45}$$

which automatically entails the condition

$$\varphi_{p+1}(2\pi)=0.$$

Since we no longer have an arbitrary constant, the condition (19.45) must be satisfied identically. In fact this is exactly what happens but it still is necessary to prove the point, which I will do in the following numbers.

208. Let us first assume that there are only two degrees of freedom and thus only four variables x_1, x_2, y_1, and y_2.

Let us refer to nos. 42, 43, and 44. There we showed that, to each system of values of the mean motions

$$n_1^0,\quad n_2^0,\quad \ldots,\quad n_n^0$$

which are mutually commensurable, there corresponds a function $[F_1]$ and that, to each maximum or to each minimum of this function, there corresponds a periodic solution.

However, in the case in question here, the mean motions are two in number

$$n_1^0, n_2^0,$$

one of which, namely, n_1^0, is zero. The values of the two mean motions thus are mutually commensurable. In addition, the function $[F_1]$ admits of an absolute maximum which it reaches for $y_1 = 0$ and which is equal to C_2. Therefore, a periodic solution must correspond to this maximum. Let

$$x_1 = \psi_1(t), \quad x_2 = \psi_2(t), \quad y_1 = \psi_1'(t), \quad y_2 = \psi_2'(t) \quad (19.46)$$

be this solution. Since n_1^0 is zero, ψ_1, ψ_2, and ψ_1' will resume their original values when t increases by one period while y_2 increases by 2π.

On eliminating t between Eqs. (19.46), we obtain

$$x_1 = \theta_1(y_2), \quad x_2 = \theta_2(y_2), \quad y_1 = \theta_3(y_2), \quad (19.47)$$

where the functions θ are periodic, with period 2π.

The characteristic exponents are two in number and, according to Chap. 4, must be equal but of opposite sign. In addition, since the periodic solution corresponds to a maximum but not to a minimum of $[F_1]$, these exponents must be real by virtue of no. 79, and the periodic solution must be unstable.

After this, we will perform a change of variables analogous to that made in no. 145.

Let

$$S^* = x_2' y_2 + \Theta + x_1' y_1 + y_1 \theta_1 - x_1' \theta_3,$$

where Θ is a function of y_2, defined by the condition

$$\frac{d\Theta}{dy_2} = \theta_2 - \theta_3 \frac{d\theta_1}{dy_2}.$$

The canonical form of the equations will not be altered if, as new variables, we take x_i' and y_i' by putting

$$x_i = \frac{dS^*}{dy_i}, \quad y_i' = \frac{dS^*}{dy_i'}.$$

We thus find

$$x_1 = x_1' + \theta_1; \quad x_2 = x_2' + \frac{d\Theta}{dy_2} + y_1 \frac{d\theta_1}{dy_2} - x_1' \frac{d\theta_3}{dy_2}; \quad (19.48)$$
$$y_1' = y_1 + \theta_3; \quad y_2' = y_2,$$

whence

$$x_2 = x_2' + \theta_2 + y_1' \frac{d\theta_1}{dy_2} - x_1' \frac{d\theta_3}{dy_2}.$$

What will be the form of the function F, expressed by means of the new variables?

Let us note first that θ_1, θ_2, and θ_3, in virtue of nos. 42–44, can be expanded in ascending powers of μ and that, for $\mu = 0$, they reduce to constants

$$x_1^0, \quad x_2^0, \quad \text{and} \quad 0.$$

This shows that x_1, y_1, and x_2 are functions of x_1', x_2', y_1', and y_2', and μ, expandable in powers of μ and periodic with respect to y_2'. For $\mu = 0$ they reduce to

$$x_1 = x_1' + x_1^0, \quad x_2 = x_2' + x_2^0, \quad y_1 = y_1'.$$

Thus F retains the same form when expressed as a function of the new variables: We state that F can be expanded in powers of μ and is periodic with respect to y_2' but that F is not periodic with respect to y_1'.

The new canonical equations

$$\frac{dx_i'}{dt} = \frac{dF}{dy_i'}, \quad \frac{dy_i'}{dt} = -\frac{dF}{dx_i'}$$

obviously admit a solution

$$x_1' = 0, \quad x_2' = 0, \quad y_1' = 0,$$

since the old equations admitted

$$x_1 = \theta_1, \quad x_2 = \theta_2, \quad y_1 = \theta_3.$$

It can be concluded from this that the three derivatives

$$\frac{dF}{dy_1'}, \quad \frac{dF}{dy_2'}, \quad \frac{dF}{dx_1'}$$

simultaneously vanish when we set

$$x_1' = x_2' = y_1' = 0.$$

Moreover, on setting $x_1' = x_2' = y_1' = 0$, the function F reduces to a constant which will be denoted here by A and which, in addition, can be expanded in powers of μ.

Let us put

$$F' = F - A.$$

Here, F' can be expanded in powers of x_1', x_2', and y_1' for small values of these variables. The series will contain no terms of degree zero and will contain no terms of the first degree other than one in x_2'. The coefficients of the expansion are functions of μ and of y_2'.

Let us then consider the equation

$$F'\left(\frac{dS'}{dy_1'}, \frac{dS'}{dy_2'}, y_1', y_2'\right) = 0,$$

and let us attempt to satisfy this by setting

$$S' = S'_0 + \sqrt{\mu} S'_1 + \mu S'_2 + \cdots . \tag{19.49}$$

We will determine the functions S'_p by recurrence, using equations completely analogous to Eqs. (19.19) of no. 204 and differing from these only by the fact that the symbols are primed and that the constants C_p are all zero.

Let us replace, in F', the function S' by its value (19.49) and let us then expand F' in ascending powers of $\sqrt{\mu}$.

Let

$$F' = \psi_0 + \sqrt{\mu}\psi_1 + \mu\psi_2 + \cdots$$

be this expansion. Then, for small values of y'_1, x'_1, and x'_2, the term ψ_p will be expandable in powers of y'_1, dS'_q/dy'_1, and dS'_q/dy'_2.

The coefficients of the series will be periodic functions of y'_2. However, the point to be emphasized here is the fact that the series will contain no zero-degree terms and that the only first-degree terms will be terms in

$$\frac{dS'_q}{dy'_2}.$$

Consequently, we will have to determine

$$S'_q - [S'_q]$$

by equations

$$n_2^0 \frac{dS'_q}{dy'_2} = \Phi \tag{19.50}$$

analogous to Eq. (19.30), and to determine $[S'_q]$ by equations

$$\frac{d^2F'_0}{dx'^2_1} \frac{d[S'_q]}{dy'_1} \frac{dS'_1}{dy'_1} = [\Phi'] \tag{19.51}$$

analogous to Eq. (19.31), where the constants analogous to C_p all are zero.

The functions Φ and Φ', entering the right-hand side of Eqs. (19.50) and (19.51), can be expanded in powers of y'_1, dS'_i/dy'_1, dS'_i/dy'_2, and the only first-degree terms will be terms in dS'_i/dy'_2.

We state that dS'_i/dy'_2 and dS'_i/dy'_1 do not assume an infinite value for $y'_1 = 0$, but that this point is actually a zero, namely, a simple zero for dS'_i/dy'_1 and a double zero for dS'_i/dy'_2.

In fact, let us demonstrate this theorem by recurrence and let us assume that it holds for the already known functions.

Then, the function Φ of Eq. (19.50) will admit the value of $y_1' = 0$ as a double zero. In fact, this value is a simple zero for each of the factors of terms of a degree higher than 1 of the expansion of Φ in powers of y_1', dS_i'/dy_1', and dS_i'/dy_2'. On the other hand, the first-degree terms in this series depend on the derivatives dS_i'/dy_2' for which $y_1' = 0$ is a double zero.

It results from this and from Eq. (19.50) that $y_1' = 0$ is a double zero for

$$\frac{dS_q'}{dy_2'},$$

and thus also for

$$S_q' - [S_q'],$$

and, finally, a simple zero for

$$\frac{dS_q'}{dy_1'} - \frac{d[S_q']}{dy_1'}.$$

One could then apply the same reasoning to the function Φ' of Eq. (19.51) as had been used for the function Φ, showing that $y_1' = 0$ is a double zero for Φ' and thus also for $[\Phi']$.

Since, on the other hand, this is a single zero for dS_i'/dy_1', it will also be a single zero for

$$\frac{d[S_q']}{dy_1'}. \qquad \text{Q.E.D.}$$

Thus the functions defined by Eqs. (19.50) and (19.51) are finite.

What relation exists between the function S defined in the preceding number and the function S' to be determined here?

We have

$$dS = x_1 dy_1 + x_2 dy_2,$$
$$dS' = x_1' dy_1' + x_2' dy_2',$$

from which, taking Eqs. (19.48) into consideration, it follows that

$$dS' - dS = d\Theta - d(y_1\theta_1),$$

whence

$$S' - S = \Theta - y_1\theta_1. \tag{19.52}$$

Since S', Θ, and θ_1 as well as their derivatives are always finite, this will also be the case for S and its derivatives.

By equating, in Eq. (19.52), the coefficients of like powers of $\sqrt{\mu}$, it is easy to calculate the functions S_p.

In fact, as we wrote above in Eq. (19.49):

$$S' = S_0' + \sqrt{\mu} S_1' + \mu S_2' + \cdots,$$

but this series is obtained by assuming that the terms S_p' are expressed as functions of the new variables x_i' and y_i'. If we return to the old variables x_i and y_i, the series will change in form and will become

$$S' = S_0'' + \sqrt{\mu} S_1'' + \mu S_2'' + \cdots$$

[see Eqs. (19.8) of no. 200].

Similarly, let

$$\Theta = \sum \mu^{p/2} \Theta_p, \quad \theta_1 = \sum \mu^{p/2} \theta_1^p.$$

We will then have

$$S_p = S_p'' - \Theta_p + y_1 \theta_1^p, \tag{19.49a}$$

which is an equation showing that the constants C_{p+1} can be chosen in such a manner that the terms dS_p/dy_i remain constantly finite.

From this, we must conclude that the conditions (19.45) are satisfied identically.

We have shown in the preceding number that the terms dS_p/dy_i are periodic functions, with period 4π with respect to y_1. This is not the case here for dS_p'/dy_i' or for dS_p''/dy_i since F, as mentioned above, is no longer periodic in y_1'. Nevertheless, Eq. (19.49a) has shown that (a) dS_p''/dy_1 is periodic; (b) dS_p''/dy_2 increases by $-4\pi\theta_1^p$ when y_1 increases by 4π.

Let us consider the equations

$$x_1 = \frac{dS}{dy_1}, \quad x_2 = \frac{dS}{dy_2} \tag{19.53}$$

which yield x_1 and x_2 as functions of y_1 and y_2. These have an interesting significance.

Let us return to Eqs. (19.46). These define the periodic solution which we used as starting point. We have shown that this solution is unstable.

Thus, based on the principles of Chap. 7, the solution gives rise to two series of asymptotic solutions whose general equations can be given the form

$$x_1 = \omega_1(t, A), \quad x_2 = \omega_2(t, A), \quad y_1 = \omega_1'(t, A), \quad y_2 = \omega_2'(t, A) \tag{19.54}$$

for the first series, or

$$x_1 = \eta_1(t,B), \quad x_2 = \eta_2(t,B), \quad y_1 = \eta_1'(t,B), \quad y_2 = \eta_2'(t,B) \tag{19.55}$$

for the second series.

Here A and B are arbitrary constants.

If t and A are eliminated among Eqs. (19.54) as one solves for x_1 and x_2, Eqs. (19.53) will be obtained; the same result is found (in which only the sign of the radical $\sqrt{C_2 - [F_1]}$ is changed) if t and B are eliminated among Eqs. (19.55).

It may be of interest to compare the above demonstration with other proofs I have given earlier (Ref. 16, pp. 211–216 and 217–219).[R16]

209. Let us now extend this proof to the case in which there are several degrees of freedom, and, for this, let us first attempt to generalize the notion used as our starting point, namely, that of the periodic solution (19.46).

Therefore, let us search for $n + 1$ functions of $n - 1$ variables

$$y_2, \quad y_3, \quad \ldots, \quad y_n,$$

functions that we denote here by

$$\eta, \quad \zeta, \quad \xi_2, \quad \xi_3, \quad \ldots, \quad \xi_n$$

and which are such that the relations

$$x_1 = \eta, \quad y_1 = \zeta, \quad x_i = \xi_i \quad (i > 1)$$

will be invariant relations in the sense given in no. 19. This entails the following conditions:

$$\begin{aligned} \sum_k \frac{d\eta}{dy_k} \frac{dF}{dx_k} &= -\frac{dF}{dy_1}, \\ \sum_k \frac{d\xi_i}{dy_k} \frac{dF}{dx_k} &= -\frac{dF}{dy_i}, \\ \sum_k \frac{d\zeta}{dy_k} \frac{dF}{dx_k} &= \frac{dF}{dx_1} \qquad (i,k = 2,3, \ldots ,n). \end{aligned} \tag{19.56}$$

There is no need to add here that, in the derivatives of F, the variables x_1, y_1, and x_i are all assumed as replaced by η, ζ, and ξ_i.

In addition, the functions η, ζ, and ξ_i must be periodic in $y_2, y_3, \ldots, y_n$. They must reduce to constants η_0, ζ_0, and ξ_i^0 for $\mu = 0$.

Finally, we will postulate one more condition here, namely that

$$x_1 dy_1 + x_2 dy_2 + \cdots + x_n dy_n = d\theta + \eta_0 d\zeta$$

be the exact differential of a function $\theta + \eta_0\zeta$ of $y_2, y_3, \ldots, y_n$. From this we derive

$$\xi_i = \frac{d\theta}{dy_i} - (\eta - \eta_0) \frac{d\zeta}{dy_i} \tag{19.57}$$

and conclude that the derivatives of θ are periodic functions. We will have, in addition,

$$F(\eta, \xi_i, \zeta, y_i) = \text{const}, \tag{19.58}$$

that is, by replacing the variables x_1, x_i, and y_1 in F by the functions η, ξ_i, and ζ, one reduces the function F to a constant.

It is easy to verify, by a calculation similar to some of those given in Chap. 15, that the second equation of system (19.56) is a necessary consequence of the two others as well as of Eqs. (19.57) and (19.58).

In fact, if Eq. (19.58) is differentiated with respect to y_k and if it is then transformed under consideration of the first and third equation (19.56) as well as of the relations

$$\frac{dx_i}{dy_k} + \frac{dy_1}{dy_k}\frac{dx_1}{dy_i} = \frac{dx_k}{dy_i} + \frac{dy_1}{dy_i}\frac{dx_1}{dy_k}$$

derived from the relations (19.57) by differentiation, then the second equation of the system (19.56) will be found.

For defining the functions η, ζ, ξ_i, and θ, we will retain the first and third equation of system (19.56) as well as Eqs. (19.57) and (19.58).

We will attempt to expand the functions η, ζ, and θ in powers of μ, in the form of

$$\begin{gathered}\eta = \eta_0 + \mu\eta_1 + \mu^2\eta_2 + \cdots, \\ \zeta = \zeta_0 + \mu\zeta_1 + \mu^2\zeta_2 + \cdots, \\ \theta = \theta_0 + \mu\theta_1 + \mu^2\theta_2 + \cdots; \quad \xi_i = \sum \mu^p \xi_i^p.\end{gathered} \tag{19.59}$$

First, setting $\mu = 0$ in the first equation of system (19.56), we will find

$$\sum n_k^0 \frac{d\eta_0}{dy_k} = 0,$$

first of all proving that η_0 does not depend on $y_2, y_3, \ldots, y_n$; this can be written in the form

$$\eta_0 = [\eta_0],$$

since $[\eta_0]$ denotes the mean value of η_0 considered as a periodic function of $y_2, y_3, \ldots, y_n$.

Setting $\mu = 0$ in the third equation of the system (19.56), we find

$$-\sum n_k^0 \frac{d\zeta_0}{dy_k} = \frac{dF_0}{dx_1}.$$

On the right-hand side, the terms x_1 and x_i must be replaced by η_0 and ξ_i^0 respectively, and these quantities must be constants such that we have

$$\frac{dF_0}{dx_1} = -n_1^0 = 0, \quad \frac{dF_0}{dx_i} = -n_i^0.$$

Let us consider n_i^0 as given in the problem, so that these equations will determine ξ_i^0 and

$$\eta_0 = [\eta_0].$$

Then, our equation becomes

$$\sum n_k^0 \frac{d\zeta_0}{dy_k} = 0,$$

whence

$$\zeta_0 = [\zeta_0].$$

Since η_0 is an absolute constant and since $d\zeta_0/dy_k$ must be zero, Eqs. (19.57) will yield

$$\xi_i^0 = \frac{d\theta_0}{dy_i}.$$

If, on the other hand, the constant of the right-hand side of Eq. (19.58) is expanded in ascending powers of μ and we write it as

$$C_0 + C_1 \mu + C_2 \mu^2 + \cdots,$$

then Eq. (19.58), setting there $\mu = 0$, will yield

$$F_0\left(\eta_0, \frac{d\theta_0}{dy_i}\right) = C_0.$$

This equation simply determines the constant C_0, and it can be shown in addition that

$$\theta_0 = \xi_2^0 y_2 + \xi_3^0 y_3 + \cdots + \xi_n^0 y_n.$$

We now have defined η_0 and θ_0. However, with respect to ζ_0 we only know that ζ_0 is a constant and that, consequently,

$$\zeta_0 = [\zeta_0],$$

but we do not know $[\zeta_0]$.

Let us equate the coefficients of μ in the first equation of system (19.56), so that, recalling that η_0 is a constant, we obtain

$$\sum n_k^0 \frac{d\eta_1}{dy_k} = \frac{dF_1}{dy_1}.$$

On the right-hand side, y_1, x_1, and x_i are assumed as being replaced by ζ_0, η_0, and ξ_i^0. This right-hand side will be a periodic function of $y_2, y_3, \ldots, y_n$, and, since $\zeta_0, \eta_0^0, \xi_i^0$ are constants, its mean value will be

$$\left[\frac{dF_1}{dy_1}\right] = \frac{d\,[F_1]}{dy_1}.$$

This mean value must be zero, which yields an equation

$$\frac{d\,[F_1]}{dy_1} = 0$$

that determines the constant ζ_0.

This leaves

$$\sum n_k^0 \frac{d\eta_1}{dy_k} = \Phi,$$

where Φ is a known periodic function whose mean value is zero, an equation from which one easily can derive

$$\eta_1 - [\eta_1]\,.$$

Let us now equate the coefficients of μ in Eq. (19.58), taking Eqs. (19.57) into consideration, which yields

$$\xi_i^1 = \frac{d\theta_1}{dy_i} - \eta_1 \frac{d\zeta_0}{dy_i} = \frac{d\theta_1}{dy_i},$$

from which we find

$$\sum n_k^0 \frac{d\theta_1}{dy_k} = \Phi - C_1.$$

Here, Φ is a known periodic function whose mean value need not be zero since we have not subjected θ_1 itself but only its derivatives to being periodic. This equation will yield θ_1, which will depend on $n-1$ constants that can be arbitrarily selected.

Let us next equate the coefficients μ in the third equation of system (19.56), so that

$$\sum n_k^0 \frac{d\zeta_1}{dy_k} = \Phi - \frac{d^2F_0}{dx_1^2}\,\eta_1. \qquad (19.60)$$

The mean value of the right-hand side must be zero, whence

$$\frac{d^2F_0}{dx_1^2}\,[\eta_1] = [\Phi]\,,$$

which yields $[\eta_1]$; Eq. (19.60) then furnishes

$$\zeta_1 - [\zeta_1]\,.$$

Let us continue in the same manner and assume that we have found

$$\begin{matrix} \eta_0, & \eta_1, & \ldots, & \eta_{p-1}, & \theta_0, & \theta_1, & \ldots, & \theta_{p-1}, \\ \zeta_0, & \zeta_1, & \ldots, & \zeta_{p-2}, & \zeta_{p-1} - [\zeta_{p-1}] \end{matrix},$$

and that we wish to find

$$\eta_p, \quad \theta_p \quad \text{and} \quad \zeta_{p-1}, \quad \zeta_p - [\zeta_p]\,.$$

Let us first equate the coefficients μ^p in the third equation of system (19.56), so that

$$\sum n_k^0 \frac{d\eta_p}{dy_k} = \Phi - \frac{d^2 F_1}{dy_1^2} \zeta_{p-1}. \tag{19.61}$$

The right-hand side must have a zero mean value, whence

$$\left[\frac{d^2 F_1}{dy_1^2} \zeta_{p-1}\right] = [\Phi]$$

or

$$\frac{d^2[F_1]}{dy_1^2} [\zeta_{p-1}] = [\Phi] - \left[\frac{d^2 F_1}{dy_1^2} (\zeta_{p-1} - [\zeta_{p-1}])\right],$$

from which we will derive $[\zeta_{p-1}]$, since $\zeta_{p-1} - [\zeta_{p-1}]$ is known. Thus ζ_{p-1} will also be known, and Eq. (19.61) will yield

$$\eta_p - [\eta_p]\,.$$

On equating the coefficients of μ^p in Eq. (19.58), and taking Eq. (19.57) into consideration, we obtain

$$\sum n_k^0 \frac{d\theta_p}{dy_k} = \Phi - C_p,$$

from which we derive θ_p.

Finally, equating the coefficients of μ^p in the third equation of system (19.56), we find an equation analogous to Eq. (19.60):

$$\sum n_k^0 \frac{d\zeta_p}{dy_k} = \Phi - \frac{d^2 F_0}{dx_1^2} \eta_p. \tag{19.62}$$

The right-hand side must have a zero mean value, and this condition

$$\Phi = \frac{d^2 F_0}{dx_1^2} [\eta_p]$$

determines $[\eta_p]$ and thus also η_p.

Next, Eq. (19.62) determines

$$\zeta_p - [\zeta_p]$$

and so on.

Consequently, we have been able to determine functions that satisfy the conditions stipulated here and we have realized a real generalization of the periodic solutions. However, whereas the series defining the periodic solutions are convergent the series of which we just demonstrated the existence is no longer convergent, so that this generalization has value only from the viewpoint of formal calculus.

210. Let us now attempt to apply the results of the preceding number to demonstrate, in the general case, that relations (19.45) are satisfied identically.

For this, let us put

$$S^* = x_2' y_2 + x_3' y_3 + \cdots + x_n' y_n$$
$$+ \theta - (\eta - \eta_0)\zeta + x_1' y_1 + y_1 \eta - x_1' \zeta$$

and let us change variables by putting

$$x_i = \frac{dS^*}{dy_i}, \quad y_i' = \frac{dS^*}{dx_i'}.$$

The canonical form of the equations will not be altered and we will find

$$x_1 = x_1' + \eta, \quad y_1' = y_1 - \zeta, \quad y_i' = y_i \quad (i > 1)$$

and, finally,

$$x_i = x_i' + \frac{d\theta}{dy_i} - (\eta - \eta_0)\frac{d\zeta}{dy_i} - \zeta\frac{d\eta}{dy_i} + y_1\frac{d\eta}{dy_i} - x_1'\frac{d\zeta}{dy_i},$$

or, taking Eq. (19.57) into consideration

$$x_i = x_i' + \xi_i + y_1'\frac{d\eta}{dy_i} - x_1'\frac{d\zeta}{dy_i}.$$

The function F retains the same form with the new variables, except that it will no longer be periodic with respect to y_1.

The new canonical equations will admit of the following, as invariant relations,

$$x_1' = x_i' = y_1' = 0,$$

which proves that, for $x_1' = x_i' = y_1' = 0$, we have

$$\frac{dF}{dy_1'} = \frac{dF}{dy_i'} = \frac{dF}{dx_1'} = 0$$

and that, in addition, F is reduced to a constant A. I thus set

$$F' = F - A$$

and I will see that, on expanding F' in powers of x_1', x_i', and y_1', there will be no terms of degree zero, and that the only first-degree terms will be terms in $x_2', x_3', \ldots, x_n'$.

Let us next consider the equation

$$F'\left(\frac{dS'}{dy_i'}, y_i'\right) = 0,$$

and let us attempt to satisfy this expression by setting

$$S' = \sum \mu^{p/2} S'_p$$

and then determine the functions S'_p by recurrence.

The calculation proceeds exactly as in no. 208.

Here again, the functions S'_q and their derivatives will be functions of y'_1, and one sees again that these functions do not become infinite for $y'_1 = 0$. On the contrary, $y'_1 = 0$ is a double zero for

$$\frac{dS'_q}{dy'_i} \quad (i > 1)$$

and a single zero for dS'_q/dy'_1.

The reasoning proceeds by recurrence as in no. 208; indeed, the equations retain their form. The details will not be given here. Let us note merely that the equation analogous to Eq. (19.50) is written as

$$\sum n_k^0 \frac{dS'_q}{dy'_k} = \Phi,$$

where

$$\Phi = \sum A \cos(m_2 y'_2 + \cdots + m_n y'_n + \omega)$$

is a periodic function of $y_2, y_3, \ldots, y_n$ whose mean value is zero. The coefficients A and ω are functions of y_1 which, naturally, are not the same for the various terms. We derive from this

$$\frac{dS'_q}{dy'_k} = \sum \frac{A m_k}{n_2^0 m_2 + \cdots + n_n^0 m_n} \cos(m_2 y'_2 + \cdots + m_n y'_n + \omega).$$

To state that $y'_1 = 0$ is a double zero for Φ is obviously to state that it also will be a double zero for each of the coefficients A and thus also for dS'_q/dy'_k.

The remainder of the reasoning is exactly as that given in no. 208.

Consequently, the functions S'_q are finite and, as in no. 208, one can conclude that this is also the case for the functions S_q and thus that relations (19.45) are satisfied identically.

Correlation with the Series of No. 125

211. In no. 125 we defined certain series S whose first terms converge in a sufficiently rapid manner if none of the combinations

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

is very small. In no. 204 *et seq.*, we defined still other series S whose convergence remains sufficient even if one of these combinations is very small.

How can one pass from one to the others? The statements made in no. 201 already predict the procedure to be followed.

The function S, defined in no. 125, depends [see the discussion following Eq. (9.7)] on an infinity of series of n arbitrary constants, namely, on

$$\begin{array}{cccc} x_1^0, & x_2^0, & \ldots, & x_n^0, \\ \alpha_{11}, & \alpha_{12}, & \ldots, & \alpha_{1n}, \\ \alpha_{21}, & \alpha_{22}, & \ldots, & \alpha_{nn}, \\ & \vdots & & \end{array}$$

However, we do not restrict the generality by assuming that all α_{ik} are zero.

In fact, let

$$S^* = S_0^* + \mu S_1^* + \mu^2 S_2^* + \cdots \tag{19.63}$$

be that of the functions S obtained on canceling all α_{ik}. This function will now contain only n arbitrary constants

$$x_1^0, \quad x_2^0, \quad \ldots, \quad x_n^0.$$

Since x_i^0 are arbitrary constants, these terms can be replaced by any expansions in powers of μ.

Thus we will replace x_i^0 by

$$x_i^0 + \mu\beta_{1i} + \mu^2\beta_{2i} + \cdots,$$

where β are some constants. The function S obtained from this substitution satisfies Eq. (9.4) of no. 125, just like S^*. However, the terms α_{ik} are no longer zero and it is obvious that the arbitrary constants β can be selected such that the values of α are whatever we please. The resultant function therefore will be the most general function S.

Let us return to S^*. This function depends on n constants x_i^0. However, on the other hand, the n mean motions n_i^0 are also functions of x_i^0, and, inversely, the terms x_i^0 are functions of n_i^0 so that we can consider S^* as being dependent on n arbitrary constants

$$n_1^0, \quad n_2^0, \quad \ldots, \quad n_n^0.$$

How do the functions S_p^* depend on these constants? Each term of S_p^* contains the sine or cosine of an angle of the following form as factor:

$$p_1 y_1 + p_2 y_2 + \cdots + p_n y_n \quad \text{(with } p \text{ as integers)}$$

and the coefficient of this sine or cosine is equal to a holomorphic function of n_i^0, divided by a product of factors of the form

$$q_1 n_1^0 + q_2 n_2^0 + \cdots + q_n n_n^0 \text{ (with } q \text{ as integers).}$$

These are factors known as "small divisors."

By using the method of reasoning of no. 201, one can demonstrate that none of the terms of S_p^* can contain more than $2p - 1$ small divisors in the denominator.

If one of these small divisors, for example,

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0,$$

were very small, then the convergence of the series S^* would become illusory. Therefore, as done in no. 202, let us replace the constants of integration n_i^0 by various expansions which no longer proceed in powers of μ but in powers of $\sqrt{\mu}$. For example, let

$$n_i^0 = \alpha_i^0 + \sqrt{\mu}\alpha_i^1 + \mu\alpha_i^2 + \cdots. \tag{19.64}$$

We assume that

$$m_1 \alpha_1^0 + m_2 \alpha_2^0 + \cdots + m_n \alpha_n^0 = 0.$$

From this it follows that the expansion of

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

starts with a term in $\sqrt{\mu}$.

Then, let

$$\frac{N}{P}\begin{matrix}\cos\\ \sin\end{matrix}(p_{12} y_1 + p_2 y_2 + \cdots + p_n y_n) \tag{19.65}$$

be any term of S_p^* where P represents the product of the small divisors.

In this case, N and P can be expanded in ascending powers of $\sqrt{\mu}$ and the exponent of μ in the first term of the expansion of P will at most be equal to $p - \frac{1}{2}$.

It follows from this that, after substituting for the n_i^0 their values (19.64), one can expand S^*in positive powers of $\sqrt{\mu}$.

Let then

$$S^* = S_0' + \sqrt{\mu}S_1' + \mu S_2' + \cdots \tag{19.66}$$

be this expansion. It is obvious that the various series (19.66) obtainable in this manner do not differ from the series discussed in this chapter and which we have learned how to form in nos. 204–207. Specifically, let us study the first terms S_0' and S_1'.

We will find

$$S_0' = S_0^* = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_n^0 y_n.$$

The terms x_i^0 are constants. These constants are themselves known functions of n_i^0, and the terms n_i^0 in S'_0 must be replaced by α_i^0 since, for $\mu = 0$, the expansion (19.64) of n_i^0 reduces to its first term, i.e., to α_i^0.

On the other hand, we find

$$S_1 = \xi_1 y_1 + \xi_2 y_2 + \cdots + \xi_n y_n + U,$$

where

$$\xi_i = \sum \frac{dx_i^0}{dn_k^0} \alpha_k^1. \tag{19.67}$$

Since the quantities x_i^0 are known functions of n_k^0, this will also be the case for their derivatives dx_i^0/dn_k^0 and one must replace there the n_k^0 by the α_k^0.

As to U, this is obtained in the following manner:

In S_p^*, let us take all terms of the form of Eq. (19.65), where the denominator P contains the small divisor

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

to the power $2p - 1$.

In the numerator N, let us replace the terms n_k^0 by α_k^0 and, in the denominator, let us replace

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0 \tag{19.68}$$

by

$$m_1 \alpha_1^1 + m_2 \alpha_2^1 + \cdots + m_n \alpha_n^1 = \gamma. \tag{19.69}$$

This term will become

$$\frac{N_0}{\gamma^{2p-1}} \begin{matrix}\cos\\ \sin\end{matrix} (p_1 y_1 + p_2 y_2 + \cdots + p_n y_n), \tag{19.70}$$

where N_0 is what becomes of N on replacing there the terms n_i^0 by α_i^0.

Let us proceed in the same manner for all terms of S_p^* that contain the small divisor (19.68) to the power $2p - 1$, and let

$$\frac{U_p}{\gamma^{2p-1}}$$

be the sum of all terms of the form (19.70) obtained in this manner.

Let us again operate similarly on all functions $S_1^*, S_2^*, \ldots$, which will successively yield

$$\frac{U_1}{\gamma}, \quad \frac{U_2}{\gamma^3}, \quad \ldots.$$

We will then have

$$U = \frac{U_1}{\gamma} + \frac{U_2}{\gamma^3} + \frac{U_3}{\gamma^5} + \cdots .$$

Assuming now that the hypothesis (19.25) of no. 204 is satisfied, we must have

$$\frac{\xi_1}{m_1} = \frac{\xi_2}{m_2} = \cdots = \frac{\xi_n}{m_n} .$$

On combining these relations with Eq. (19.67) and with Eq. (19.69), we can write

$$\xi_i = m_i A \gamma,$$

where A is a coefficient easy to calculate and depends on the integers m_i and on the derivatives dx_i^0/dn_k^0.

From this we derive

$$\frac{dS_1'}{dy_1} = m_1 A \gamma + \frac{1}{\gamma}\frac{dU_1}{dy_1} + \frac{1}{\gamma^3}\frac{dU_2}{dy_1} + \cdots , \tag{19.71}$$

and we conclude that the square of the series on the right-hand side of Eq. (19.71), which just like the series itself must proceed in decreasing powers of γ, will reduce to its two first terms

$$m_1^2 A^2 \gamma^2 + m_1 A \frac{dU_1}{dy_1} .$$

This results in a series of identities

$$2m_1 A \frac{dU_2}{dy_1} + \left(\frac{dU_1}{dy_1}\right)^2 = 0,$$

$$2m_1 A \frac{dU_3}{dy_1} + 2\frac{dU_1}{dy_1}\frac{dU_2}{dy_2} = 0,$$

$$\cdots ,$$

which, even independent of the applications in view of which this chapter has been written, are peculiar and unexpected properties of expansion (19.63).

Divergence of the Series

212. The series obtained in this chapter are divergent, just as those developed by Newcomb and Lindstedt.

Let us consider one of the series S defined in no. 204. This series will depend on a certain number of arbitrary constants.

In the first place, we have

$$x_1^0, \quad x_2^0, \quad \ldots, \quad x_n^0.$$

These constants are connected by the relation

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0 = 0.$$

As in nos. 205 *et seq.*, let us assume that

$$m_1 = 1, \quad m_2 = m_3 = \cdots = m_n = 0.$$

We have shown that this is always permissible. Our relation becomes

$$-n_1^0 = \frac{dF_0}{dx_1^0} = 0.$$

This equation can be solved for x_1^0, yielding

$$x_1^0 = \varphi(x_2^0, x_3^0, \ldots, x_n^0), \tag{19.72}$$

and, in addition, these n constants are linked to C_0 via the relation

$$F_0(x_1^0, x_2^0, \ldots, x_n^0) = C_0.$$

We next have, besides C_0,

$$C_2, \quad C_4, \quad C_6, \quad \ldots .$$

Finally, we have

$$\begin{array}{cccc} \alpha_1, & \alpha_2, & \ldots, & \alpha_n \\ & \vdots & & \\ \alpha_1^p, & \alpha_2^p, & \ldots, & \alpha_n^p, \\ & \vdots & & \end{array}$$

However, without restricting the generality, we can assume that these quantities are interconnected by relations (19.25) and (19.26) of no. 204 or, even better, we can assume without restriction of generality that all these quantities α_i and α_i^p are zero.

The constants C_4, C_6, . . . are connected by certain relations with the arbitrary constants α_i and α_i^p. Thus assuming that α_i and α_i^p are zero, C_4, C_6, . . . become wholly determined functions of x_i^0 and of C_2.

Consequently, a total of n arbitrary constants

$$x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0, \quad \text{and} \quad C_2,$$

remains, since x_1^0 is connected with the other x_i^0 by a relation.

Let us now consider the relations

$$x_1 = \frac{dS}{dy_1}, \quad x_2 = \frac{dS}{dy_2}, \quad \ldots, \quad x_n = \frac{dS}{dy_n}. \tag{19.73}$$

The right-hand sides are functions of

$$y_1, \quad y_2, \quad \ldots, \quad y_n, \quad x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0, \quad C_2.$$

Let us then solve Eqs. (19.73) for

$$x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0, \quad C_2,$$

so that

$$x_i^0 = \psi_i(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \quad (i = 2,3, \ldots n),$$
$$C_2 = \theta(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n). \tag{19.74}$$

If the series S were convergent, then ψ_i and θ would be integrals of differential equations.

Let us define their form.

Let us first return to the case of no. 204 and, consequently, assume that C_2 is larger than the maximum of $[F_1]$.From this it follows that

$$\sqrt{C_2 - [F_1]}, \quad \frac{1}{\sqrt{C_2 - [F]}},$$

will be holomorphic functions of $C_2, y_1, y_2, \ldots, y_n$ for all real values of y_i and for values of C_2 close to the value considered here.

Above, we have assumed that F is a holomorphic function of x_i and y_i for all real values of y_i and for values of x_i close to x_i^0.

In addition, the second derivative of F_0 with respect to x_1^0 will generally not be zero, so that x_1^0 will be a holomorphic function of the other x_i^0.

From all this, it follows that the terms dS_p/dy_i will be holomorphic functions for all real values of y_i and for values of

$$x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0, \quad C_2$$

close to those considered here.

Consequently, let

$$\lambda_2, \quad \lambda_3, \quad \ldots, \quad \lambda_n, \quad \gamma$$

be values of these constants close to those considered here. Let us put

$$\lambda_1 = \varphi(\lambda_2, \lambda_3, \ldots, \lambda_n).$$

The two sides of Eqs. (19.73) can be expanded in powers of

$$\sqrt{\mu}, \quad x_1 - \lambda_1, \quad x_2 - \lambda_2, \quad \ldots, \quad x_n - \lambda_n, \quad x_1^0 - \lambda_1, \quad x_2^0 - \lambda_2, \quad \ldots,$$
$$x_n^0 - \lambda_n, \quad C_2 - \gamma,$$

and in sines and cosines of multiples of y_i.

However, before applying the theorem of no. 30 to Eqs. (19.73), we will transform one of these equations. For this purpose, let us put

$$x_1 = \varphi(x_2, x_3, \ldots, x_n) + \sqrt{\mu}\, x_1'.$$

Then, the first equation of system (19.73) becomes

$$\varphi(x_2,x_3,\ldots,x_n)+\sqrt{\mu}x_1'=\varphi(x_0,x_3^0,\ldots,x_4^0)+\sqrt{\mu}\frac{dS_1}{dy_1}+\mu\frac{dS_2}{dy_1}+\cdots,$$

or, taking the other equations of system (19.73) into consideration

$$\sqrt{\mu}x_1'=\varphi\left(\frac{dS}{dy_2},\frac{dS}{dy_3},\ldots,\frac{dS}{dy_n}\right)-\varphi(x_2^0,x_3^0,\ldots,x_n^0)$$

$$+\sqrt{\mu}\frac{dS_1}{dy_1}+\mu\frac{dS_2}{dy_1}+\cdots.$$

However, we know that dS_1/dS_2, dS_1/dy_3, $\ldots$, dS_1/dy_n are zero, which means that the differences

$$\frac{dS}{dy_i}-x_i^0\quad(i>1),$$

and, consequently, the difference

$$\varphi\left(\frac{dS}{dy_i}\right)-\varphi(x_i^0),$$

are divisible by μ. Therefore, we can put

$$\varphi\left(\frac{dS}{dy_2},\frac{dS}{dy_3},\ldots,\frac{dS}{dy_n}\right)-\varphi(x_2^0,x_3^0,\ldots,x_n^0)=\mu H,$$

whence

$$x_1'=\frac{dS_1}{dy_1}+H\sqrt{\mu}+\frac{dS_2}{dy_1}\sqrt{\mu}+\frac{dS_3}{dy_1}\mu+\cdots. \tag{19.75}$$

Let us add the $n-1$ last equations of system (19.73) to Eq. (19.75). This will yield a system of n equations whose sides can be expanded in powers of

$$\sqrt{\mu},\ x_1',\ x_2-\lambda_2,\ \ldots,\ x_n-\lambda_n,\ x_2^0-\lambda_2,\ \ldots,\ x_n^0-\lambda_n,\ C_2-\gamma$$

and in sines and cosines of multiples of y_i.

For $\mu=0$, this system reduces to

$$x_1'=\frac{dS_1}{dy_1},\quad x_i=x_i^0.$$

Therefore, we must prove that, for

$$x_1'=0,\quad x_i=x_i^0=\lambda_i,$$

the functional determinant of $x_i^0-x_i$ and of dS_1/dy_1-x_1' with respect to x_i^0 and with respect to C_2 does not vanish. However, this determinant reduces to the derivative of dS_1/dy_1 with respect to C_2, or, if

$$\frac{dS_1}{dy_1}=A\sqrt{C_2-[F_1]},$$

to

$$\frac{A}{2\sqrt{C_2 - [F_1]}}.$$

Thus, this determinant is not zero and the theorem of no. 30 becomes applicable. Consequently, if our series were convergent, our differential equations would admit of n integrals ψ_i and θ uniform with respect to x and y and periodic with respect to y. However, this is impossible, which means that the series diverge. Q.E.D.

The same result exists in the case of libration. To prove this, it is only necessary to recall that, in no. 206, we reduced our equations, by a suitable change of variables, to the form of the equations in no. 134. Using the same method of reasoning as in Chap. 13, we can again demonstrate that the convergence of the series will entail the existence of uniform integrals, in contradiction to the theorem of Chap. 5.

Even in the limiting case, the series are still divergent; but I will be able to prove this rigorously only later.

One can then ask, by what mechanism, so to say, are the terms of these series are susceptible to increase such as to prevent convergence.

In the particular case in which there are only two degrees of freedom, no small divisors are introduced.

In fact, the equations to be integrated then have one of two forms:

$$n_2^0 \frac{dS_p}{dy_2} = \Phi,$$

$$\frac{dS_1}{dy_1} \frac{d[S_p]}{dy_1} = \Phi$$

and the only divisors introduced here, namely, $n_2^0 m_2$ and dS_1/dy_1 are not very small.

In exchange, however, one must perform differentiations and, in differentiating a term containing the sine or cosine of

$$p_1 y_1 + p_2 y_1 + \cdots + p_n y_n,$$

one of the integers p_i, which might be very large, is introduced into the multiplier.

Consequently, what prevents convergence from taking place is not the presence of small divisors introduced by the integration but the presence of large multipliers introduced by the differentiation.

This can also be presented in a different manner.

In the case of no. 125 and in the presence of only two degrees of freedom, we have small divisors of the form

$$m_1 n_1^0 + m_2 n_2^0.$$

Let us replace there n_1^0 and n_2^0 by expansions analogous to series (19.64) of the previous number. For example, let

$$n_2^0 = \alpha_2, \quad n_1^0 = \alpha_1 \sqrt{\mu}.$$

Our small divisors then become

$$m_2 \alpha_2 + \mu_1 \alpha_1 \sqrt{\mu}.$$

The expression

$$\frac{1}{m_2 \alpha_2 + m_1 \alpha_1 \sqrt{\mu}}$$

can be expanded in powers of $\sqrt{\mu}$, yielding

$$\frac{1}{m_2 \alpha_2} + \sqrt{\mu} \frac{m_1 \alpha_1}{(m_2 \alpha_2)^2} + \mu \frac{(m_1 \alpha_1)^2}{(m_2 \alpha_2)^3} + \cdots . \tag{19.76}$$

None of the terms of this series contains a very small divisor in the denominator, since $m_2 \alpha_2$ never is very small.

It is obvious, however, that no matter how small μ might be, whole numbers m_1 and m_2 can be found such that

$$\sqrt{\mu} \frac{m_1 \alpha_1}{m_2 \alpha_2} > 1$$

and such that, consequently, series (19.76) diverges. This furnishes an explanation for the fact that, on replacing—as I did in the preceding number—the mean motions by their expansions (19.73) and developing the series in powers of $\sqrt{\mu}$, one will arrive at divergent series.

One can connect what I have just said with what I said in nos. 109, *et seq.*

CHAPTER 20
Bohlin Series

213. In the preceding chapter, we demonstrated a procedure for constructing the function S. To derive from this the coordinates as a function of time, it is sufficient to apply the Jacobi method.

To simplify matters somewhat, let us assume that the integer denoted by m_1 is equal to 1 and that the other integers m_i are zero. This had been done by us in nos. 205 and 206, and we also know that the general case can be reduced to this particular case by the change of variables given in Eq. (19.16) of no. 202.

The function S, defined in nos. 204 *et seq.* depends on n variables y_i and, in addition, contains n arbitrary constants $x_2^0, x_3^0, \ldots, x_n^0$ and C_2. Other constants could be introduced into our calculations, namely, x_1^0, C_p, α_i^p. However, we will assume that

(a) x_1^0 is connected to the other x_i^0 by relation (19.20) of no. 204;

(b) α_i^p satisfies the condition (19.26) of no. 204;

(c) C_p is expressed in some manner and is arbitrary up to a new order, as a function of the other constants.

Thus S will be a function of

$$y_1, \quad y_2, \quad \ldots, \quad y_n; \quad C_2, \quad x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0 .$$

Let us then put

$$x_i = \frac{dS}{dy_i}; \quad w_1 = \frac{dS}{dC_2}; \quad w_k = \frac{dS}{dx_k^0} \tag{20.1}$$

$$(i = 1,2, \ldots ,n; K = 2,3, \ldots ,n) .$$

From this, the quantities x_i and y_i are derived as functions of w, x_k^0, and C_2; if, in the resultant expressions, the terms x_k^0 and C_2 are considered arbitrary constants while the terms w are considered linear functions of time, then we will have the coordinates x_i and y_i expressed as a function of time. This is actually what the theorem of no. 3 teaches us.

However, it is preferable to modify the form of Eqs. (20.1) somewhat and to write

$$x_i = \frac{dS}{dy_i}, \quad \theta_1 w_1 = \frac{dS}{dC_2}, \quad w_k + \theta_k w_1 = \frac{dS}{dx_k^0}, \tag{20.2}$$

where θ are arbitrary functions of C_2 and x_k^0.

It is obvious that, on replacing Eqs. (20.1) by Eqs. (20.2), the terms w

will remain linear functions of time since the quantities θ, depending only on C_2 and on x_k^0, will be constants.

Let us see what use we will make of these arbitrary functions θ; they will be chosen such that the quantities x_i, $\cos y_i$, and $\sin y_i$ will be periodic functions of w, with period 2π.

Let us first place ourselves in the first case, namely, that in which dS_1/dy_1 is always real and never vanishes; let us then see what is the form of the series obtained in this fashion.

In this case, the quantities dS_q/dy_i are periodic functions of y, with period 2π. As to S, this will be a function of the form

$$S = S' + \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n ,$$

where S' is a periodic function of y while β are functions of C_2 and of x_k^0.

In addition, S' and β can be expanded in powers of $\sqrt{\mu}$.

Since, according to the hypotheses made with respect to the integers m_i, the conditions (19.26) of no. 204 reduce to

$$\alpha_i^p = 0 \quad (i = 2,3, \ldots ,n) ,$$

we will simply have

$$\beta_2 = x_2^0, \quad \beta_3 = x_3^0, \quad \ldots, \quad \beta_n = x_n^0 .$$

If one expands β_1 in powers of $\sqrt{\mu}$, the first term similary reduces to x_1^0.

We stipulate that, when

$$w_1, \quad w_2, \quad \ldots, \quad w_n$$

change into

$$w_1 + 2k_1\pi, \quad w_2 + 2k_2\pi, \quad \ldots, \quad w_n + 2k_n\pi ,$$

where the terms k_i are integers, the quantities x_i and y_i will change into

$$x_i \text{ and } y_i + 2k_i\pi .$$

This result is obtained by setting

$$\theta_1 = \frac{d\beta_1}{dC_2}; \quad \theta_k = \frac{d\beta_1}{dx_k^0} .$$

From this it follows that θ_1 and θ_k can be expanded in powers of $\sqrt{\mu}$. For $\mu = 0$, the quantity β_1 reduces to x_1^0. However, x_1^0 is connected with the other x_k^0 by relation (19.20) of no. 204, which, in the case in question here, reduces to

$$n_1^0 = 0 .$$

Thus, for $\mu = 0$, the quantities θ_1 and θ_k reduce to

$$\theta_1 = \frac{dx_1^0}{dC_2} = 0, \quad \theta_k = \frac{dx_1^0}{dx_k^0} .$$

From Eqs. (20.2) we derive first the terms y_i and then the terms x_i in the form of functions of w, x_k^0, C_2, and μ that can be expanded in powers of $\sqrt{\mu}$. For $\mu = 0$, the first and third equation of system (20.2) become

$$x_i = x_i^0, \quad w_1 + \frac{dx_1^0}{dx_k^0} w_k = y_1 + \frac{dx_1^0}{dx_k^0} y_k .$$

So far as the second equation is concerned, it reduces to $0 = 0$. However, if one first divides this equation by $\sqrt{\mu}$ and then sets $\mu = 0$, it will change into

$$\theta_1' w_1 = \frac{dS_1}{dC_2} ,$$

where $\theta_1' \sqrt{\mu}$ is the first term of the expansion of θ_1. Returning to the notations of the previous chapter, we can write

$$\theta_1' w_1 = \frac{d}{dC_2} \int \sqrt{\frac{C_2 - [F_1]}{A}}\, dy_1 = \frac{1}{2\sqrt{A}} \int \frac{dy_1}{\sqrt{C_2 - [F_1]}} .$$

The right-hand side can be expanded in the form

$$\gamma y_1 + \psi(y_1) ,$$

where γ is a constant depending on C_2 and on x_k^0, while ψ is a periodic function.

We then determine θ_1' in accordance with the convention made above, by setting

$$\theta_1' = \gamma ,$$

whence

$$\gamma(w_1 - y_1) = \psi .$$

On the other hand, we obtain

$$\frac{dw_1}{dy_1} = \frac{q}{2\gamma\sqrt{A}} \frac{1}{\sqrt{c_2 - [F_1]}} = \frac{1}{2\gamma A \dfrac{dS_1}{dy_1}} .$$

Since dS_1/dy_1 always has the same sign, dw_1/dy_1 will always be positive so that w_1 will be a monotonically increasing function of y_1 which increases by 2π when y_1 increases by 2π.

Inversely, it follows from this that y_1 is a monotonically increasing function of w_1, increasing by 2π when y_1 increases by 2π.

Consequently, we can write

$$y_1 = w_1 + \eta ,$$

where η is a function of w_1, with period 2π.

Thus if we no longer assume $\mu = 0$, the first terms of the expansion of

$$x_i, \quad y_1, \quad y_k$$

will be, respectively,

$$x_i^0, \quad w_1 + \eta, \quad w_k - \frac{dx_1^0}{dx_k^0}\,\eta\,.$$

The next terms will be periodic with respect to w, such that x_i and $y_i - w_i$ will be periodic functions of w_i.

We have seen above that the quantities w must be linear functions of time, such that

$$w_i = n_i t + \overline{\omega}_i\,,$$

where $\overline{\omega}_i$ are arbitrary constants of integration.

This leaves the quantities n_i to be determined.

For this, let us return to Eq. (19.18) of no. 204. The right-hand side C is equal to

$$C = C_0 + C_2\,\mu + C_4\,\mu^2 + \cdots\,.$$

Here, C_0 is a function f x_k^0 while C_4, C_6, . . . are functions of C_2 and of x_k^0, which were selected arbitrarily, but once and for all.

It results from this that C is a function of our constants C_2 and x_k^0.

Now, the Jacobi method indicates that we have

$$\begin{aligned} \theta_1 n_1 &= -\frac{dC}{dC_2}\,, \\ n_k + \theta_k n_1 &= -\frac{dC}{dx_k^0}\,. \end{aligned} \tag{20.3}$$

Since the quantities θ and C are given as functions of C_2 and of x_k^0, these equations will yield the terms n_i as a function of these same variables.

Let us note first that, since C and θ can be expanded in powers of $\sqrt{\mu}$, this must be the same for n_i.

The first term of the expansion of θ_1 is $\gamma\sqrt{\mu}$. The first term of the expansion of dC/dC_2 is μ. The first term of the expansion of n_1 will be

$$\frac{\sqrt{\mu}}{\gamma}\,,$$

such that n_1 vanishes for $\mu = 0$, as was to be expected. On the other hand, for $\mu = 0$ the second equation of system (20.3) yields

$$n_k = -\frac{dC_0}{dx_k^0} = n_k^0\,.$$

The first term of the expansion of n_k will thus be n_k^0.

Case of Libration

214. Let us now pass to the second case, namely the case in which dS_1/dy_1 may vanish and is not always real.

Let us first see what might be the form of the function S, a point that will be especially useful in the next number. I say that the derivatives

$$\frac{dS_p}{dy_1}, \quad \frac{dS_p}{dy_2}, \quad \ldots, \quad \frac{dS_p}{dy_n}$$

will have the form

$$\sum \frac{A}{\left(\dfrac{dS_1}{dy_1}\right)^q} \begin{matrix}\cos \\ \sin\end{matrix} (p_2 y_2 + p_3 y_3 + \cdots + p_n y_n), \tag{20.4}$$

where $q, p_2, p_3, \ldots, p_n$ are integers while A is a periodic function of y_1 that does not become infinite.

First, it is obvious that

(a) the sum or the product of two functions of the form (20.4) will again be of the form (20.4);

(b) the derivative of a function of the form (20.4), either with respect to y_1 or with respect to $y_2, y_3, \ldots, y_n$, will again be of the form (20.4).

Therefore, let us assume that the derivatives

$$\frac{dS_1}{dy_i}, \quad \frac{dS_2}{dy_i}, \quad \ldots, \quad \frac{dS_{p-1}}{dy_i}$$

all have the form (20.4) and let us attempt to prove that this will be the same for dS_p/dy_i.

In fact, these derivatives are given by an equation of the form

$$\sum_{k=2}^{k=n} n_k^0 \frac{dS_p}{dy_k} = \Phi, \tag{20.5}$$

where Φ, being a combination of functions of the form (20.4), will also have the same form. From this equation, we deduce

$$\frac{dS_p}{dy_2}, \quad \frac{dS_p}{dy_3}, \quad \ldots, \quad \frac{dS_p}{dy_n}, \quad S_p - [S_p],$$

showing that all these functions are of the form (20.4).

Next, we have

$$\frac{d[S_p]}{dy_1} \frac{dS_p}{dy_1} = [\Phi], \tag{20.6}$$

where $[\Phi]$ has the form (20.4). It will be the same for $d[S_p]/dy_1$ and thus also for dS_p/dy_1.

Despite the complexity of the form of S, one could directly construct Eqs. (20.2) of the foregoing number and derive from these the quantities x and y as a function of w; however, it is simpler to proceed in a different manner.

We have actually shown in no. 206 that, by making a change of variables and by passing from the variables x_i and y_i to the variables u_1, v_1, x'_k, and z_k, we obtain equations that have entirely the same form as those in no. 134. Consequently, the conclusions of this number are applicable, which also holds for all statements made in Chaps. 14 and 15 with respect to the problem of no. 134.

It follows from this that one can solve these equations by equating u_1, v_1, x'_i, and z_i with functions of n integration constants and n linear functions of time

$$w_1, \quad w_2, \quad \ldots, \quad w_n \, .$$

Ths can be done in such a manner that

$$u_1, \quad v_1 - w_1, \quad x'_k \quad \text{and} \quad z_k - w_k$$

become periodic functions of w which, in addition, can be expanded in powers of $\sqrt{\mu}$.

Returning to the original variables, we see that

$$x_i, \quad y_1, \quad \text{and} \quad y_k - w_k \quad (k > 1)$$

are periodic functions of w.

Further, we will have

$$w_i = n_i v + \overline{\omega}_i \, ,$$

where $\overline{\omega}_i$ are constants of integration while n_i can be expanded in powers of $\sqrt{\mu}$.

The first term of the expansion of n_i is n_i^0 and, since n_1^0 is zero, the expansion of n_1 will start with a term in $\sqrt{\mu}$.

All these series are deduced from the function V, defined in no. 206.

This function V itself depends on the variables of the second series

$$v_1, \quad z_2, \quad z_3, \quad \ldots, \quad z_n \, ,$$

and, in addition, on n constants of integration

$$\lambda_1, \quad \lambda_2, \quad \lambda_3, \quad \ldots, \quad \lambda_n \, ,$$

in such a manner that

$$V - \lambda_1 v_1 - \lambda_2 z_2 - \lambda_3 z_3 - \cdots - \lambda_n z_n \, ,$$

becomes a periodic function of v_1 and of z_k.

This will yield the variables u_1, v_1, x'_k, and z_k as a function of λ and w, by means of the equations

$$u_1 = \frac{dV}{dv_1}, \quad x'_k = \frac{dV}{dz_k}, \quad w_i = \frac{dV}{d\lambda_i}. \tag{20.7}$$

The manner of deducing Eqs. (20.7) from Eqs. (20.2) of the preceding number is sufficiently complicated to warrant a more detailed discussion.

We have

$$dV = u_1 dv_1 + \sum x'_k dz_k + \sum w_i d\lambda_i .$$

We still agree that the subscript k varies from 2 to n and the subscript i, from 1 to n.

On the other hand,

$$dT = \sum x_i dy_i + \sqrt{\mu} \sum z_i dx'_i ,$$

and

$$v_1 du_1 = z_1 dx'_1 ,$$

whence

$$dT = \sum x_i dy_i + \sqrt{\mu} \sum z_k dx'_k + \sqrt{\mu} v_1 du_1 .$$

Putting

$$S = V\sqrt{\mu} + T - \sqrt{\mu}\left(\sum z_k x'_k + u_1 v_1\right), \tag{20.8}$$

we obtain, by an easy computation,

$$dS = \sum x_i dy_i + \sqrt{\mu} \sum w_i d\lambda_i ,$$

such that, expressing S as a function of y_i and λ_i, we will have

$$x_i = \frac{dS}{dy_i}, \quad w_i \sqrt{\mu} = \frac{dS}{d\lambda_i}. \tag{20.9}$$

Since these continual changes of variables may lead to some confusion, we will define this in some detail:

V is expressed as a function of v_1, z_k, λ_i;

T is expressed as a function of u_1, x'_k, y_i;

S is expressed as a function of λ_i and y_i.

Thus we have $6n$ variables, namely,

$$x_i, \quad y_i, \quad u_1, \quad v_1, \quad x'_k, \quad z_k, \quad \lambda_i, \quad w_i .$$

However, since these variables are connected by the $4n$ relations

$$x_i = \frac{dT}{dy_i}, \quad \sqrt{\mu} z_k = \frac{dT}{dx'_k}, \quad \sqrt{\mu} v_1 = \frac{dT}{du_1},$$

$$u_1 = \frac{dV}{dv_1}, \quad x'_k = \frac{dV}{dz_k}, \quad w_i = \frac{dV}{d\lambda_i},$$

we have in reality only $2n$ independent variables, which permits expressing each of our functions S, V, T by means of $2n$ properly chosen variables.

The function V exhibits the following characteristic property: When one of the quantities z_k increases by 2π, the other variables v_1, z, and λ remain unchanged while V increases by $2\pi\lambda_k$.

We actually know that the derivatives of V with respect to v_1 and z_k are periodic with respect to these variables.

Now, when z_k thus changes into $z_k + 2\pi$, the other z, v_1, and λ do not change; what happens here?

Since the derivatives of V are periodic as stated above, u_1 and x'_k will not change.

To see what becomes of y_i, we will use the following equations:

$$\sqrt{\mu} z_k = \frac{dT}{dx'_k}, \quad \sqrt{\mu} v_1 = \frac{dT}{du_1}.$$

These equations, which are nothing else but Eqs. (19.32) of no. 206, demonstrate that, if z_k increases by 2π, then y_k increases by 2π while the other y_i do not change.

Under the same conditions, T increases by $2\pi(x_k^0 + \sqrt{\mu} x'_k)$ and $z_k x'_k$ increases by $2\pi x'_k$, so that S will increase by

$$2\pi(x_k^0 + \sqrt{\mu}\lambda_k).$$

It follows from this that the derivatives of S with respect to y are periodic with respect to $y_2, y_3, \ldots, y_n$.

The function S, defined by Eq. (20.8), thus exhibits the characteristic property of the functions studied in nos. 204, 205, and 207.

Nevertheless, it differs from those in one important point:

The function S of the foregoing number depends not only on the variables y_i but also on n constants

$$x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0, \quad C_2.$$

In addition, the analysis of nos. 204 and 205 proves that all functions S whose derivatives are periodic can be derived from this, on replacing these n constants by arbitrary functions of n other constants.

The function S defined by Eq. (20.8) depends on the variables y_i and on the n constants λ_i; but it also depends on the constants x_i^0, since the x_i^0 are contained in the function T and, consequently, in the change of variables of no. 206. However, in no. 206 as well as in the above calculation, we treated the quantities x_i^0 as absolute constants. This is the reason why the differentials $d\lambda_i$ appear in the expression of dV whereas the differentials dx_i^0 do not appear there.

We note, in addition, that an increase in y_k by 2π will cause the function S of the preceding number to increase by $2\pi x_k^0$ whereas the function defined by Eq. (20.8) will increase by $2\pi(x_k^0 + \sqrt{\mu}\lambda_k)$.

We conclude from this that the function S, derived from Eq. (20.8), is obtained on replacing, in that of the preceding number, the constants x_k^0 by $x_k^0 + \sqrt{\mu}\lambda_k$ and the constant C_2 by a certain function.

$$\varphi(x_2^0, x_3^0, \ldots, x_n^0, \lambda_2, \lambda_3, \ldots, \lambda_n) .$$

Let us now compare Eqs. (20.2) with Eqs. (20.9). We find

$$\frac{dS}{d\lambda_1} = \frac{dS}{dC_2}\frac{d\varphi}{d\lambda_1},$$

$$\frac{dS}{d\lambda_k} = \frac{dS}{dC_2}\frac{d\varphi}{d\lambda_k} + \sqrt{\mu}\frac{dS}{dx_k^0},$$

whence, taking Eqs. (20.2) and (20.9) into consideration,

$$w_1\sqrt{\mu} = \theta_1 w_1 \frac{d\varphi}{d\lambda_1},$$

$$w_k\sqrt{\mu} = \theta_1 w_1 \frac{d\varphi}{d\lambda_k} + \sqrt{\mu}(w_k + \theta_k w_1) ,$$

from which it follows that

$$\theta_1 = \frac{\sqrt{\mu}}{\left(\dfrac{d\varphi}{d\lambda_i}\right)}, \quad \theta_k = -\frac{\left(\dfrac{d\varphi}{d\lambda_k}\right)}{\left(\dfrac{d\varphi}{d\lambda_1}\right)}. \tag{20.10}$$

This means that we pass from Eqs. (20.2) to Eqs. (20.9) by replacing the quantities x_k^0 and C_2 by $x_k^0 + \sqrt{\mu}\lambda_k$ and φ and the quantities θ by their values (20.10).

Limiting Case

215. Let us finally pass to the limiting case, namely the case in which C_2 is equal to the maximum of $[F_1]$.

Let us note first that we can still assume that, for

$$x_1 = x_i = y_1 = 0 ,$$

we have

$$F = \frac{dF}{dy_1} = \frac{dF}{dy_i} = \frac{dF}{dx_1} = 0 ,$$

and that, consequently, the expansion of $F_1, F_2, \ldots$ in powers of x_1, x_i, and y_1 contains no terms of degree zero and no terms of the first degree other than terms in $x_2, x_3, \ldots, x_n$.

If, indeed, this were not the case, we would carry out the change of variables of nos. 208 and 210, which would return us to the case in which this assumption holds.

It results from this that, if the following values are assigned to the arbitrary constants

$$x_2^0 = x_3^0 = \cdots = x_n^0 = 0 \quad (\text{whence } x_1^0 = C_0 = 0) ,$$
$$C_2 = C_4 = \cdots = C_6 = 0 ,$$

exactly the limiting case will occur, and the function S will be such that the quantities dS_p/dy_1 have a single zero while the quantities dS_p/dy_i $(i > 1)$ have a double zero for $y_1 = 0$. To prove this, it is sufficient to recall that, in the calculation of nos. 208 and 210, the change of variables led to equations completely analogous to Eqs. (19.19) of no. 204 differing from these only by the fact that the symbols are primed and that all constants C_p are zero [see the discussion at Eq. (19.49)].

Let us now assign other values, close to zero, to the constants x_2^0, x_3^0, $\ldots$, x_n^0. It is still possible to select $C_2, C_4, C_6, \ldots$ in such a manner that C_2 is equal to the maximum of $[F_1]$ and that, since the conditions (19.44) of no. 207 are satisfied, the functions $S_1, S_2, \ldots, S_p$ remain finite.

The values of C_2, C_4, C_6, $\ldots$ which satisfy these conditions will be holomorphic functions of $x_2^0, x_3^0, \ldots, x_n^0$ such that

$$C_p = \varphi_p(x_2^0, x_3^0, \ldots, x_n^0) .$$

According to our above statements, these functions must vanish for

$$x_2^0 = x_3^0 = \cdots = x_n^0 = 0 .$$

We have thus defined a function S depending on $n - 1$ arbitrary constants

$$x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0 .$$

This function has the form

$$S = \beta_1 y_1 + x_2^0 y_2 + x_3^0 y_3 + \cdots + x_n^0 y_n + S' , \qquad (20.11)$$

where β_1 is a constant and S' can be expanded in sines and cosines of multiples of

$$\frac{y_1}{2}, \quad y_2, \quad y_3, \quad \ldots, \quad y_n .$$

This function, in addition, is holomorphic with respect to x_k^0, and, if one sets

$$x_2^0 = x_3^0 = \cdots = x_n^0 = 0\,,$$

the derivative dS/dy_1 will admit of a single zero for $y_1 = 0$ while the other derivatives dS/dy_i will admit of a double zero.

To obtain a function S depending on n arbitrary constants, we will set

$$\begin{aligned} C_0 &= \varphi_0(x_2^0, \ldots, x_n^0)\,, \\ C_2 &= \lambda + \varphi_2(x_2^0, \ldots, x_n^0)\,, \\ C_4 &= \varphi_4(x_2^0, \ldots, x_n^0)\,, \\ C_6 &= \varphi_6(x_2^0, \ldots, x_n^0)\,, \\ &\vdots \end{aligned}$$

Thus we have a function containing the n constants

$$\lambda, \quad x_2^0, \quad \ldots, \quad x_n^0\,.$$

According to our statements at the beginning of the preceding number, the derivatives of this function S will have the form (20.4).

However, there is more to this. Let

$$\frac{A}{\left(\dfrac{dS_1}{dy_1}\right)^q} \begin{matrix}\cos \\ \sin\end{matrix} (p_2 y_2 + p_3 y_3 + \cdots + p_n y_n)$$

be a term of one of these derivatives brought to the form (20.4). I say that the numerator A does not depend on λ.

This has to do with the fact that the constants $C_4, C_6, \ldots$ do not depend on λ.

To demonstrate the point in question, let us agree, for abbreviating the notation, to say that an expression has the form (20.4′) whenever it has the form (20.4), and that, in addition, the numerators A are independent of λ.

Assuming that

$$\frac{dS_1}{dy_i}, \quad \frac{dS_2}{dy_i}, \quad \ldots, \quad \frac{dS_{p-1}}{dy_i}$$

is of the form (20.4′), I say that this must be so also for dS_p/dy_i.

In fact, in Eq. (20.5) of the preceding number, the right-hand side will have the form (20.4′); consequently, this will also be the case for

$$\frac{dS_p}{dy_2}, \quad \frac{dS_p}{dy_3}, \quad \ldots, \quad \frac{dS_p}{dy_n}, \quad S_p - [S_p]\,.$$

I say that it will also be the case for

$$\frac{dS_p}{dy_1} - \frac{d\,[S_p]}{dy_1},$$

in other words, the derivative with respect to y_1 of an expression of the form (20.4′) will again be of the form (20.4′). Let, in fact,

$$\sum A \left(\frac{dS_1}{dy_1}\right)^{-q} \frac{\cos}{\sin}(\omega)$$

be this expression where, for brevity, we used ω instead of

$$p_2 y_2 + \cdots + p_n y_n \,.$$

Its derivative is

$$\sum \frac{dA}{dy_1}\left(\frac{dS_1}{dy_1}\right)^{-q} \frac{\cos}{\sin}(\omega) - \sum_2^q A \frac{d}{dy_1}\left(\frac{dS_1}{dy_1}\right)^2 \left(\frac{dS_1}{dy_1}\right)^{-q-2} \frac{\cos}{\sin}(\omega)\,. \tag{20.12}$$

If A is independent of λ, this will also be so for dA/dy_1. Moreover, to within a constant factor, $(dS_1/dy_1)^2$ is equal to

$$\lambda + \varphi_2 - [F_1]\,.$$

Its derivative

$$\frac{d}{dy_1}\left(\frac{dS_1}{dy_1}\right)^2$$

thus is independent of λ in such a manner that expression (20.12) will have the form (20.4′).

Q.E.D.

Then, in Eq. (20.6) of the preceding number, the right-hand side will have the form (20.4′). Consequently, this is also the case for

$$\frac{d\,[S_p]}{dy_1} \quad \text{and} \quad \text{for} \quad \frac{dS_p}{dy_1}\,.$$ Q.E.D.

Therefore, the function S will have the form

$$S = \sum A \left(\frac{dS_1}{dy_1}\right)^{-q} \frac{\cos}{\sin}(\omega)$$
$$+ \int \sum A_1 \left(\frac{dS_1}{dy_1}\right)^{-q} dy_1 + x_2^0 y_2 + x_3^0 y_3 + \cdots + x_n^0 y_n\,. \tag{20.13}$$

If, in this expression, the constants

$$\lambda, \quad x_2^0, \quad x_3^0, \quad \ldots, \quad x_n^0\,,$$

are made to vanish, then A will admit of a zero of order $q + 2$ and A_1 will admit of a zero of the roder $q + 1$ for $y_1 = 0$. This is necessary for dS/dy_k to have a double zero and for dS/dy_1 to have a single zero.

After this, we must consider the following equations, analogous to Eqs. (20.2):

$$x_i = \frac{dS}{dy_i},$$

$$\theta_1 w_1 = \frac{dS}{d\lambda}, \tag{20.2a}$$

$$w_k + \theta_k w_1 = \frac{dS}{dx_k^0}.$$

In these expressions, after differentiation, we must set

$$\lambda = x_k^0 = 0\,.$$

However, it is also possible, even before differentiation, to set

$$\lambda = x_k^0 = 0$$

in the first equation of the system (20.2a), and

$$x_k^0 = 0$$

in the second equation, and

$$\lambda = 0$$

in the third equation.

The essential point here is not to cancel, before the differentiation, the variable with respect to which one differentiates.

The first equation of the system (20.2a) indicates that the terms x_i can be expanded in sines and cosines of multiples of

$$\frac{y_1}{2},\quad y_2,\quad y_3,\quad \ldots,\quad y_n\,.$$

Let us now consider the third equation of the system (20.2a). If we set there $\lambda = 0$, we will see that S is of the form (20.11); on differentiating Eq. (20.11), we obtain

$$\frac{dS}{dx_k^0} = y_1 \frac{d\beta_1}{dx_k^0} + y_k + \frac{dS'}{dx_k^0},$$

whence

$$w_1\theta_k + w_k = y_1 \frac{d\beta_1}{dx_k^0} + y_k + \frac{dS'}{dx_k^0}. \tag{20.14}$$

The last term on the right-hand side can be expanded in sines and cosines of multiples of

$$\frac{y_1}{2},\quad y_2,\quad y_3,\quad \ldots,\quad y_n\,.$$

Let us next pass to the second equation of the system (20.2a). To obtain $dS/d\lambda$, we differentiate Eq. (20.13) after having set $x_k^0 = 0$.

This yields

$$\frac{dS_1}{dy_1} = D\sqrt{\lambda - [F_1]}$$

(where D is a constant), since φ_2 becomes zero.

Consequently, we have

$$\frac{dS}{d\lambda} = -\sum_q \frac{AD^2}{2}\left(\frac{dS_1}{dy_1}\right)^{-q-2} \frac{\cos}{\sin}(\omega) + \int \sum_q \frac{A_1 D^2}{2}\left(\frac{dS_1}{dy_1}\right)^{-q-2} dy_1 . \tag{20.15}$$

After differentiation, we set $\lambda = 0$. Then, for $y_1 = 0$, the quantity dS_1/dy_1 admits of a single zero, while A admits of a zero of the order $q + 2$ and A_1 of a zero of the order $q + 1$.

It follows from this that the first term on the right-hand side of Eq. (20.15) remains finite but that, in the second term, the quantity under the integral sign admits a simple pole for $y_1 = 2k\pi$ such that it can be brought to the form

$$\frac{\alpha}{2\sin(y_1/2)} + f(y_1) ,$$

where $f(y_1)$ is a finite and periodic function.

Thus the integral itself diverges logarithmically for $y_1 = 0$, $y_1 = 2\pi$, . . . ; that is to say, this integral can be brought to the form

$$\alpha \log \tan \frac{y_1}{4} + \psi ,$$

where ψ is a function of y_1 that remains finite for all values of y_1, while α is a constant.

Thus we have

$$\frac{dS}{d\lambda} = \alpha \log \tan \frac{y_1}{4} + \gamma y_1 + \Theta ,$$

where γ is a new constant while Θ is a function expanded in sines and cosines of multiples of

$$\frac{y_1}{2}, \quad y_2, \quad y_3, \quad \ldots, \quad y_n ,$$

whence

$$w_1\theta_1 = \alpha \log \tan \frac{y_1}{4} + \gamma y_1 + \Theta . \tag{20.16}$$

It is now a question of using Eqs. (20.14) and (20.16) for finding y as a function of w.

Since the right-hand side of Eqs. (20.14) and (20.16) can be expanded in powers of $\sqrt{\mu}$, let us seek the first terms of the series.

The term independent of $\sqrt{\mu}$ reduces to zero on the right-hand side of Eq.(20.16) and to

$$\frac{dx_1^0}{dx_k^0} y_1 + y_k$$

on the right-hand side of Eq. (20.14).

As for the term in $\sqrt{\mu}$, it must reduce in Eqs. (20.14) and (20.16) to, respectively,

$$\sqrt{\mu}\,\frac{dS}{dx_k^0} \quad \text{and} \quad \sqrt{\mu}\,\frac{dS_1}{d\lambda}.$$

With respect to the first of these quantities, we merely remark that it depends only on y_1 but not on $y_2, y_3, \ldots, y_n$.

With respect to the second of these quantities, setting

$$\lambda = x_k^0 = 0,$$

and differentiating, we find

$$\frac{dS_1}{d\lambda} = \frac{D}{2} \int \frac{dy_1}{\sqrt{-F_1}}.$$

That said, , let us consider the right-hand sides of Eqs. (20.14) and (20.16).

These are n functions of $y_1, y_2, \ldots, y_n$. Their functional determinant Δ with respect to $y_1, y_2, \ldots, y_n$ is divisible by $\sqrt{\mu}$. However, if one divides by $\sqrt{\mu}$ and if, after this division, one sets $\mu = 0$, this functional determinant will reduce to

$$\frac{D}{2\sqrt{-F_1}}.$$

The expression does not vanish for any system of values of y, since F_1 never is infinite.

Consequently, if μ is sufficiently small, Δ will not vanish.

On the contrary, Δ may become infinite. In fact, the right-hand sides of Eqs. (20.14) and (20.16) become infinite for

$$y_1 = 2k\pi.$$

As a result, if one assigns to $y_2, y_3, \ldots, y_n$ all possible values and varies y_1 from zero to 2π, Δ will not change sign.

For simplification, we will take

$$\theta_1 = 1, \quad \theta_k = 0,$$

such that Eqs. (20.14) and (20.16) can be written as

$$w_k = y_1 \frac{d\beta_1}{dx_k^0} + y_k + \frac{dS'}{dx_k^0}, \tag{20.14a}$$

$$w_1 = 2 \log \tan \frac{y_1}{4} + \gamma y_1 + \Theta. \tag{20.16a}$$

Because of the presence of the logarithmic term, a variation of y_1 from zero to 2π will cause w_1 to vary from $-\infty$ to $+\infty$.

Consequently, if one assigns all possible values to y_k and varies y_1 from zero to 2π, the quantities w will take all possible values. In addition, we have seen that Δ will not change sign under these conditions.

Therefore, the y are uniform functions of w for all real values of w. In fact, starting from Eqs. (20.14) and (20.16) and applying the theorem of no. 30, one can expand the quantities y in powers of

$$w_1 - h_1, \quad w_2 - h_2, \quad \ldots, \quad w_n - h_n,$$

where $h_1, h_2, \ldots, h_n$ are any constants, since the functional determinant never vanishes.

We should add that

$$y_1, \quad y_2 - w_2, \quad y_3 - w_3, \quad \ldots, \quad y_n - w_n$$

are periodic functions of

$$w_2, \quad w_3, \quad \ldots, \quad w_n,$$

and, indeed, when y_k increases by 2π, w_k increases by 2π. The first equation of the system (20.2a) then shows that the terms x_i are also uniform functions of w, periodic with respect to

$$w_2, \quad w_3, \quad \ldots, \quad w_n.$$

When w_1 tends to $\pm\infty$, y_1 tends to zero or to 2π. We then have to determine what becomes of Eqs. (20.14) and (20.16) when one sets there, for example,

$$w_1 = \infty, \quad y_1 = 0.$$

Equation (20.16) becomes illusory, and Eq. (20.14) can be written as

$$w_k = y_k + \frac{dS'}{dx_k^0}.$$

From this one obtains $y_2, y_3, \ldots, y_n$ as functions of the $n-1$ arguments

$$w_2, \quad w_3, \quad \ldots, \quad w_n.$$

It is obvious that $y_k - w_k$ is periodic with respect to $w_2, w_3, \ldots, w_n$.

Consequently, let

$$y_k = w_k + \eta_k(w_2, w_3, \ldots, w_n) \,.$$

If, in the first equation of the system (20.2a), we set $y_1 = 0$, this expression reduces to

$$x_i = 0 \,.$$

Thus, we can find a particular solution of Eqs. (20.2a) by setting

$$x_1 = x_i = y_1 = 0 \,, \quad y_k = w_k + \eta_k \,. \tag{20.17}$$

The meaning of these equations (20.17) is evident.

In no. 209, we found a generalization of the periodic solutions. In fact, we formed the invariant relations

$$x_1 = \eta, \quad y_1 = \zeta, \quad x_i = \xi_i \,.$$

Because of the hypothesis made at the beginning of this number, these invariant relations reduce here to

$$x_1 = y_1 = x_i = 0 \,.$$

We recognize here the first three equations of Eqs. (20.17).

These four equations (20.17) thus furnish, in a new form, the generalization of the periodic solutions. It is obvious that x_i, y_i, and $y_k - w_k$ are expressed as periodic functions of $n - 1$ arguments of the form

$$w_k = n_k t + \overline{\omega}_k \,.$$

In the particular case in which there are two degrees of freedom, only a single argument w_2 remains.

Then x_1, x_2, y_1, and $y_2 - w_2$ are expressed as periodic functions of w_2 and thus also of time. This readily yields the periodic solutions, as they had been defined in Chap. 3.

An interesting consequence is the fact that, if there are only two degrees of freedom, series (20.17) are convergent whereas they are of value only from the viewpoint of formal calculus as soon as the number of degrees of freedom exceeds two.

216. Let us specifically investigate what happens when w_1, for example, is negative and very large. The corresponding values of y_1 will then be very small so that the right-hand side of Eq. (20.14) can be expanded in ascending powers of y_1.

So far as Eq. (20.16a) is concerned, we will transform it as

$$e^{w_1/\alpha} = \tan \frac{y_1}{4} \, e^{\gamma y_1/\alpha} \, e^{\Theta/\alpha} \,. \tag{20.16b}$$

If α is positive, as I will suppose, to be specific, and if w_1 is negative and very large, then the exponential

$$e^{w_1/\alpha}$$

will be very small. The right-hand side of Eq. (20.16b) can be expanded in powers of y_1.

Thus let us write our equations in the forms

$$w_k = y_k + \psi_k \,, \tag{20.14b}$$

$$e^{w_1/\alpha} = \psi_1 \,. \tag{20.16c}$$

The terms ψ can be expanded in powers of y_1 and of $\sqrt{\mu}$, while each of the terms of the series will be periodic with respect to

$$y_2, \quad y_3, \quad \ldots, \quad y_n \,.$$

Consequently, both sides of Eqs. (20.14a) and (20.16b) can be considered as having been expanded in powers of y_1, $\sqrt{\mu}$, and $e^{w_1/\alpha}$.

Let us note that α can be expanded in powers of $\sqrt{\mu}$ and let

$$\alpha_1\sqrt{\mu}$$

be the first term of the series.

On the other hand, the first term of the expansion of γ and of Θ will be a term in $\sqrt{\mu}$, such that the expansion of γ/α and of Θ/α will start with a term independent of $\sqrt{\mu}$.

If, in Eqs. (20.14b) and (20.16c), we set $\mu = 0$ they will become

$$w_k = y_k + \frac{dx_1^0}{dx_k^0} y \,,$$

$$e^{w_1/\alpha} = e^{dS_1/\alpha_1 d\lambda} \,.$$

The functional determinant of the right-hand sides of these equations, with respect to $y_1, y_2, \ldots, y_n$, reduces to 1 for $y_1 = 0$.

This permits application of the theorem of no. 30.

It follows from this that, for all values of

$$w_2, w_3, \ldots, w_n \,,$$

the quantities y can be expanded in powers of $\sqrt{\mu}$ and of

$$e^{w_1/\alpha} \,.$$

The coefficients of the series are functions of

$$w_2, \quad w_3, \quad \ldots, \quad w_n \,.$$

To ascertain the form of these functions, let us note that an increase in y_k by 2π will cause an increase in w_k by 2π.

We conclude from this that

$$y_1 \quad \text{and} \quad y_k - w_k$$

can be expanded in series developed in powers of

$$\sqrt{\mu} \quad \text{and} \quad e^{w_1/\alpha},$$

whose coefficients are periodic functions of

$$w_2, \quad w_3, \quad \ldots, \quad w_n .$$

The first equation of the system (20.2a) then directly demonstrates that the quantities x_i can be expanded in series of the same form.

If, instead of assuming w_1 to be negative and very large and y_1 to be close to zero, we had assumed w_1 to be positive and very large and y_1 very close to 2π, we would have arrived at the same result. However, instead of series proceeding in powers of

$$\sqrt{\mu} \quad \text{and} \quad e^{w_1/\alpha},$$

we would then have had series proceeding in power of

$$\sqrt{\mu} \quad \text{and} \quad e^{-w_1/\alpha}.$$

Let us return to the case in which w_1 is negative and very large and in which y_1 is very close to zero, and let us assume that there are only two degrees of freedom.

In that event, we have only two arguments

$$w_1 \quad \text{and} \quad w_2 ,$$

and our series will proceed in powers of $\sqrt{\mu}$ and of $e^{w_1/\alpha}$ as well as in sines and cosines of multiples of w_2. Since the arguments w_1 and w_2 are linear functions of time, our series proceed in powers of $\sqrt{\mu}$ and of an exponential whose exponent is proportional to time, since the coefficients of the various terms are periodic functions of time. Consequently, these do not differ from the series studied in Chap. 7 defining the asymptotic solutions.

From this, a consequence is obtained which has been demonstrated in Chap. 7.

If the series remain arranged in powers of $\sqrt{\mu}$ and of the exponential, they will not converge and will be of value only from the viewpoint of formal calculus. If the series are arranged in ascending powers of the exponential alone (thus combining, into a single term, all terms containing the same power of exponential but differing powers of μ), the series will become convergent. If, conversely, this operation is performed for the case in which more than two degrees of freedom exist, then the series will not become convergent.

217. At the beginning of no. 215, we made certain hypotheses as to the function F. We assumed that we had

$$F = \frac{dF}{dy_1} = \frac{dF}{dy_i} = \frac{dF}{dx_1} = 0$$

for

$$x_1 = x_i = y_1 = 0 .$$

We added there that, if the function F did not satisfy these conditions, it would be sufficient to carry out the change of variables as indicated in nos. 208 and 210.

Let us thus assume that the function F does not satisfy these conditions. Let x_i and y_i be the old variables and let us make the change of variables as given in no. 210; in addition, let x_i' and y_i' be the new variables. We then have

$$\begin{gathered} x_1 = x_1' + \eta, \quad y_1' = y_1 - \xi, \quad y_i' = y_i , \\ x_i = x_i' + \xi_i - y_1' \frac{d\eta}{dy_i} - x_1' \frac{d\xi}{dy_i} \end{gathered} \tag{20.18}$$

(see no. 210).

With the new variables, the conclusions of the last two sections become applicable so that

$$x_1', \quad x_k', \quad y_1', \quad y_k' - w_k$$

can be represented by series arranged in powers of $\sqrt{\mu}$ and of cosines and sines of multiples of

$$w_2, \quad w_3, \quad \ldots, \quad w_n$$

whose coefficients are uniform functions of w_1. These uniform functions can be expanded in powers of $e^{w_1/\alpha}$ if w_1 is negative and sufficiently large, and in powers of $e^{-w_1/\alpha}$ if w_1 is sufficiently large.

From the relations (20.18) that connect x_i and y_i with x_i' and y_i', it is permissible to conclude that

$$x_1, \quad x_k, \quad y_1, \quad y_k - w_k$$

are still expandable in series of the same form.

The only difference here is the fact that, for $w_1 = -\infty$, the terms x_1', x_k', and y_1' reduce to zero, whereas x_1, x_k, and y_1 do not vanish.

Setting $w_1 = -\infty$, from which $e^{w_1/\alpha} = 0$ follows, we find

$$x_1 = \varphi_1, \quad x_k = \varphi_k, \quad y_1 = \varphi_1', \quad y_k = w_k + \varphi_k', \tag{20.19}$$

where φ_i and φ_i' represent series arranged in powers of $\sqrt{\mu}$ and in trigonometric terms of multiples of

$$w_2, \quad w_3, \quad \ldots, \quad w_n .$$

On eliminating $w_2, w_3, \ldots, w_n$ from the relations (20.19), we necessarily must find

$$x_1 = \eta, \quad y_1 = \xi, \quad x_k = \xi_k ,$$

i.e., the relations given in no. 209. If there are only two degrees of freedom, the relations (20.19) will simply represent a periodic solution (see no. 208).

Setting

$$w_1 = +\infty, \quad \text{whence } e^{-w_1/\alpha} = 0,$$

we similarly obtain

$$x_1 = \varphi_1, \quad x_k = \varphi_k, \quad y_1 = \varphi'_1 + 2\pi, \quad y_k = w_k + \varphi'_k .$$

The series studied in this chapter could be obtained directly by procedures analogous to those given in Chaps. 14 and 15. Despite the obvious interest of this topic, we cannot pursue it further since this would lead too far. We will restrict ourselves here to recalling that, by the change of variables given in no. 206, one returns to the problem of no. 134 to which the methods given in Chaps. 14 and 15 are directly applicable.

Comparison with the Series of No. 127

218. We have seen in no. 211 that the series of nos. 204 *et seq.* can be derived from those of no. 125. We propose here to see how the series of the present chapter can be deduced from those given in no. 127.

Let us start by treating the simplest case, namely, that of no. 199. In this case, our equations (omitting the subscript 1, which has become useless) can be written as

$$x = \sqrt{C - \mu \cos y}\,,$$
$$\theta w = \int \frac{dy}{\sqrt{C - \mu \cos y}}, \tag{20.20}$$

where $2\theta\pi$ denotes the real period of the integral on the right-hand side. These equations permit calculating x and y as functions of the argument w, of the constant C, and of μ.

If we first assume that μ is very small with respect to C,the terms can be expanded in ascending powers of μ, resulting in the series of no. 127. If, conversely, C is comparable to μ, we will put $C = C_1 \mu$ and thus return to the series studied in the present chapter.

Let us now investigate this in somewhat more detail. Equations (20.20) prove that x, $\cos y$, and $\sin y$ are doubly periodic functions of θw or, which comes to the same, of w. Let ω_1 and ω_2 be the two periods (considering θw as the independent variable). For example, ω_i will be equal to the integral on the right-hand side, taken between zero and 2π, while ω_2 will be equal to twice this integral taken between $\pm \cos^{-1} C/\mu$. In addition, when θw increases by ω_2, the quantity y does not change whereas, when θw increases by ω_1, the quantity y increases by 2π.

If $C > |\mu|$, then ω_1 is real and we must use $\theta = \omega_1/2\pi$. In that case, x and $y - w$ are periodic functions of w, with period 2π. If μ is small with respect to C, the terms can be expanded in powers of μ (which leads to the series of no. 127), and each of the terms will be periodic, with period 2π with respect to w.

However, if C is of the same order of magnitude as μ and if we put $C = C_1\mu$, it will happen that, for one and the same value of C_1, the period ω_1 and the coefficient θ will be proportional to $1/\sqrt{\mu}$. If we then put

$$\theta_0 = \frac{\theta}{\sqrt{\mu}},$$

Eqs. (20.20) become

$$x = \sqrt{\mu}\,\sqrt{C_1 - \cos y}\,,$$
$$\theta_0 w = \int \frac{dy}{\sqrt{C_1 - \cos y}}\,. \tag{20.20a}$$

The second of these equations no longer depends on μ. We derive from this $y - w$ and $x/\sqrt{\mu}$ as series expanded in sines and cosines of multiples of w, depending on C_1 but not depending on μ. These are the series of the present chapter.

The series obtained first, expanded in powers of μ and analogous to those of no. 127, were—as is easy to demonstrate—of the following form:

$$x = \sqrt{C} + \mu C^{-1/2}\varphi_1 + \mu^2 C^{-3/2}\varphi_2 + \mu^3 C^{-5/2}\varphi_3 + \cdots\,,$$
$$y = w + \frac{\mu}{C}\psi_1 + \left(\frac{\mu}{C}\right)^2 \psi_2 + \cdots\,, \tag{20.21}$$

where φ and ψ are independent of both μ and C but are periodic, with period 2π with respect to w.

If we next set $C = C_1\,\mu$, then y and $x/\sqrt{\mu}$, as I have said, will no longer depend either on C_1 or on μ.

Thus to pass from the series of no. 127 to those in this chapter, it is necessary to set $C = C_1\,\mu$ and to again develop the series in ascending powers of μ. In the particular case of interest here, the new expansions obtained in this manner reduce to a single term since x contains only terms in $\sqrt{\mu}$ and since y contains only terms independent of μ.

As soon as C_1 is larger than 1, the quantity ω_1 is real and x and $y - w$ are periodic, with period 2π with respect to w. However, if C_1 is smaller than 1 then ω_1 becomes imaginary and ω_2 is real. Thus, we must take $\theta = \omega_2/2\pi$. Then, x and y (but no longer $y - w$) are periodic, with period 2π with respect to w.

If we take w as the independent variable, a discontinuity will occur due to the fact that the definition of θ changes when C_1 passes from a value larger than 1 to a value smaller than 1. This drawback can be avoided by taking $\theta w\sqrt{\mu}$ as the independent variable.

In fact, if x and y are expressed as functions of $\theta w\sqrt{\mu}$ and of C_1, the expressions obtained for $C_1 < 1$ are the analytic continuation of those obtained for $C_1 > 1$.

Let us start from the series (20.21), i.e., from the series of no. 127, and let us set there $C = C_1\,\mu$, so that

$$\frac{x}{\sqrt{\mu}} = C_1^{1/2} + C_1^{-1/2}\varphi_1 + C_1^{-3/2}\varphi_2 + \cdots,$$
$$y = w + \frac{1}{C_1}\psi_1 + \frac{1}{C_1^2}\psi_2 + \cdots. \tag{20.21a}$$

These series are convergent if C_1 is sufficiently large. In this case it is sufficient to sum them. When the series become divergent, the functions $x/\sqrt{\mu}$ and y can still be extended by analytic continuation. It happens that, as one continues to values of C_1 smaller than 1, the form of these functions will be completely modified since the real period becomes imaginary and vice versa.

Consequently, the double periodicity constitutes the explanation for the widely differing cases encountered in our study. The period which is real in the ordinary case becomes imaginary in the case of libration and vice versa. In the limiting case, one of the periods becomes infinite.

However, this raises the question as to how these results can be extended to the case in which $F = x^2 + \mu F_1$, where F_1 is an arbitrary function depending only on y and being periodic in y; then Eqs. (20.20) become

$$x = \sqrt{C - \mu F_1}\,,$$
$$\theta w = \int \frac{dy}{\sqrt{C - \mu F_1}}. \tag{20.20b}$$

Let A be the maximum of F_1.

This represents the ordinary case if

$$C > A\mu$$

and the case of libration if

$$C < A\mu\,.$$

However, here x, $\cos y$, and $\sin y$ are no longer elliptic functions of w. They are no longer uniform and doubly periodic for all real and imaginary values of w (despite the fact that they naturally remain uniform for all real values of w).

The foregoing results nevertheless remain valid.

It is sufficient to restrict ourselves to a domain D such that the imaginary part of θw be sufficiently small and that, on the other hand, C be sufficiently close to $A\mu$.

If x, $\cos y$, and $\sin y$ are considered as functions of θw and of C (or of $\theta w\sqrt{\mu}$ and of $C_1 = C/\mu$), then these functions will be uniform and doubly periodic, provided that one does not leave the domain D. One of the periods is equal to the integral on the right-hand side [Eq. (20.20b)] taken between 0 and 2π while the other period is equal to twice this same integral taken between two values of y_1 that render μF_1 equal to C.

This is sufficient for having the conditions under which one passes from the ordinary case to the libration case become the same as in the particular case studied first.

To facilitate extension of these results to the general case, it is possible to introduce the mean motion n_1 which we will denote here simply by n since we have everywhere eliminated the now useless subscript 1.

According to the princples of no. 3, it then follows that

$$n = \frac{1}{\theta} = \frac{2\pi}{\omega_1}.$$

On the other hand, if n is expanded in powers of μ, as had been done above, such that

$$n = n^0 + \mu n^1 + \cdots,$$

we will obtain, for $\mu = 0$,

$$\omega_1 = \int \frac{dy}{\sqrt{C}} = \frac{2\pi}{\sqrt{C}},$$

whence

$$n^0 \Phi = \sqrt{C}.$$

Thus we can take n^0 and μ as variables instead of C and μ.

In that case, series (20.21) will proceed in powers of μ and of $1/n^0$, which makes them analogous to the series considered in no. 201 which contained terms in

$$\frac{\mu^p}{(n_1^0)^q} \; (q \leqslant 2p - 1).$$

Finally, let us pass to the general case.

Let us consider the series of no. 127. These express the $2n$ variables x_i and y_i as functions of n arguments

$$w_1, \quad w_2, \quad \ldots, \quad w_n$$

and of n constants of integration. For example, we will select, for the n integration constants, the quantities that we had previously denoted by

$$n_1^0, \quad n_2^0, \quad \ldots, \quad n_n^0 .$$

In our series, which proceed in integral powers of μ, the small divisors

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

appear in the denominators.

Let us now assume that one of the small divisors becomes very small. For example, let us assume that this divisor is n_0^1 (since, if it were another, it would be impossible to carry out the change of variables of no. 202). Let us first see what the maximum exponent of n_1^0 in the denominator of each of the terms of our series will be.

According to what we have seen in nos. 201 and 211, the expansion of S contains only terms in

$$\frac{\mu_p}{(n_1^0)^q}$$

where

$$q \leqslant 2p - 1 .$$

If we then form the equations

$$x_i = \frac{dS}{dy_i}, \quad w_i = \frac{dS}{dx_i^0},$$

we will again find that the derivative dS/dy_i contains only terms in $\mu^q/(n_1^0)^q$. However, in the derivative dS/dx_i^0, there also will appear terms in

$$\mu^p \frac{d}{dx_i^0}\left[(n_1^0)^{-q}\right] = -q\mu^p \frac{dn_1^0}{dx_i^0}(n_1^0)^{-q-1},$$

i.e., terms in

$$\frac{\mu^p}{(n_1^0)^{q+1}} \quad (q \leqslant 2p - 1) .$$

From the equations

$$w_i = \frac{dS}{dx_i^0}$$

we then obtain the y_i as functions of w_i and of x_i^0 or, if preferred, as functions of w_i and of n integration constants

$$n_1^0, \quad n_2^0, \quad \ldots, \quad n_n^0 .$$

Accordingly, we see that the expansion of y will contain only terms in

$$\frac{\mu^p}{(n_1^0)^{q+1}} \quad (q \leqslant 2p-1) .$$

Let us then substitute the resultant values of y into the equations

$$x_i = \frac{dS}{dy_i} . \tag{20.22}$$

Before the substitution, the right-hand side of Eq. (20.22) contains only terms in

$$\frac{\mu^p}{(n_1^0)^q} .$$

Let

$$\frac{\mu^p}{(n_1^0)^q} \varphi_\alpha (y_1, y_2, \ldots, y_n, n_1^0, n_2^0, \ldots, n_n^0)$$

be one of these terms, since φ_α does not become infinite for $n_1^0 = 0$. After the substitution, we have

$$\varphi_\alpha = \sum \frac{\mu^\lambda}{(n_1^0)^h} \psi(w_i, n_i^0) \quad (h \leqslant 2\lambda) ,$$

where ψ does not become infinite for $n_1^0 = 0$.

The general term on the right-hand side of Eq. (20.22), after the substitution, will thus have the form

$$\frac{\mu^{p+\lambda}}{(n_1^0)^{q+h}} \psi ,$$

and it will be obvious that

$$q + h \leqslant 2(p+\lambda) - 1 .$$

This yields the general conclusion that, in the series of no. 127, the expressions of x_i contain only terms in

$$\frac{\mu^p}{(n_1^0)^q}$$

while those of y_i contain only terms in

$$\frac{\mu^p}{(n_1^0)^{q+1}} ,$$

where

$$q \leqslant 2p - 1 .$$

That said, let us assume that n_1^0 is very small and of the same order of magnitude as $\sqrt{\mu}$. Let us then put

$$n_1^0 = \alpha_1\sqrt{\mu} + \alpha_2\mu + \alpha_3\,\mu^{2/3} + \cdots ,$$

where α represent new constants. This is what we had done in no. 211. For example, one can just simply put

$$n_1^0 = \alpha_1\sqrt{\mu}\ .$$

After this substitution, a term in

$$\frac{\mu^p}{(n_1^0)^q}$$

is no longer of the order p in μ but of the order $p - q/2$.

Let us then group those of the terms of our series that have thus become of the same order in μ. Each of the resultant groups will form a subseries; the total series will be the sum of all these subseries.

To obtain the series of the present chapter, it is sufficient to form the sum of each of these subseries.

If α_1 is sufficiently large, the subseries will be convergent (the main series naturally will remain divergent and will be of value only from the viewpoint of formal calculus). However, if α_1 is too small for having the subseries become convergent, one can still proceed by analytic continuation, which is easy to understand.

In this manner, the function

$$\frac{1}{1+x},$$

defined by the series

$$1 - x + x^2 - \cdots ,$$

continues to exist after the series has ceased to converge.

Let us therefore consider the sum of one of these subseries. This sum will first be periodic, with period 2π, in $w_2, w_3, \ldots, w_n$. In addition, it will be a function of another argument $\theta_1 w_1$, uniform for the real values of this argument and for the values whose imaginary part is sufficiently small—or, in other words, as long as the argument $\theta_1 w_1$ remains in the interior of a certain domain containing the entire real axis. If α_1 varies between certain limits, this function, at the interior of this domain, will be uniform and doubly periodic, with one of the periods being real and the other imaginary. For a certain value of α_1, one of the periods becomes infinite after which the real period becomes imaginary, and vice versa.

This is the manner in which one passes from the ordinary case to the case of libration.

CHAPTER 21

Extension of the Bohlin Method

Extension to the Problem of No. 134

219. At the beginning of Chap. 11 we explained the particular difficulties encountered in the three-body problem. These difficulties are due to the fact that all variables of the first series, i.e., the variables x_i, do not figure in the function F_0.

In Chaps. 11 and 13 we showed how to overcome this difficulty and still derive a function S expanded in powers of μ, satisfying the Jacobi equation

$$F = C,$$

and having a form such as to have its derivatives with respect to y_i be periodic functions of y_i.

In addition, this function S depends on n constants of integration, for example, on the n quantities

$$n_1^0, n_2^0, \ldots, n_n^0.$$

If one of the linear combinations

$$m_1 n_1^0 + m_2 n_2^0 + \cdots + m_n n_n^0$$

is very small and of the order of magnitude of $\sqrt{\mu}$, we can put, as we had done in no. 211,

$$n_i = \alpha_i^0 + \alpha_i^1 \sqrt{\mu} + \alpha_i^2 \mu + \cdots,$$

where α_i^p are new constants, and then assume that

$$m_1 \alpha_1^0 + m_2 \alpha_2^0 + \cdots + m_n \alpha_n^0 = 0.$$

Let us then arrange each of the terms of S in ascending powers of $\sqrt{\mu}$ and let us group all terms containing like powers of $\sqrt{\mu}$ as factor. Each of the groups of terms obtained in this manner must exhibit the same property as the function S itself, i.e., the derivatives must be periodic functions of y.

Thus it can be predicted that Bohlin's method is still applicable to cases in which F_0 does not depend on all variables of the first series and, particularly, to the three-body problem. However, this application raises several delicate questions, which need be discussed in greater detail.

220. Therefore, let us imagine that F_0 does not depend on all variables of the first series. To show this explicitly, we will denote the variables of the first series by

$$x_1, x_2, \ldots, x_p, z_1, z_2, \ldots, z_q,$$

and the corresponding variables of the second series by

$$y_1, y_2, \ldots, y_p, u_1, u_2, \ldots, u_q,$$

and then assume that F_0 depends on all x_i but not on z_i.

We propose forming a function S of y and of u which would satisfy the Jacobi equation

$$F\left(\frac{dS}{dy_i}, \frac{dS}{du_i}, y_i, u_i\right) = C, \tag{21.1}$$

where it is assumed that, on the left-hand side, the variables of the first series x_i and z_i have been replaced by the corresponding derivatives dS/dy_i and dS/du_i.

We stipulate also that the function S be expandable in powers of $\sqrt{\mu}$ and that its derivatives be periodic with respect to y and to u.

Setting $u = 0$, Eq. (21.1) becomes

$$F_0\left(\frac{dS_0}{dy_1}, \frac{dS_0}{dy_2}, \ldots, \frac{dS_0}{dy_p}\right) = C_0, \tag{21.2}$$

which indicates that S_0 is of the form

$$S_0 = x_1^0\, y_1 + x_2^0\, y_2 + \cdots + x_p^0\, y_p + T_0,$$

where T_0 depends only on u.

We put

$$n_i^0 = -\frac{dF_0}{dx_i^0}.$$

If no linear relation with integral coefficients exists between the terms n_i^0, no difficulty arises and the calculations of Chap. 11 become applicable; one can form the function S_0, which will contain only integral powers of μ since the terms containing odd powers of $\sqrt{\mu}$ vanish.

Therefore, let us assume that a linear relation exists between the quantities n_i^0 and let

$$n_1^0 = 0$$

be this relation. This can be assumed to be so since, in the contrary case, the change of variables of no. 202 could be applied.

Before going further, let us introduce a new notation. Let U be any periodic function of y depending also on u. We will denote by

$$[U]$$

the mean value of U considered as a function of $y_2, y_3, \ldots, y_n$ and by

$$[[U]]$$

the mean value of U considered as a function of $y_1, y_2, \ldots, y_n$.

From this definition it results that $[U]$ is a function of y_1 and of u whereas $[[U]]$ is a function of only u.

Assuming that U, instead of being a periodic function of y, is a function such that its derivatives are periodic and that

$$U = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_p^0 y_p + U',$$

with U' being periodic and x_i^0 being constants, then we can put

$$[U] = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_p^0 y_p + [U'],$$

and

$$[[U]] = x_1^0 y_1 + x_2^0 y_2 + \cdots + x_p^0 y_p + [[U']].$$

After this, let us return to Eqs. (19.19) of no. 204. The first of these equations is nothing else but Eq. (21.2), which we had considered above.

The second equation indicates that

$$\frac{dS_1}{dy_2}, \quad \frac{dS_1}{dy_3}, \quad \ldots, \quad \frac{dS_1}{dy_n}$$

are constants. Without restricting the generality, we can assume that these constants are zero. This actually means returning to the hypotheses (19.25) of no. 204.

Then, S_1 will be a function only of y_1 and of u such that

$$S_1 = [S_1].$$

Let us now consider the third equation of system (19.19).

The function Φ, which appears on the right-hand side, is nothing else but $-F_1$.

The second term on the left-hand side reduces to

$$\frac{1}{2} \frac{d^2F_0}{dx_1^{0\,2}} \left(\frac{dS_1}{dy_1}\right)^2,$$

since the other dS_1/dy_i are zero.

On putting

$$\frac{1}{2} \frac{d^2F_0}{dx_1^{0\,2}} = A,$$

the equation will become

$$-\sum n_i^0 \frac{dS_2}{dy_i} + A\left(\frac{dS_1}{dy_1}\right)^2 = C_2 - F_1. \tag{21.3}$$

However, it is important to note that here the function F_1 is not known. In fact, this function depends on x, z, y, and u and the terms x_i must there be replaced by the terms x_i^0 which are known, while the terms z_i must be replaced by

$$\frac{dS_0}{du_i} = \frac{dT_0}{du_i}$$

which are not known.

Let us now take the mean values of both sides with respect to $y_2, y_3, \ldots, y_n$. First, the terms $[dS_2/dy_i]$ reduce to constants and, without restricting the generality, it can then be assumed that these constants are zero since this means returning to the hypotheses (19.26) of no. 204.

On the other hand,

$$\left[\frac{dS_1}{dy_1}\right]^2 = \left(\frac{dS_1}{dy_1}\right)^2,$$

since S_1 does not depend on $y_2, y_3, \ldots, y_n$.

Finally, it is of importance to mention that, in calculating the mean value of F_1, one can operate as though the functions dT_0/du_i (which must here be substituted for z_i) were constants since these functions do not depend on $y_2, y_3, \ldots, y_n$.

Thus we obtain

$$A\left(\frac{dS_1}{dy_1}\right)^2 = C_2 - [F_1], \tag{21.3a}$$

whence

$$\frac{dS_1}{dy_1} = \sqrt{\frac{C_2 - [F_1]}{A}}.$$

Let us now take the mean-square values of both sides with respect to y_1, so that

$$\left[\left[\frac{dS_1}{dy_1}\right]\right] = \left[\left[\sqrt{\frac{C_2 - [F_1]}{A}}\right]\right].$$

If S_1 is a function whose derivatives are periodic, then the left-hand side will reduce to a constant which we denote by h. Consequently, we must have

$$\left[\left[\sqrt{\frac{C_2 - [F_1]}{A}}\right]\right] = h$$

or

$$\int_0^{2\pi} \sqrt{C_2 - [F_1]}\, dy_1 = 2\pi Ah. \tag{21.4}$$

The left-hand side depends on u_i and, in addition, on the derivatives dT_0/du_i that enter F_1. Consequently, a partial differential equation defines T_0. We will define this function T_0 in such a manner that its derivatives become periodic. Then, Eq. (21.4) can be written in the form

$$\Theta\left(\frac{dT_0}{du_i}, u_i\right) = 2\pi Ah. \tag{21.4a}$$

Thus everything reduces to an integration of this equation (21.4a). We will turn to this later. Let us first assume that the integration is possible and let

$$T_0 = z_1^0 u_1 + z_2^0 u_2 + \cdots + z_q^0 u_q + T'_0,$$

be a complete solution of this equation, containing the q constants of integration z_i^0. Naturally, we assume that T'_0 is a function of u_i and of the constants z_i^0, periodic with respect to u_i.

After T_0 has been determined in this manner, it becomes possible to calculate dS_1/dy_1 and thus also

$$S_1 - [[S_1]].$$

Consequently, we can write

$$S_1 = S'_1 + T_1,$$

where S'_1 is a known function of y_1 and of u, while T_1 is a still unknown function of u.

Equation (21.3) will then yield

$$-\sum n_i^0 \frac{dS_2}{dy_i} = [F_1] - F_1,$$

from which we deduce

$$\frac{dS_2}{dy_2}, \quad \frac{dS_2}{dy_2}, \quad \ldots, \quad \frac{dS_2}{dy_n}, \quad S_2 - [S_2].$$

Let us now consider the fourth equation of system (19.19).

In the second term on the left-hand side, the quantities dS_2/dy_i are known except for dS_2/dy_1. Therefore, this second term can be written as follows:

$$2A \frac{dS_2}{dy_1}\frac{dS_1}{dy_1} + \Phi.$$

On the other hand, in Eq. (19.19) we designated the right-hand side by Φ

since it was completely known. However, here this is no longer the case since the right-hand side depends on dS_1/du_i and thus also on dT_1/du_i which are unknown. It is easy to demonstrate that the right-hand side will have the form

$$-\sum \frac{dF_1}{dz_i}\frac{dT_1}{du_i} + \Phi,$$

where Φ is known.

Therefore, our equation is written as

$$-\sum n_i^0 \frac{dS_3}{dy_i} + 2A\frac{dS_2}{dy_1}\frac{dS_1}{dy_1} + \sum \frac{dF_1}{dz_i}\frac{dT_1}{du_i} = \Phi. \qquad (21.5)$$

It is obvious that, in dF_1/dz_i, the terms x_i and z_i must be replaced by x_i^0 and dT_0/du_i, respectively.

Let us take the mean values of both sides with respect to $y_2, y_3, \ldots, y_n$. As above, we can assume that the mean values of dS_3/dy_i $(i > 1)$ are zero. This yields

$$2A\frac{d\,[S_2]}{dy_1}\frac{dS_1}{dy_1} + \sum \frac{d\,[F_1]}{dz_i}\frac{dT_1}{du_i} = \Phi. \qquad (21.6)$$

From this, we derive that

$$\frac{d\,[S_2]}{dy_1} = \frac{\Phi - \sum \dfrac{d\,[F_1]}{dz_i}\dfrac{dT_1}{du_i}}{2A\dfrac{dS_1}{dy_1}}.$$

Both sides of this equation depend on y_1 and on u. The mean value of the left-hand side must reduce to a constant to which, without restricting the generality, we can assign an arbitrary value, for example, the value zero. Consequently, we must have

$$\left[\left[\frac{\Phi - \sum \dfrac{d\,[F]}{dz_i}\dfrac{dT_1}{du_i}}{2A\dfrac{dS_1}{dy_1}}\right]\right] = 0,$$

which can be written in the form

$$-\int_0^{2\pi} \frac{\sum \dfrac{d\,[F_1]}{dz_i}\dfrac{dT_1}{du_i}}{2\sqrt{C_2 - [F_1]}}\,dy = \Phi, \qquad (21.7)$$

or else

$$\sum \frac{d\Theta}{dz_i}\frac{dT_1}{du_i} = \Phi. \tag{21.7a}$$

Here, Θ is a function of z_i and of u_i, periodic with respect to u_i. On replacing there the terms z_i by dT_0/du_i, one obtains the left-hand side of Eq. (21.4a); similarly, in Eq. (21.7a) we assume that, in the derivatives $d\Theta/dz_i$, the terms z_i have been replaced by dT_0/du_i.

Equation (21.7a) must determine T_1. We will demonstrate that its integration is easy if one knows how to integrate Eq. (21.4a).

Actually, if we know how to integrate Eq. (21.4a), we will have defined a function T_0 depending on u_i and on q constants z_i^0 such that, if its derivatives in Θ are substituted for the terms z_i, this function Θ will reduce to a constant with respect to u_i, i.e., to a function of z_i^0 which we will call

$$\theta(z_1^0, z_2^0, \ldots, z_q^0).$$

On the other hand, we will put

$$z_i = \frac{dT_0}{du_i}; \quad u_i^0 = \frac{dT_0}{dz_i^0}. \tag{21.8}$$

This will yield $2q$ relations between the $4q$ quantities z_i, u_i, z_i^0, u_i^0, in such a manner that we can take, as independent variables, either z_i and u_i or z_i^0 and u_i or z_i^0 and u_i^0.

To avoid any confusion, we will represent the derivatives by the symbol d if we use z_i and u_i or z_i^0 and u_i^0 as variables and by the symbol ∂ if we use z_i^0 and u_i as variables.

In Eq. (21.7a), Θ must be considered as expressed by means of z_i and u_i (since it is only after the differentiation that z_i is replaced by dT_0/du_i). Conversely, T_1 is a function of u_i depending also on the integration constants z_i^0. With our new notation, Eq. (21.7a) must then be written in the form

$$\sum \frac{d\Theta}{dz_i}\frac{\partial T_1}{\partial u_i} = \Phi.$$

On the other hand, we identically have

$$\Theta = \theta$$

and, since θ depends only on z_i^0,

$$\frac{\partial \Theta}{\partial u_i} = 0.$$

This equation can also be written as

$$\frac{d\Theta}{du_i}+\sum\frac{d\Theta}{dz_k}\frac{\partial z_k}{\partial u_i}=0.$$

We have, in addition

$$\frac{\partial T_1}{\partial u_i}=\frac{dT_1}{du_i}+\sum\frac{dT_1}{dz_k}\frac{\partial z_k}{\partial u_i}=0.$$

After this, one can successively find, on transforming Eq. (21.7a),

$$\sum\frac{d\Theta}{dz_i}\frac{dT_1}{du_i}+\sum\frac{d\Theta}{dz_i}\frac{dT_1}{dz_k}\frac{\partial z_k}{\partial u_i}=\Phi,$$

or, on permuting the indices,

$$\sum\frac{d\Theta}{dz_i}\frac{dT_1}{du_i}+\sum\frac{d\Theta}{dz_k}\frac{dT_1}{dz_i}\frac{\partial z_k}{\partial u_i}=\Phi;$$

since

$$\frac{\partial z_k}{\partial u_i}=\frac{\partial z_i}{\partial u_k}=\frac{\partial^2 T_0}{\partial u_i \partial u_k},$$

whence

$$\sum\left(\frac{d\Theta}{dz_i}\frac{dT_1}{du_i}-\frac{d\Theta}{du_i}\frac{dT_1}{dz_i}\right)=\Phi$$

or, using u_i^0 and z_i^0 as variables,

$$\sum\left(\frac{d\Theta}{dz_i^0}\frac{dT_1}{du_i^0}-\frac{d\Theta}{du_i^0}\frac{dT_1}{dz_i^0}\right)=\Phi.$$

Since Θ reduces to θ which does not depend on u_i^0, we finally obtain

$$\sum\frac{d\theta}{dz_i^0}\frac{dT_1}{du_i^0}=\Phi, \qquad (21.7b)$$

where Φ must be expressed as a function of the variables u_i^0 and of the constants of integration z_i^0. Since the derivatives of θ depend only on the constants z_i^0, they also are constants. It follows from this that, since Eq. (21.7b) is an expression with constant coefficients, it can be directly integrated.

The quantity Φ is periodic with respect to u_i. It happens often that the form of the function T_0 and of Eqs. (21.8) will be such that the terms u_i are uniform functions of u_i^0 and vice versa. Then, the differences $u_i - u_i^0$ will be periodic functions either of u_i or of u_i^0.

Thus Φ, which is periodic with respect to u_i, will also be periodic with respect to u_i^0. In that case, Eq. (21.7b) can be integrated in such a manner that the derivatives dT_1/du_i^0 are periodic with respect to u_i^0 or, what amounts to the same thing, in such a manner that the derivatives $\partial T_1/\partial u_i$ are periodic with respect to u_i or else that T_1 increases by a constant when u_i increases by 2π.

Since Eq. (21.7) has been integrated in this manner, Eq. (21.6) will yield $d[S_2]/dy_1$, so we can write

$$S_2 = S'_2 + T_2,$$

where S'_2 is a completely known function of y and u while T_2 is an unknown function which depends only on u.

Then, Eq. (21.5) can be written as

$$\sum n_i^0 \frac{dS_3}{dy_i} = \Phi$$

which then determines

$$\frac{dS_3}{dy_2}, \quad \frac{dS_3}{dy_3}, \quad \ldots, \quad \frac{dS_3}{dy_n}, \quad S_3 - [S_3]$$

and so on.

Extension to the Three-Body Problem

221. Everything is thus brought back to an integration of Eq. (21.4). Let us see what the form of this equation will be in the case of the three-body problem. The equation is written in the form

$$\int_0^{2\pi} \sqrt{C_2 - [F_1]}\, dy_1 = 2\pi A h.$$

But what is the form of $[F_1]$?

As variables, we will select the quantities

$$\begin{array}{cccc} \Lambda_1, & \Lambda'_1, & \xi_1, & \xi'_1, \\ \lambda_1, & \lambda'_1, & \eta_1, & \eta'_1 \end{array}$$

defined in no. 145 [see Eq. (12.16)], to which, if the three bodies do not move in the same plane, we must add the variables

$$\begin{array}{cc} p, & p', \\ q, & q', \end{array}$$

defined in Vol. I, Eqs. (1.28).

After this, the function F will be expanded in positive powers of μ, ξ_1, ξ'_1, η_1, η'_1, p, q, p', q' and in sines and cosines of multiples of λ_1 and λ'_1. A term in

$$\begin{matrix}\cos \\ \sin\end{matrix}(m\lambda_1 + m'\lambda_1')$$

must contain as a factor a mononomial whose degree with respect to the variables $\xi_1, \eta_1, p, q, \ldots$ is at least equal to $|m + m'|$ and could differ from this only by an even number. Finally, F_0 will depend only on Λ_1 and Λ_1'.

Let us then imagine that we have

$$m\frac{dF_0}{d\Lambda_1^0} + m'\frac{dF_0}{d\Lambda_1'^0} = 0,$$

where m and m' are two integers, while Λ_1^0 and $\Lambda_1'^0$ are two constants to which we will equate $dS_0/d\lambda_1$ and $dS_0/d\lambda_1'$, and which consequently are analogous to the constants that we denoted by x_i^0 in the previous number. We then put

$$m\lambda_1 + m'\lambda_1' = y_1.$$

For forming $[F_1]$ it is only necessary to eliminate, in F_1, all terms depending on λ_1 or on λ_1' except those that depend only on y_1.

To show explicitly the degree of each term with respect to the eccentricities or inclinations, let us replace everywhere

$$\xi_1, \quad \eta_1, \quad \xi_1', \quad \eta_1', \quad p, \quad q, \quad p', \quad q'$$

by

$$\epsilon\xi_1, \quad \epsilon\eta_1, \quad \epsilon\xi_1', \quad \epsilon\eta_1', \quad \epsilon p, \quad \epsilon q, \quad \epsilon p', \quad \epsilon q'$$

and let us account for the degree of each of the terms of F_1 with respect to ϵ.

We will have

$$[F_1] = R + R',$$

where R represents the set of terms that are simultaneously independent of λ_1 and of λ_1', such that

$$R = [[F_1]]$$

and where R' is the set of terms depending on y_1, and on y_1 alone.

Then, R can be expanded in powers of ϵ^2 so that

$$R = R_0 + \epsilon^2 R_2 + \epsilon^4 R_4 + \cdots.$$

As to R', it is divisible by

$$\epsilon^{|m+m'|}.$$

In general, we will have

$$|m + m'| > 2,$$

in such a manner that we can put

$$R' = \epsilon^3 R''.$$

Here, R_0 depends only on Λ_1^0 and on $\Lambda_1'^0$ and may be considered as being a constant. Consequently, we can put

$$C_2 = R_0 + k_0^2 + \epsilon^2 k_1$$

and, at the same time,

$$Ah = k_0 + \epsilon^2 k_1',$$

such that Eq. (21.4) becomes

$$\int_0^{2\pi} \sqrt{k_0^2 + \epsilon^2 k_1 - \epsilon^2 R_2 - \epsilon^3 R'' - \epsilon^4 R_4 - \cdots}\; dy_1 = 2\pi(k_0 + \epsilon^2 k_1'),$$

where, on expanding the radical in powers of ϵ, reducing, and dividing by $2\pi\epsilon^2$,

$$\frac{k_1 - R_2}{2k_0} + \epsilon Z = k_1'.$$

Here, Z represents a function expandable in positive powers of ϵ, ξ, η, p, and q.

Finally, putting

$$k_1 - 2k_0 k_1' = K,$$

we obtain

$$R_2 - 2\epsilon k_0 Z = K.$$

The function R_2 is the same as that denoted by this symbol in no. 131 (except that the symbols ξ and η carry the subscript 1). Therefore, just as in no. 131, we can define the variables ρ_i and ω_i (but forming them with ξ_1 and η_1 instead of with ξ and η), and we will take as new variables,

$$\Lambda_1, \quad \Lambda_1', \quad \rho_i,$$
$$\lambda_1, \quad \lambda_1', \quad \omega_i.$$

Then, R_2 reduces to

$$2A_1\rho_1 + 2A_2\rho_2 + 2A_3\rho_3 + 2A_4\rho_4$$

(see the concluding remarks of no. 131).

On replacing ρ_i by $dT_0/d\omega_i$, we finally will have to integrate the equation

$$\Theta\left(\frac{dT_0}{d\omega_i}, \omega_i\right) = R_2 - 2\epsilon k_0 Z = K. \tag{21.4b}$$

The left side Θ is periodic with respect to ω_i and can be expanded in powers of ϵ; if we set there $\epsilon = 0$, this will reduce to R_2 and no longer depend on ω_i but only on $dT_0/d\omega_i$. Consequently, we can apply the procedures used in no. 125.

Thus the integration of Eq. (21.4) to which we have reduced our problem is entirely feasible.

The case in which

$$m + m' = \pm 1 \quad \text{or} \quad \pm 2$$

is treated in exactly the same manner. The case in which

$$m + m' = 0,$$

whence

$$\Lambda_1^0 = \Lambda_1'^0,$$

i.e., the case in which the two major axes differ very little, presents special difficulties.

Discussion of the Series

222. Let us return to the notations of no. 220 and assume that we have determined the function S by the procedures of that section. The problem is not yet completely solved. For this, it is still necessary to form the equations

$$\frac{dS}{dx_k^0} = w_k + \theta_k w_1 \quad (k > 1),$$
$$\frac{dS}{dC_2} = \theta_1 w_1, \quad \frac{dS}{dz_i^0} = w_i' + \theta_i' w_1, \quad x_i = \frac{dS}{dy_i}, \quad z_i = \frac{dS}{du_i}; \tag{21.9}$$

where θ and θ' will be properly chosen functions of the constants x_i^0 and z_i^0. After this, it is necessary to solve these equations to obtain x_i, y_i, and u_i as functions of x_i^0, z_i^0, w_i, and w_i'. Finally, it is necessary to replace w_i and w_i' by linear functions of time whose coefficients will be properly chosen. This will yield the expressions of the coordinates x, y, z, u as a function of time.

Let us first see what the form of Eqs. (21.9) will be.

The function S, since its derivatives are periodic, can be written in the form

$$S = \beta y_1 + x_2^0 y_2 + x_3^0 y_3 + \cdots + x_p^0 y_p + z_1^0 u_1 + \cdots + z_q^0 u_q + S',$$

where β is a constant independent of the y and the u, while S' is periodic with respect to y and u. The coefficients of y_k $(k > 1)$ and of u_i, with restricting the generality, can be assumed as being equal to x_k^0 and to z_i^0. In fact, this returns us exactly to hypotheses (19.26) of no. 204.

As to β, this can be expanded in powers of $\sqrt{\mu}$:

$$\beta = \beta_0 + \sqrt{\mu}\beta_1 + \mu\beta_2 + \cdots,$$

where β_0 is equal to x_1^0, and β_1 is equal to the constant h of Eq. (21.4) in no. 220.

Similarly, S' can be expanded in powers of $\sqrt{\mu}$

$$S' = S'_0 + S'_1\sqrt{\mu} + \cdots$$

with

$$S'_0 = T'_0,$$

$$S'_1 = \int \left(\sqrt{\frac{C_2 - [F_1]}{A}} - h \right) dy_1 + T'_1.$$

Then, Eqs. (21.9) become

$$\begin{aligned} w_k + \theta_k w_1 &= y_k + \frac{d\beta}{dx_k^0} y_1 + \frac{dS'}{dx_k^0}, \\ \theta_1 w_1 &= \frac{d\beta}{dC_2} y_1 + \frac{dS'}{dC_2}, \\ w'_i + \theta'_i w_1 &= u_i + \frac{d\beta}{dz_i^0} y_1 + \frac{dS'}{dz_i^0}. \end{aligned} \tag{21.10}$$

We are thus led to take

$$\theta_k = \frac{d\beta}{dx_k^0}, \quad \theta_1 = \frac{d\beta}{dC_2}, \quad \theta'_i = \frac{d\beta}{dz_i^0}.$$

However, a difficulty is produced by the following fact: Since $\beta_0 = x_1^0$ is independent of both C_2 or z_i^0, the quantities θ_1 and θ'_i vanish for $u = 0$ and are divisible by $\sqrt{\mu}$. Conversely, dS'/dC_2, for $\mu = 0$, reduces to

$$\frac{dT'_0}{dC_2}$$

and does not vanish.

Next, we must set

$$w_i = n_i t + \varpi_i; \quad w'_i = n'_i t + \varpi'_i,$$

where n are determined constants while ϖ are arbitrary constants. For determining n, we proceed in the following manner:

On replacing, in F, the terms x_i and z_i by dS/dy_i and dS/du_i, this function F, according to the very definition of the function S, must reduce to a constant or, rather, to a function of the integration constants x_k^0, C_2, and z_i^0. Thus let

$$F = \varphi(x_k^0, C_2, z_i^0),$$

so that

$$n_k + \theta_k n_1 = -\frac{d\varphi}{dx_k^0} \quad (k > 1),$$

$$\theta_1 n_1 = -\frac{d\varphi}{dC_2}, \qquad (21.11)$$

$$n_i' + \theta_i' n_1 = -\frac{d\varphi}{dz_i^0}.$$

It is obvious that the quantities n can be expanded in powers of $\sqrt{\mu}$. To ascertain the form of the series, let us expand the function φ itself in powers of μ. This yields

$$\varphi = C_0 + \mu C_2 + \mu^2 C_4 + \cdots .$$

We have, in addition,

$$C_0 = F_0(x_1^0, x_2^0, \ldots, x_p^0),$$

whence

$$\frac{dC_0}{dx_k^0} = \frac{dF_0}{dx_k^0} + \frac{dF_0}{dx_1^0}\frac{dx_1^0}{dx_k^0} = -n_k^0 - n_1^0\frac{dx_1^0}{dx_k^0} = -n_k^0,$$

since n_1^0 is zero.

In addition, it can be seen that

$$\frac{d\varphi}{dC_2} = \mu$$

and that the expansion of $d\varphi/dz_i^0$ starts with a term in μ^2.

The second equation of system (21.11) where the coefficient θ_1 is divisible by $\sqrt{\mu}$ and the right-hand side by μ, indicates that the expansion of n_1 starts with a term in $\sqrt{\mu}$. Since θ_i' is also divisible by $\sqrt{\mu}$ and $\theta_i' n_i$ by μ, and the right-hand by μ^2, the third equation of system (21.11) indicates that n_i' is divisible by μ.

Let us note, on the other hand, that Eqs. (21.10) can be considerably simplified. Until now, we had assumed that S and S' were expressed as functions of the variables y and u and of the constants x_k^0, C_2, and z_i^0. Let us now put

$$\beta = x_1^0 + \gamma\sqrt{\mu}$$

and assume, which amounts to the same thing, that S and S' are expressed as functions of y and u as well as of the constants x_k^0, γ, and z_i^0. In this manner, our Eqs. (21.10) become

$$(w_k - y_k) + \frac{dx_1^0}{dx_k^0}(w_1 - y_1) = \frac{dS'}{dx_k^0},$$

$$\sqrt{\mu}(w_1 - y_1) = \frac{dS'}{d\gamma}, \qquad (21.10a)$$

$$w_i' - u_i = \frac{dS'}{dz_i^0}.$$

Nevertheless, it still is true that, although Eqs. (21.10) and (21.10a) implicitly yield our coordinates in functions of w, they can no longer be solved by the method given in no. 30 and that, consequently, the correlations between these coordinates and w are much more complicated than in no. 127 or in Chaps. 11 and 20.

We will restrict ourselves to the following remark: What becomes of our equations in the case of $\mu = 0$? Do they imply a contradiction? Since n_1 and n_i' vanish for $\mu = 0$, the quantities w_1 and w_i' reduce to constants $\overline{\omega}_1$ and $\overline{\omega}_i'$, such that we first will have

$$\overline{\omega}_i' - u_i = \frac{dT_0'}{dz_i^0}.$$

Since T_0' contains no variables other than u_i, these equations indicate that the terms u_i are constants. Let us pass to the second equation of the system (21.10a) and, since $\overline{\omega}_1$ is an arbitrary constant, let us set this equal to $\alpha_1/\sqrt{\mu}$ where α_1 is a given and finite constant. The second equation then becomes

$$\alpha_1 = \frac{dT_0'}{d\gamma} \quad \text{or} \quad \frac{dT_0'}{d\gamma} = \text{const.}$$

and, since T_0' depends only on u which are constants, this equation is satisfied identically.

Let us now see what becomes of the first equation. Again, let us put

$$\overline{\omega}_k = \frac{\alpha_k}{\sqrt{\mu}} + \alpha_k',$$

where α_k and α_k' are finite constants. Let us replace $w_1 - y_1$ by its value derived from the second equation and let us write the terms in $1/\sqrt{\mu}$ as well as the terms independent of $\sqrt{\mu}$.This yields

$$\frac{\alpha_k}{\sqrt{\mu}} + (\alpha_k' + n_k t - y_k) + \frac{dx_1^0}{dx_k^0}\left(\frac{1}{\sqrt{\mu}}\frac{dT_0'}{d\gamma} + \frac{dS_1'}{d\gamma}\right) = \frac{dS_0'}{dx_k^0} = \frac{dT_0'}{dx_k^0},$$

whence

$$\alpha_k + \frac{dx_1^0}{dx_k^0}\frac{dT_0'}{d\gamma} = 0 \quad \text{or} \quad \frac{dT_0'}{d\gamma} = \text{const.},$$

$$\alpha_k' + n_k t - y_k + \frac{dx_1^0}{dx_k^0}\frac{dS_1'}{d\gamma} = \frac{dT_0'}{dx_k^0}.$$

The first is satisfied identically, and the second yields y_k.

Second Method

223. The calculations can also be arranged differently and, instead of using Eq. (21.4) of no. 220, one can directly apply Eq. (21.3a), which is written in the form

$$A\left(\frac{dS_1}{dy_1}\right)^2 = C_2 - [F_1]. \tag{21.3a}$$

Let us return to the notations of no. 221 and let us select as variables the quantities

$$\begin{matrix} \Lambda_1, & \Lambda_1', & \rho_i, \\ \lambda_1, & \lambda_1', & \omega_i, \end{matrix}$$

as they had been defined in no. 221. Let us see what the form of Eq. (21.3a) will be.

(i) Both sides of this equation will no longer depend in an arbitrary manner on λ_1 and on λ_1' but only on

$$m\lambda_1 + m'\lambda_1' = y_1,$$

where m and m' are integers defined in no. 221. In effect, we have obtained $[F_1]$ by eliminating in F_1 all terms that depend on λ_1 and on λ_1' in a manner other than via the combination $m\lambda_1 + m'\lambda_1'$.

(ii) The two sides also depend on Λ_1 and Λ_1'. However, these quantities must there be replaced by the constants Λ_1^0 and $\Lambda_1'^0$, analogous to x_i^0. Thus A becomes a constant.

(iii) Both sides are periodic with respect to y_1 and ω_i.

(iv) Both sides can be expanded in integral powers of ϵ and in fractional powers of ρ_i, which must be replaced by $dT_0/d\omega_i$.

Thus Eq. (21.3a) can be written as

$$H\left(\frac{dS_1}{dy_1}, \frac{dT_0}{d\omega_i}, y_1, \omega_i\right) = C_2. \tag{21.3b}$$

Let us consider the expansion of H in powers of ϵ. The term independent of ϵ reduces to

$$A\left(\frac{dS_1}{dy_1}\right)^2 + R_0,$$

where R_0, defined as in no. 221, is a constant that depends only on Λ_1^0 and $\Lambda_1'^0$.

The term in ϵ is zero (except if $m + m' = \pm 1$, a case which we will disregard here).

The term in ϵ^2 reduces to

$$R_2 + 2A_1\rho_1 + 2A_2\rho_2 + 2A_3\rho_3 + 2A_4\rho_4.$$

The first term that depends on y_1 is the term in

$$\epsilon^{|m+m'|}.$$

The manner in which Eq. (21.3b) can be treated is as follows: Let us attempt to expand S_1 in powers of ϵ and let

$$S_1 = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots .$$

Let us similarly expand C_2 and T_0 and let

$$C_2 = \gamma_0 + \epsilon\gamma_1 + \epsilon^2\gamma_2 + \cdots, \quad T_0 = V_0 + \epsilon V_1 + \cdots .$$

On replacing T_0 by this value in R and expanding R, we obtain

$$R = R_0' + \epsilon^2 R_2' + \epsilon^3 R' + \cdots .$$

We first find

$$A\left(\frac{dU_0}{dy_1}\right)^2 + R_0' = \gamma_0,$$

which demonstrates that dU_0/dy_1 is a constant. Thus, let

$$U_0 = \alpha y_1,$$

where α is a constant that depends on the integration constant γ_0. From this it follows that

$$2\alpha A \frac{dU_1}{dy_1} = \gamma_1,$$

which indicates that dU_1/dy_1 again is a constant. Without restricting the generality, we can assume that U_1 and γ_1 are zero.

This yields

$$2\alpha A \frac{dU_2}{dy_1} + 2\sum A_i \frac{dV_0}{d\omega_i} = \gamma_2.$$

This equation shows that dU_2/dy_1 again is a constant which we can again consider as zero without restricting the generality; this leaves the following equation to be treated:

$$2\sum A_i \frac{dV_0}{d\omega_i} = \gamma_2,$$

which indicates that $dV_0/d\omega_i$ are constants that can be arbitrarily selected since γ_2 is arbitrary.

From this, we obtain

$$2\sum A_i \frac{dT_1}{d\omega_i} + 2\alpha A \frac{dU_3}{dy_1} = \gamma_3.$$

We can also assume U_3 and γ_3 to be zero without restricting the generality, since

$$2\alpha A \frac{dU_4}{dy_1} + R'_4 = \gamma_4.$$

In addition, we will assume U_4 to be zero, which leaves

$$R'_4 = \gamma_4,$$

which readily permits determining T_2 since y_1 does not enter there.

We can continue in this manner up to the term in $\epsilon^{|m+m'|}$. Let us put $|m + m'| = q$, we then obtain

$$2\alpha A \frac{dU_q}{dy_1} + R'_q + M_q \cos y_1 + N_q \sin y_1 = \gamma_q.$$

Here, M_q and N_q, which depend on the ω_i and on $V_0, V_1, \ldots, V_{q-3}$ which are known functions of ω_i, can be considered as known.

So far as R'_q is concerned, we will have

$$R'_q = 2\sum A_i \frac{dV_{q-2}}{d\omega_i} + L_q,$$

where L_q is a known function of ω_i.

Then, the above equation can be decomposed into two equations and we can write

$$2\alpha A \frac{dU_q}{dy_1} = -M_q \cos y_1 + N_q \sin y_1,$$

$$2\sum A_i \frac{dV_{q-2}}{d\omega_i} = \gamma_q - L_q.$$

The right-hand sides are known so that it is easy to derive from these equations the values of U_q and V_{q-2}. It is obvious that the derivatives of V_{q-2} are periodic with respect to ω_i. Again, without restricting the generality, we can select γ_q in such a manner as to cause the mean value of $\gamma_q - L_q$ to vanish. Then V_{q-2} will itself be periodic. So far as U_q is concerned, it is obvious that it will be periodic with respect to y_1 and ω_i.

We will continue in this manner. Equating the coefficients of $\epsilon^p (p > q)$, we will find

$$2\alpha A \frac{dU_p}{dy_1} + 2\sum A_i \frac{dV_{p-2}}{d\omega_i} = \gamma_p + \Phi, \qquad (21.12)$$

where Φ is a known function, periodic with respect to y_1 and ω_i. We will assume the function Φ as being expanded in a trigonometric series and will select γ_p in such a manner as to cancel the mean value of the right-hand side.

We then put

$$\gamma_p + \Phi = \Phi' + \Phi'',$$

where Φ' represents the set of terms that depend on y_1, while Φ'' represents the set of terms that do not depend on y_1, such that

$$\Phi'' = [[\gamma_p + \Phi]].$$

Then, Eq. (21.12) will be decomposed into two expressions by writing

$$2\alpha A \frac{dU_p}{dy_1} = \Phi',$$

$$2\sum A_i \frac{dV_{p-2}}{d\omega_i} = \Phi''.$$

These two equations will determine V_{p-2} and U_p. The resultant two functions will be periodic.

Since Eq. (21.3a) of no. 220 has been integrated in this manner, Eq. (21.3) will yield $S_2 - [S_2]$ from which Eqs. (21.5) and (21.6) can then be formed.

We will treat Eq. (21.6) as we had treated Eq. (21.3a). Since the two sides of Eq. (21.6) can be expanded in powers of ϵ, we will similarly expand $[S_2]$ and T_1, writing

$$[S_2] = U_0' + \epsilon U_1' + \epsilon^2 U_2' + \cdots,$$

$$T_1 = V_0' + \epsilon V_1' + \epsilon^2 V_2' + \cdots.$$

Then the coefficients of like powers of ϵ will be equated on both sides of Eq. (21.6), yielding a series of equations which will permit determining, by recurrence, the quantities U_i' and V_i'.

On equating the coefficients of ϵ^p, one obtains an equation that can be used for determining U_p' and V_{p-2}'. This equation will have the same form as Eq. (21.12) except that U_p and V_{p-2} would be replaced there by U_p' and V_{p-2}'. Consequently, this equation can be treated in exactly the same manner.

After Eq. (21.6) has thus been integrated, one can continue in the same manner.

Case of Libration

224. How can the case of libration be presented?

Let us return to the equations of the previous section and assume that

$$\alpha = U_0 = 0.$$

The calculation is continued as above until one arrives at the equation obtained by equating the coefficients of ϵ^q. This will yield

$$U_i = 0 \quad \left(i = 1,2,3, \ldots, \frac{q}{2} - 1\right)$$

and, if q is even, the equation in ϵ^q can be written as

$$A\left(\frac{dU_{q/2}}{dy_1}\right)^2 + 2\sum A_i \frac{dV_{q-2}}{d\omega_i} + L_q + M_q \cos y_1 + N_q \sin y_1 = \gamma_q. \tag{21.13}$$

If, for abbreviation, we put

$$2\sum A_i \frac{dV_{q-2}}{d\omega_i} = X$$

and if we eliminate, for the moment, the subscript $q/2$ of U as well as the subscripts q of L, M, N, and γ, we obtain

$$\frac{dU_{q/2}}{dy_1} = \sqrt{\frac{\gamma - L - X - M\cos y_1 - N\sin y_1}{A}} = \sqrt{Z}\,,$$

calling, for abbreviation, Z the quantity under the radical.

The integral

$$\int \sqrt{Z}\,dy_1$$

is an elliptic integral of the second kind. One of its periods is

$$\int_0^{2\pi} \sqrt{Z}\,dy_1.$$

If γ and X are selected in such a manner that Z is always positive, this period will always be real. We stipulate that it is constant and independent of ω_i. Thus we equate this period with a given constant h, yielding an equation

$$\int_0^{2\pi} \sqrt{Z}\,dy_1 = h. \tag{21.14}$$

Solving this equation for X will furnish

$$X = \gamma + \psi(\omega_i),$$

where ψ is a function of ω_i which can be considered as given and which is periodic.

This yields

$$2\sum A_i \frac{dV_{q-2}}{d\omega_i} = \gamma + \psi(\omega_i),$$

which is an equation determining V_{q-2}, after which it is easy to derive $U_{q/2}$ from Eq. (21.13).

This represents the ordinary case.

However, it might happen that γ and X are chosen in such a manner that Z can vanish. In this event, it is the second period of our elliptic integral that is real. Equating this second period to a given constant h will yield an equation (21.14a) analogous to Eq. (21.14). If we solve the equation for X, we obtain

$$X = \gamma + \psi'(\omega_i)$$

or

$$2\sum A_i \frac{dV_{q-2}}{d\omega_i} = \gamma + \psi'(\omega_i),$$

which will determine V_{q-2} since ψ' is known and periodic.

This represents the case of libration.

The limiting case will be obtained by writing that one of the periods of the corresponding elliptic integral of the first kind is infinite, which yields the following equation for determining V_{q-2}:

$$2\sum A_i \frac{dV_{q-2}}{d\omega_i} = L\gamma_q - L_q + \sqrt{M_q^2 + N_q^2}\,.$$

The drawback of this manner of operating lies in the fact that the expressions obtained in the two cases are not an analytic continuation of each other.

On equating the coefficients of ϵ^{q+1}, we obtain

$$2A\frac{dU_{q/2}}{dy_1}\frac{dU_{q/2+1}}{dy_1} + 2\sum A_i \frac{dV_{q-1}}{d\omega_i} = \gamma_{q+1} + \Phi, \quad (21.15)$$

where Φ is known and periodic.

If, for example, we have to do with the ordinary case, we must write that

$$\int_0^{2\pi} \frac{dU_{q/2+1}}{dy_1}\, dy_1$$

is equal to a given constant h independent of ω_i. Thus (by putting, for abbreviation, $dU_{q/2}/dy_1 = W$) we will find

$$2\sum A_i \frac{dV_{q-1}}{d\omega_i} = \gamma + \frac{\displaystyle\int_0^{2\pi} \frac{\Phi dy_1}{2AW} - h}{\displaystyle\int_0^{2\pi} \frac{dy_1}{2AW}}. \tag{21.16}$$

This equation yields V_{q-1}, and Eq. (21.15) will then furnish

$$U_{q/2+1}.$$

The equations obtained by equating the coefficients of the other powers of ϵ will have the same form as Eq. (21.15). This will be exactly the same for the equations obtained by equating, on both sides of Eq. (21.16), all coefficients of different powers of ϵ.

All these equations can thus be treated in precisely the same manner.

The results would be exactly the same if q were odd; in that event, it would only be necessary to modify the form of the expansion of S_1 and to write

$$S_1 = \epsilon^{q/2} U_{q/2} + \epsilon^{q/2+1} U_{q/2+1} + \epsilon_{q/2+2} U_{q/2+2} + \cdots,$$

where S_1 is thus expanded in odd powers of $\sqrt{\epsilon}$.

All results obtained since the beginning of this chapter are rather incomplete, and new studies will be necessary. These would be premature.

Divergence of the Series

225. We have seen in no. 212 that the series resulting from application of the Bohlin method generally are divergent; we have attempted to explain the mechanism of this divergence. It might be useful to return to this subject and to study in some detail a simple example which would yield a better understanding of this mechanism. Let

$$-F = p + q^2 - 2\mu \sin^2 \frac{y}{2} - \mu\epsilon\varphi(y)\cos x,$$

where $(p,x; q,y)$ are two pairs of conjugate variables, while $\varphi(y)$ is a periodic function of y, with period 2π, and where μ and ϵ are two constants which are assumed to be very small.

Let us then form the canonical equations

$$\frac{dx}{dt} = -\frac{dF}{dp} = 1; \quad \frac{dy}{dt} = -\frac{dF}{dq} = 2q,$$

$$\frac{dp}{dt} = \frac{dF}{dx} = -\mu\epsilon\varphi(y)\sin x; \tag{21.17}$$

$$\frac{dq}{dt} = \frac{dF}{dy} = \mu \sin y + \mu\epsilon\varphi'(y)\cos x,$$

whence

$$\frac{d^2y}{dx^2} = 2\mu \sin y + 2\mu\epsilon\varphi'(y)\cos x.$$

Integration of these equations is almost direct when $\epsilon = 0$. Let us write the Jacobi partial differential equation, and let

$$\frac{dS}{dx} + \left(\frac{dS}{dy}\right)^2 = 2\mu \sin^2 \frac{y}{2} + \mu\epsilon\varphi(y)\cos x + C, \qquad (21.18)$$

where C is a constant. Let us expand S and C in powers of ϵ and let

$$S = S_0 + S_1\epsilon + S_2\epsilon^2 + \cdots,$$
$$C = C_0 + C_1\epsilon + C_2\epsilon^2 + \cdots .$$

For $\epsilon = 0$, Eq. (21.18) becomes

$$\frac{dS_0}{dx} + \left(\frac{dS_0}{dy}\right)^2 = 2\mu \sin^2 \frac{y}{2} + C_0. \qquad (21.19)$$

As stated above, the integration is almost immediate and, in fact, to obtain the complete integral of Eq. (21.19) it is sufficient to take, calling A_0 a constant,

$$\frac{dS_0}{dx} = A_0; \quad C_0 = A_0 + 2h\mu; \quad \frac{dS_0}{dy} = \sqrt{2\mu}\sqrt{h + \sin^2 \frac{y}{2}},$$

$$S_0 = A_0x + \sqrt{2\mu}\int\sqrt{h + \sin^2 \frac{y}{2}}\, dy .$$

On the whole, this practically returns us, except for the notations, to the example treated in no. 199. The case of $h > 0$ corresponds to the ordinary case; the case of $h < 0$ to the case of libration; the case of $h = 0$ to the limiting case.

Let us show explicitly the remarkable particular solutions.

First, we have the simple solution

$$x = t, \quad p = 0, \quad y = 0, \quad q = 0,$$

which is a periodic solution. Let us see what the corresponding asymptotic solutions will be.

We obtain these on setting $A_0 = h = 0$ in S_0, which yields

$$S_0 = \mp 2\sqrt{2\mu} \cos \frac{y}{2},$$

whence

$$p = 0, \quad q = \pm\sqrt{2\mu} \sin \frac{y}{2}; \quad \tan \frac{y}{4} = Ce^{\pm t\sqrt{2\mu}}, \quad x = t,$$

which shows that the characteristic exponents are equal to $\pm\sqrt{2\mu}$.

Let us now calculate $S_1, S_2, \ldots$.

On equating the coefficients of ϵ, in Eq. (21.18), we find

$$\frac{dS_1}{dx} + 2\frac{dS_0}{dy}\frac{dS_1}{dy} = \mu\varphi(y)\cos x + C_1,$$

where C_1 is a constant which can be assumed as zero without restriction of generality; in other words,

$$\frac{dS_1}{dx} + 2\sqrt{2\mu}\sqrt{h + \sin^2\frac{y}{2}}\,\frac{dS_1}{dy} = \mu\varphi(y)\cos x. \qquad (21.20)$$

Then, S_1 is the real part of the function Σ defined by the equation

$$\frac{d\Sigma}{dx} + 2\sqrt{2\mu}\sqrt{h + \sin^2\frac{y}{2}}\,\frac{d\Sigma}{dy} = \mu\varphi(y)e^{ix}; \qquad (21.20a)$$

We obtain this by putting

$$\Sigma = \psi e^{ix},$$

whence

$$i\psi + 2\sqrt{2\mu}\sqrt{h + \sin^2\frac{y}{2}}\,\frac{d\psi}{dy} = \mu\varphi. \qquad (21.20b)$$

To integrate this linear equation, let us first integrate the homogeneous equation; it can be written as

$$\alpha\psi + \sqrt{h + \sin^2\frac{y}{2}}\,\frac{d\psi}{dy} = 0,$$

by putting

$$\alpha = \frac{i}{2\sqrt{2\mu}};$$

whence

$$\psi = K\exp\left(-\alpha\int\frac{dy}{\sqrt{h + \sin^2\frac{y}{2}}}\right),$$

where K is a constant. We establish the elliptic integral

$$\int\frac{dy}{\sqrt{h + \sin^2\frac{y}{2}}} = u,$$

yielding

$$\psi = Ke^{-\alpha u}$$

for the general integral of the homogeneous equation. To integrate the inhomogeneous equation, we will consider K as a function of y, which gives

$$2\sqrt{2\mu}\,\frac{dK}{dy}\,e^{-\alpha u}\sqrt{h+\sin^2\frac{y}{2}}=\mu\varphi,$$

whence

$$K=\sqrt{\frac{\mu}{8}}\int e^{\alpha u}\varphi\;dy\Big/\sqrt{h+\sin^2\frac{y}{2}}=\sqrt{\frac{\mu}{8}}\int e^{\alpha u}\varphi\;du,$$

and, finally,

$$\psi=e^{-\alpha u}\sqrt{\frac{\mu}{8}}\int e^{\alpha u}\varphi\;du. \tag{21.21}$$

If we put $\alpha=\beta i$, then β will be real and we will have

$$S_1=\sqrt{\frac{\mu}{8}}\Big[\cos(\beta u-x)\int\varphi\cos\beta u\,du$$

$$+\sin(\beta u-x)\int\varphi\sin\beta u\,du\Big]. \tag{21.22}$$

We will discuss below expressions (21.21) and (21.22). Let us demonstrate first how the subsequent approximations would be carried out.

We would find

$$\frac{dS_2}{dx}+2\sqrt{2\mu}\sqrt{h+\sin^2\frac{y}{2}}\,\frac{dS_2}{dy}=\Phi, \tag{21.23}$$

where Φ is a known function of y and of x, which is periodic with respect to x and which thus can be put in the form

$$\Phi=\sum\varphi_n e^{nix},$$

where n is a positive or negative integer and φ_n is a known function of y. In the sum of the right-hand side, the number of terms is limited. If we then put

$$S_2=\sum\psi_n e^{nix},$$

where ψ_n depends only on y, then the function ψ_n must satisfy the differential equation

$$in\psi_n+2\sqrt{2\mu}\sqrt{h+\sin^2\frac{y}{2}}\,\frac{d\psi_n}{dy}=\varphi_n.$$

Since this equation has exactly the same form as Eq. (21.20b), it can be treated in the same manner.

The functions S_3, S_4, . . . would then be given by an equation of the same form as Eq. (21.23) and treated in exactly the same manner.

This method has been used in quite a different form by Gyldén.[9]

Let us now discuss expressions (21.21) and (21.22).

Let us first consider the ordinary case in which $h > 0$. Since then $\varphi(y)$ is a periodic function of y it will also be a periodic function of u whose period will be equal to the real period of the elliptic integral of u. Consequently, we can write

$$\varphi = \sum A_m e^{im\lambda u},$$

where λ is a real constant depending on the period of the integral u, while m is an integer.

From this, we derive

$$\psi = \sqrt{\frac{\mu}{8}} \sum A_m \frac{e^{im\lambda u}}{\alpha + im\lambda}$$

or

$$\psi = \sum \frac{\mu A_m}{i} \frac{e^{im\lambda u}}{1 + m\lambda\sqrt{8\mu}},$$

and finally, if ρ_m and ω_m are the modulus and the argument of A_m,

$$S_1 = \sum \mu\rho_m \frac{\sin(m\lambda u + x + \omega_m)}{1 + m\lambda\sqrt{8\mu}}. \tag{21.24}$$

It is obvious that each of the terms of S_1 can be expanded in powers of $\sqrt{\mu}$. One can attempt to carry out the expansion and then combine all terms that contain the same power of $\sqrt{\mu}$ as factor. From the formal viewpoint, this will yield the expansion of S_1 in powers of $\sqrt{\mu}$. Let

$$S_1 = \sum T_p \, \mu^{p/2}. \tag{21.25}$$

We have

$$T_{p+2} = (-\lambda\sqrt{8})^p \sum m^p \rho_m \sin(m\lambda u + x + \omega_m).$$

This is the same result as would have been obtained with the Bohlin method. In that case, S would have been expanded in powers of $\sqrt{\mu}$ and we would have found

$$S = S_0 + S_1'\sqrt{\mu} + S_2'\,\mu + \cdots + S_p'\,\mu^{p/2} + \cdots.$$

In turn, the function S_p' would have been expandable in ascending powers of ϵ, and the coefficient of ϵ^p would have been precisely T_p.

The series T_p would have been convergent. In fact, if—as we are assuming here—the function $\varphi(y)$ is holomorphic for all real values of y, we will have

$$\rho_m < k h_0^{|m|},$$

where k and h_0 are two positive constants ($h_0 < 1$), from which it follows that the series

$$\sum m^p \rho_m$$

converges absolutely and, *a fortiori*, also the series T_{p+2}.

On the other hand, series (21.24) converges while this is not so for series (21.25).

To ascertain this, it is sufficient to consider a highly specific example. Setting

$$x = \frac{\pi}{2}, \quad u = 0, \quad \omega_m = 0, \quad \rho_m = A^{|m|}, \quad 0 < A < 1, \quad \lambda = \frac{1}{\sqrt{8}},$$

we obtain

$$T_{p+2} = \sum (-m)^p A^{|m|} \quad (m \text{ varying from } -\infty \text{ to } +\infty)$$

which shows that T_{p+2} is zero if p is odd, and is equal to

$$2 \sum m^p A^m \quad (m \text{ varying from } 1 \text{ to } +\infty).$$

However, we obviously have

$$\sum m^p A^m > \sum m(m-1)\cdots(m-p+1)A^m = \frac{p!\, A^p}{(1-A)^{p+1}},$$

from which, for example at $A = \frac{1}{2}$, it follows that

$$T_{p+2} > 2(p!).$$

The terms of series (21.25) are then pairwise zero, and those that remain are larger than the corresponding terms of the expansion

$$2 \sum q!\, \mu^{q+1},$$

which evidently is divergent.

The statements made above with respect to the expansion of S_1 obviously are applicable to that of S_2 and of other analogous functions.

Practically no changes need be made in the above calculation for the case of $h < 0$, i.e., for the case of libration. The only difference here is that the real period of the integral u no longer is

$$u_0 = \int_0^{2\pi} \frac{dy}{\sqrt{R}},$$

but

$$u_1 = \int_{-\beta}^{+\beta} \frac{dy}{\sqrt{R}},$$

denoting by $\sqrt{R}$ the radical $\sqrt{h + \sin^2(y/2)}$ and putting

$$\beta = 2 \sin^{-1}\sqrt{-h}\ .$$

The quantity λ must then no longer be equal to $2\pi/u_0$ but to $2\pi/u_1$.

226. The limiting case in which $h = 0$ is of greater interest. In this case, we have

$$u = \int \frac{dy}{\sin(y/2)} = 2 \log \tan(y/4)\ ,$$

and, on putting

$$\tan(y/4) = t,$$

$$du = \frac{2\,dt}{t}.$$

First, for example, let

$$\varphi(y) = \sin y,$$

from which we obtain

$$\varphi(y) = \frac{4t(1-t^2)}{(1+t^2)^2},$$

whence

$$\psi = t^{-2\alpha}\sqrt{\frac{\mu}{8}} \int \frac{4t^{2\alpha}(1-t^2)dt}{(1+t^2)^2}.$$

Now, integrating by parts, we find

$$\int \frac{t^{2\alpha}(1-t^2)dt}{(1+t^2)^2} = \frac{t^{2\alpha+1}}{1+t^2} - 2\alpha \int \frac{t^{2\alpha}dt}{1+t^2},$$

whence

$$\psi = 4\sqrt{\frac{\mu}{8}}\,\frac{t}{1+t^2} - it^{-2\alpha} \int \frac{t^{2\alpha}dt}{1+t^2}. \qquad (21.26)$$

One could suggest expanding at least from the formal viewpoint, the function ψ in powers of $\sqrt{\mu}$. However, it is preferable for this to return to the general case.

When y varies from 0 to 2π, the quantity u will vary from $-\infty$ to

$+\infty$. Here, $\varphi(y)$ is a function of u; let us assume that it can be represented by the Fourier integral in the form

$$\varphi = \int_{-\infty}^{+\infty} e^{iqu}\theta(q)dq.$$

For this, since $\varphi(y)$ is analytic and periodic for all real values of y, it is sufficient that

$$\varphi(0) = 0.$$

This will yield

$$\psi = \sqrt{\frac{\mu}{8}}\, e^{-\alpha u}\int du \int_{-\infty}^{+\infty} e^{(\alpha + iq)u}\theta(q)dq.$$

This formula, in reality, contains an arbitrary constant since the integration limits with respect to u are indeterminate. This constant can be disposed of in the following manner:

Let us interchange the order of the integrations and let us perform the integration with respect to u, so that

$$\psi = \sqrt{\frac{\mu}{8}}\, e^{-\alpha u}\int_{-\infty}^{+\infty}\left(\frac{e^{(\alpha + iq)u}}{\alpha + iq}\theta(q) + \eta(q)\theta(q)\right)dq,$$

where $\eta(q)$ is an arbitrary function of q introduced by the integration. In certain cases, one could first assume this function to be zero, which would leave

$$\psi = \sqrt{\frac{\mu}{8}}\, e^{-\alpha u}\int_{-\infty}^{+\infty}\frac{e^{(\alpha + iq)u}\theta(q)dq}{\alpha + iq}$$

or

$$\psi = \frac{\mu}{i}\int_{-\infty}^{+\infty}\frac{e^{iqu}\theta(q)dq}{1 + q\sqrt{8\mu}} \tag{21.27}$$

or, on denoting by ρ and ω the modulus and the argument of $\theta(q)$,

$$S_1 = \mu\int_{-\infty}^{+\infty}\frac{\rho\sin(qu + x + \omega)dq}{1 + q\sqrt{8\mu}}, \tag{21.28}$$

where ρ and ω are functions of q.

However, so that Eq. (21.27) will have meaning, it is necessary that the integral is finite and, for that, that the function under the integral sign does not become infinite for $q = -1/\sqrt{8\mu}$, i.e., that

$$\theta\left(-\frac{1}{\sqrt{8\mu}}\right) = 0.$$

Since this can generally not take place, one could replace Eq. (21.27) by the following equation [which is only another way of disposing of the arbitrary function $\eta(q)$]:

$$\psi = \frac{\mu}{i} \int_{\infty}^{+\infty} \frac{e^{iqu} - e^{iqu_0 + \alpha(u_0 - u)}}{1 + q\sqrt{8\mu}} \theta(q) dq, \qquad (21.27a)$$

where u_0 is an arbitrary constant, from which it follows that

$$S_1 = \mu \int_{-\infty}^{+\infty} \frac{\rho\, dq}{1 + q\sqrt{8\mu}} \left[\sin(qu + x + \omega) - \sin\left(qu_0 + x + \omega + \frac{u_0 - u}{\sqrt{8\mu}}\right)\right]. \qquad (21.28a)$$

However, this can also be overcome in a different manner. In general, $\theta(q)$ will be a function of q which will remain holomorphic if q is real or if the imaginary part of q is not too large. For example, let

$$\varphi = \sin y = \frac{4t(1 - t^2)}{(1 + t^2)^2}.$$

Since, according to the Fourier formula, we have

$$\theta(q) = \int_{-\infty}^{+\infty} \frac{\varphi}{2\pi} e^{-iuq} du,$$

it follows, on replacing φ and u as functions of t, that

$$2\pi\theta(q) = \int_0^{\infty} \frac{4t^{-2iq}(1 - t^2) dt}{(1 + t^2)^2}.$$

On applying, to this integral, the transformation which led to Eq. (21.26), we find

$$2\pi\theta(q) = 8qi \int_0^{\infty} \frac{t^{-2iq} dt}{1 + t^2} = \frac{8qi\pi}{e^{\frac{q\pi}{2}} + e^{-\frac{q\pi}{2}}},$$

from which we finally obtain

$$\theta(q) = \frac{4qi}{e^{\frac{q\pi}{2}} + e^{-\frac{q\pi}{2}}}.$$

It is obvious that $\theta(q)$ ceases being holomorphic only when q is equal to $\sqrt{-1}$, multiplied by an odd integer.

Accordingly, the formula

$$\varphi = \int_{-\infty}^{+\infty} e^{iqu}\theta(q) dq$$

will remain valid when the integral is no longer taken along the real axis but along a curve C remaining above this axis but deviating so little from it that no singular point of $\theta(q)$ exists between this curve and the axis.

Then Eqs. (21.27) and (21.28) will also be valid when taking the integrals along C; they will hold without restriction since, no matter what $\theta(q)$ might be, the quantity under the integral sign does not become infinite along the integration path.

This immediately shows an important property of the function ψ defined by the function (21.27). Under the integral sign we have the exponential e^{iqu}. Since the imaginary part of q is positive, if u is real, positive, and very large the modulus of this exponential is very small. Thus for $u = +\infty$, i.e., for $y = 2\pi$, the quantities ψ and S_1 vanish. The integration path C can also be replaced by another path C' which remains below the real axis without deviating much from this axis, in such a manner that no singular point of θ exists between C' and this axis.

The integrals (21.27) and (21.28), taken along C', will yield other values of ψ and S_1 which I will denote by ψ' and S_1' so as to distinguish them from the former.

Since the imaginary part of q is negative, when u is real, negative, and very large, the exponential e^{iqu} will have a very small modulus. Thus, for $u = -\infty$, i.e., for $y = 0$, the quantities ψ' and S_1' vanish.

This raises the question whether ψ is equal to ψ'. It is obvious that the quantity under the integral sign presents a singular point between the two integration paths C and C', which is the point

$$q = -\frac{1}{\sqrt{8\mu}}.$$

This singular point is a pole. The difference of the two integrals thus will be equal to $2i\pi$, multiplied by the residue, which yields

$$\psi' - \psi = \pi\sqrt{\frac{\mu}{2}}\, e^{\frac{-iu}{\sqrt{8\mu}}}\, \theta\left(-\frac{1}{\sqrt{8\mu}}\right),$$

and, denoting by ρ_0 and ω_0 the modulus and the argument of $\theta(-1/\sqrt{8\mu})$,

$$S_1' - S_1 = \pi\sqrt{\frac{\mu}{2}}\,\rho_0 \cos\left(x - \frac{u}{\sqrt{8\mu}} + \omega_0\right).$$

It is obvious that ψ' is not equal to ψ, unless

$$\theta\left(-\frac{1}{\sqrt{8\mu}}\right) = \theta(i\alpha) = \int_{-\infty}^{+\infty} \frac{\varphi}{2\pi} e^{\alpha u} du = 0.$$

Let us now attempt to expand ψ and ψ' in powers of $\sqrt{\mu}$. This yields the following: Let

$$\psi = \sum \mu^{\frac{p}{2}} \psi_p, \quad \psi' = \sum \mu^{\frac{p}{2}} \psi'_p,$$

so that

$$\psi_p \quad \text{and} \quad \psi'_p = \int \frac{e^{iqu}}{i} \theta(q)(-q\sqrt{8})^{p-2} dq,$$

integral being taken along C for ψ_p and along C' for ψ'_p.

However, this time, the quantity under the integral sign presents no singular point between C and C', from which it follows that

$$\psi_p = \psi'_p.$$

Thus, despite the fact that the functions ψ and ψ' are not equal, their formal expansions in powers of $\sqrt{\mu}$ are identical. This is as much as saying that these series are not convergent.

However, this also shows that, if μ is considered an infinitesimal of the first order, then the difference $\psi - \psi'$ will be an infinitesimal of infinite order such as, for example, $e^{-1/\mu}$.

In fact, in the particular case in which $\varphi(y) = \sin y$, we have

$$\theta\left(-\frac{1}{\sqrt{8\mu}}\right) = \frac{-i\sqrt{\mu/2}}{e^{\frac{\pi}{4\sqrt{2\mu}}} + e^{-\frac{p}{4\sqrt{2\mu}}}},$$

which demonstrates that the difference $\psi - \psi'$ and $S_1 - S'_1$ are of the same order of magnitude as

$$\frac{1}{\sqrt{\mu}} e^{-\frac{\pi}{4\sqrt{2\mu}}}.$$

227. Below, we will obtain the same results by simpler means; however, I wished to present the results under the form given here so as to better understand the passage from the ordinary case to the limiting case.

Let us now compare Eqs. (21.24) and (21.28). In the formula (21.24) we have a series into which the quantity $m\lambda$ enters. Since m is an integer, $m\lambda$ can take only certain values, which will be closer to one another the smaller λ becomes. When h tends to zero, the period of the integral u increases indefinitely and λ tends to zero. The values of $m\lambda$ steadily approach more closely and, at the limit, the series is transformed into an integral which finally leads to Eq. (21.28).

However, when λ decreases thus in a continuous manner, it will pass through certain values for which a circumstance arises that requires some discussion.

If $-1/(\lambda\sqrt{8\mu})$ becomes a whole number, one of the denominators of Eq. (21.24)

$$1 + m\lambda\sqrt{8\mu}$$

vanishes and the formula becomes illusory. In fact, one of the terms of this formula becomes infinite. In this case, it is easy to see that the term which thus becomes infinite must be replaced by

$$\sqrt{\frac{\mu}{8}} A_m u e^{-\alpha u} = \sqrt{\frac{\mu}{8}} A_m u e^{im\lambda u} \tag{21.29}$$

and we will then have

$$\psi = \sqrt{\frac{\mu}{8}} e^{-\alpha u} \sum A_m \int e^{(im\lambda + \alpha)u} du.$$

If $im\lambda + \alpha$ is not zero, the integral on the right-hand side will be equal to

$$\frac{e^{(im\lambda + \alpha)u}}{im\lambda + \alpha}$$

plus a constant which can be assumed to be zero. However, if $im\lambda + \alpha$ is zero, this integral will be equal to u plus a constant which can be assumed to be zero.

On thus substituting Eq. (21.29) in ψ for the term which would become infinite, the function ψ no longer becomes infinite but ceases being periodic with respect to u.

228. Let us return to the limiting case in which h is zero and let us first assume that

$$\varphi(y) = \sin y.$$

Then, Eq. (21.26) will yield

$$\psi = 4\sqrt{\frac{\mu}{8}} \frac{t}{1+t^2} - it^{-2\alpha} \int_0^t \frac{t^{2\alpha} dt}{1+t^2} + Ct^{-2\alpha},$$

where C is a constant of integration. The first term can be expanded in ascending powers of t, provided that t is smaller than 1. This is also the case for the second term, since

$$\frac{t^{2\alpha}}{1+t^2} = \sum t^{(2\alpha+2n)}(-1)^n.$$

From this we conclude, on carrying out the integration, that

$$\psi = \sqrt{2\mu} \sum t^{(2n+1)}(-1)^n - i \sum \frac{t^{2n+1}(-1)^n}{2\alpha + 2n + 1} + Ct^{-2\alpha}.$$

It is also obvious that, for $t = 0$, the expression $\psi - Ct^{-2\alpha}$ vanishes. On the other hand, since the real part of α is zero, the expression $t^{-2\alpha}$ does not vanish for $t = 0$.

To have the function ψ vanish for $t = 0$, i.e., for $u = -\infty$, it is thus necessary and sufficient to have the constant C vanish. The function denoted by ψ' in no. 226 then is equal to

$$\psi' = \sqrt{2\mu}\,\frac{t}{1+t^2} - it^{-2\alpha}\int_0^t \frac{t^{2\alpha}dt}{1+t^2}.$$

Equation (21.26) can also be written in the form

$$\psi' = \sqrt{2\mu}\,\frac{t}{1+t^2} + it^{-2\alpha}\int_t^\infty \frac{t^{2\alpha}dt}{1+t^2} + C't^{-2\alpha},$$

where C' is a new constant.

Assuming that t is larger than 1 and that we have expanded in descending powers of t, it follows that

$$\psi = \sqrt{2\mu}\sum t^{-(2n+1)}(-1)^n + i\sum\frac{t^{-(2n+1)}(-1)^n}{2n+1-2\alpha} + C't^{-2\alpha}.$$

The first and second term vanish for $t = \infty$, but this is not so for the third term.

To have the function ψ vanish for $t = \infty$, i.e., for $u = +\infty$, it is necessary and sufficient that the constant C' vanish. The function, denoted by ψ in no. 226 is thus equal to

$$\psi = \sqrt{2\mu}\,\frac{t}{1+t^2} + it^{-2\alpha}\int_t^\infty \frac{t^{2\alpha}dt}{1+t^2}.$$

To have ψ become equal to ψ', it would thus be necessary to have

$$\int_0^\infty \frac{t^{2\alpha}dt}{1+t^2} = 0,$$

which, as demonstrated above, does not take place.

Expressed more generally, let us assume that $\varphi(y)$ vanishes for $y = 0$, so that

$$\psi = t^{-2\alpha}\sqrt{\frac{\mu}{8}}\int \varphi t^{2\alpha-1}dt.$$

Here, φ vanishes for $y = 0$, i.e., for $t = 0$, and for $y = 2\pi$, i.e., for $t = \infty$. First, then, let t be small and let us expand φ in powers of t; let

$$\varphi = \sum A_n t^n,$$

whence

$$\begin{aligned}\psi &= \sqrt{\frac{\mu}{8}}\,t^{-2\alpha}\int_0^t \varphi t^{2\alpha-1}dt + Ct^{-2\alpha}\\ &= \sqrt{\frac{\mu}{8}}\sum A_n\frac{t^n}{2\alpha+n} + Ct^{-2\alpha},\end{aligned}$$

where C is a constant of integration. So as to have this expression vanish for $t = 0$, it is necessary and sufficient that C is zero. The function ψ' of no. 226 will then be equal to

$$\psi' = \sqrt{\frac{\mu}{8}}\, t^{-2\alpha} \int_0^t \varphi t^{2\alpha - 1} dt = \sqrt{\frac{\mu}{8}} \sum A_n \frac{t^n}{2\alpha + n}. \qquad (21.30)$$

Now, let t be very large. Let us expand φ in descending powers of t and let

$$\varphi = \sum B_n t^{-n},$$

so that

$$\psi = -\sqrt{\frac{\mu}{8}}\, t^{-2\alpha} \int_t^\infty \varphi t^{2\alpha - 1} dt + C' t^{-2\alpha}$$
$$= \sqrt{\frac{\mu}{8}} \sum \frac{B_n t^{-n}}{2\alpha + n} + C' t^{-2\alpha},$$

where C' is an integration constant. So as to have this expression vanish for $t = \infty$, it is necessary and sufficient that C' is zero. The function ψ of no. 226 thus is equal to

$$\psi = -\sqrt{\frac{\mu}{8}}\, t^{-2\alpha} \int_t^\infty \varphi t^{2\alpha - 1} dt = \sqrt{\frac{\mu}{8}} \sum \frac{B_n t^{-n}}{2\alpha - n}. \qquad (21.31)$$

So as to have ψ be equal to ψ', it would be necessary that

$$\int_0^\infty \varphi t^{2\alpha - 1} dt = \frac{1}{2} \int_{-\infty}^{+\infty} \varphi e^{\alpha u} du = 0,$$

i.e., that

$$\theta(i\alpha) = 0,$$

which generally does not take place.

Let us now expand the expressions (21.30) and (21.31) in powers of $\sqrt{\mu}$. We then find

$$\psi' = \frac{\mu}{2i} \sum A_n \frac{t^n}{1 - in\sqrt{2\mu}},$$

which, for the formal expansion of ψ', yields

$$\psi' = \sum T'_p \mu^{(p+2)/2}, \quad T'_p = \frac{1}{2i} \sum A_n t^n n^p (\sqrt{-2})^p. \quad (21.32)$$

Similarly, Eq. (21.31) furnishes

$$\psi = \frac{\mu}{2i} \sum B_n \frac{t^{-n}}{1 + in\sqrt{2\mu}},$$

whence

$$\psi = \sum T_p \mu^{(p+2)/2}, \quad T_p = \frac{1}{2i} \sum B_n t^n n^p (-\sqrt{-2})^p. \tag{21.32a}$$

Under this form, the identity of the two expansions is not as directly manifest as under the form which we had given it previously.

229. However, it is easy to pass from one to the other.

In fact, we have

$$\theta(q) = \int_{-\infty}^{+\infty} \frac{\varphi}{2\pi} e^{-iqu} du.$$

We state that $\theta(q)$ is a meromorphic function of q which has no singularities other than poles and whose poles are equal to $\frac{1}{2}i$, multiplied by a positive or negative integer. In fact, let us write

$$2\pi\theta(q) = \int_0^\infty \varphi e^{-iqu} du + \int_{-\infty}^0 \varphi e^{-iqu} du.$$

If the imaginary part of q is positive, the second integral will be a holomorphic function of q, presenting no singularity since, for $u = -\infty$, the quantities φ and e^{-iqu} vanish. This cannot also be true for the first integral.

If, conversely, the imaginary part of q is negative, the first integral will be a holomorphic function of q, but this cannot be so for the second integral.

Therefore, let us study the singularities that the second integral might exhibit when the imaginary part of q is negative. We will assume that this imaginary part is larger than $-n/2$. Let us return to the series

$$\varphi = \sum A_n t^n = \sum A_n e^{+nu/2}.$$

We can then write

$$\varphi = A_1 e^{+u/2} + A_2 e^{+u} + \cdots + A_n e^{+nu/2} + R_n,$$

and as u tends to $-\infty$, $R_n e^{-nu/2}$ will tend to zero. The second integral can then be written as

$$J_1 + J_2 + \cdots + J_n + S_n,$$

where

$$J_K = A_K \int_{-\infty}^0 e^{(K/2 - iq)u} du, \quad S_n = \int_{-\infty}^0 R_n e^{-iqu} du.$$

The integral J_K has no meaning in itself as soon as the imaginary part of q becomes smaller than $-K/2$ and can be given a meaning only by analytic continuation. We then find

$$J_K = \frac{A_K}{\frac{K}{2} - iq}.$$

So far as S_n is concerned, as long as the imaginary part of q is larger than $-n/2$, this will be a function of q presenting no singularity since the quantity under the integral sign vanishes for $u = \infty$.

This demonstrates that the second integral is a meromorphic function of q, admitting as poles

$$q = -i\frac{K}{2} \quad (K \text{ a positive integer})$$

with the residues

$$iA_K.$$

One can similarly see that the first integral is a meromorphic function of q, admitting as poles

$$q = i\frac{K}{2} \quad (K \text{ positive integer})$$

with the residues

$$-iB_K.$$

Therefore, the poles of $\theta(q)$ are

$$q = \pm i\frac{K}{2}$$

with the respective residues

$$\frac{B_K}{2i\pi},$$

when using the upper sign and

$$-\frac{A_K}{2i\pi},$$

when using the lower sign.

Let us then return to Eq. (21.27) and assume that the integral is taken along the curve C.

Let us construct a circle K, having the origin as center and $(2m + 1)/4$ as radius where m is very large. Let K_1 be the sector of this circle located above the curve C. Let C_1 be the segment of the curve C which is within the circle K.

The two arcs C_1 and K_1 will form a closed contour while the integral (21.27), taken along this contour, will be equal to $2i\pi$ multiplied by the sum of the residues relative to the poles interior to the contour, i.e., to the sum of the m first terms of series (21.31).

We can demonstrate that integral (21.27), taken along K_1, tends to zero when m increases indefinitely. The calculation can be carried out without difficulty but is useless since we know in advance that series (21.31) is convergent.

The integral taken along C_1 tends to ψ; thus, ψ is equal to the sum of series (21.31).

This returns us to series (21.30) as well as to series (21.32) and (21.32a).

The above statements are sufficient to understand how one can pass from the series in no. 226 to those in no. 228.

230. We can now propose to connect the series in no. 228 with those in Chap. 7.

It has been shown in no. 225 that, when $\epsilon = 0$, the equations admit of a simple periodic solution

$$x = t, \quad p = y = q = 0$$

with the characteristic exponents $\pm\sqrt{2\mu}$ and that the corresponding asymptotic solutions will be

$$p = 0, \quad q = \pm\sqrt{2\mu}\sin\frac{y}{2}, \quad \tan\frac{y}{4} = Ce^{\pm t\sqrt{2\mu}}, \quad x = t.$$

The third of these equations can also be written as

$$\cot\frac{y}{4} = Ce^{t\sqrt{2\mu}}$$

or

$$\tan\frac{y}{4} = Ce^{t\sqrt{2\mu}},$$

depending on whether the upper or lower sign is used.

Since the characteristic exponents are not zero, the principles of Chaps. 3 and 4 show that, for small values of ϵ, a periodic solution will again exist. We again will have $x = t$, while p, y, and q will be functions of t and of ϵ, expandable in ascending powers of ϵ, vanishing with ϵ, and periodic with respect to t, with period 2π.

Similarly, the characteristic exponents, which are equal but of opposite sign and which are denoted by $\pm\beta$, can be expanded in ascending powers of ϵ (see Chap. 4); β will reduce to $+\sqrt{2\mu}$ for $\epsilon = 0$.

For small values of ϵ, two series of asymptotic solutions will also exist, which are presented in the following form: For the first series, we have

$$x = t, \quad p = \eta_1, \quad q = \eta_2, \quad \cot\frac{y}{4} = \eta_3, \tag{21.33}$$

where η_1, η_2, and η_3 are series that can be expanded in powers of $Ce^{-\beta t}$ and whose coefficients are periodic in t.

For the second series, we obtain

$$x = t, \quad p = \eta_1', \quad q = \eta_2', \quad \tan\frac{y}{4} = \eta_3'; \tag{21.33a}$$

where η_1', η_2', and η_3' are series expandable in powers of $Ce^{+\beta t}$ and whose coefficients are periodic in t.

If we now consider these quantities as functions of ϵ, then no. 106 will show that the six functions η can be expanded in ascending powers of ϵ.

If we consider these as functions of μ, no. 104 will show that each of the terms of the six functions η will have a coefficient of the form

$$\frac{N}{\Pi},$$

where N is a polynomial expanded in ascending powers of $\sqrt{\mu}$ and β, while Π is a product of factors of the form

$$m\sqrt{-1} + n\beta,$$

where m and n are positive or negative integers.

As we have seen in no. 108, N/Π can be expanded in powers of $\sqrt{\mu}$; however, the expansion generally is purely formal since the characteristic exponents vanish for $\mu = 0$.

Let us now transform the expressions (21.33) and (21.33a). Let us begin by replacing t everywhere by x. Let us then solve the equation

$$\cot\frac{y}{4} = \eta_3$$

for C, from which we find

$$Ce^{\beta x} = \zeta.$$

Noting that, for $\epsilon = 0$, η_3 reduces to $Ce^{\beta x}$ we will find that ζ can be expanded in powers of ϵ and of $\cot(y/4)$ and that its coefficients are periodic in x.

Let us substitute ζ for $Ce^{\beta x}$ in η_1 and η_2; then, η_1 and η_2 will become functions of x and of y, and the expression

$$\eta_1 dx + \eta_2 dy$$

will be an exact differential dS. We will integrate this differential, obtaining a certain function S which exhibits the following properties:

(i) Its derivatives are periodic with respect to x.

(ii) It can be expanded in powers of ϵ and of $\cot(y/4)$.
(iii) An arbitrary term of

$$\frac{dS}{dx} = \eta_1 \quad \text{or of} \quad \frac{dS}{dy} = \eta_2$$

will be composed of the cosine or sine of a multiple of x, multiplied by a power of $\cot(y/4)$, by a power of ϵ, and by a coefficient having the form

$$\frac{N}{\Pi},$$

where N can be expanded in powers of ϵ, of $\sqrt{\mu}$, and of β and where Π is a product of factors of the form

$$m\sqrt{-1} + n\beta.$$

(iv) The expression N/Π can be expanded in powers of ϵ and of $\sqrt{\mu}$. Consequently, this is true also for S. However, whereas the expansion of S in powers of ϵ is convergent, the expansion in powers of $\sqrt{\mu}$ has value only from the formal viewpoint.

We could have operated similarly on the expression (21.33a) and would then have obtained a function S' completely analogous to the function S, with the only difference that, instead of being expanded in powers of ϵ and of $\cot(y/4)$, it would have been expanded in powers of ϵ and of $\tan(y/4)$.

We have stated that S (and S') can be expanded in powers of ϵ; therefore, let

$$S = S_0 + S_1\epsilon + S_2\epsilon^2 + \cdots .$$

Then S_1 is nothing else than the real part of $\psi\, e^{ix}$, and ψ is presented in the form of a series arranged in powers of $\cot(y/4)$, i.e., in descending powers of the variable which was denoted by t in no. 228.

This expansion is nothing else but series (21.31).

Let us see what becomes of the expressions N/Π in this transformation.

The quantity N can be expanded in powers of ϵ. Furthermore, since β can also be expanded in powers of ϵ, so can

$$\frac{1}{m\sqrt{-1} + n\beta},$$

and the first term of the series will be

$$\frac{1}{m\sqrt{-1} + n\sqrt{2\mu}}.$$

Let us thus assume that we have an expression N/Π in which the first term of the expansion of N in powers of ϵ reduces to $(\epsilon\mu/2)\ B_n$ and in which the product Π reduces to a single factor

$$\sqrt{-1} - n\beta.$$

Then, the expansion of N/Π will have, as first term,

$$\frac{\epsilon\mu}{2} \frac{B_n}{\sqrt{-1} - n\sqrt{2\mu}} = \epsilon\sqrt{\frac{\mu}{8}} \frac{B_n}{2\alpha - n}.$$

This explains the presence in the series (21.31) the presence of the coefficient

$$\frac{B_n}{2\alpha - n}.$$

Similarly, S' can be expanded in powers of ϵ, which yields

$$S' = S'_0 + S'_1\epsilon + \cdots.$$

Here, S'_1 is the real part of $\psi' e^{ix}$, while ψ' is presented in the form of a series arranged in powers of t and being nothing else but series (21.30).

231. The functions S and S' present themselves in the form of series. The expansion of S in powers of $\cot(y/4)$ is convergent only if y is close to 2π. The expansion of S', proceeding in powers of $\tan(y/4)$, is convergent only if y is close to zero. However, one can define by analytic continuation S and S' for all values of y. Thus, these functions can be "continued" in such a manner that they both will be defined for values of y between y_0 and y_1, with y_0 and y_1 themselves between zero and 2π.

This raises the question as to whether, in the region where both are defined, the functions S and S' might be equal. The question must be answered in the negative. Actually, if we had identically

$$S = S',$$

then the terms of the convergent series of S and S' in powers of ϵ would be equal and one would have specifically

$$S_1 = S'_1,$$

and, consequently,

$$\psi = \psi'.$$

However, we have demonstrated in the preceding numbers that ψ is not equal to ψ'.

Thus S is not equal to S'. From this an important consequence can be drawn. We know that S and S' can be formally expanded in powers of $\sqrt{\mu}$; let

$$S = T_0 + T_1\sqrt{\mu} + T_2\mu + \cdots,$$
$$\vdots \tag{21.34}$$
$$S' = T'_0 + T'_1\sqrt{\mu} + T'_2\mu + \cdots.$$

These series can be obtained either by the methods given in nos. 207–210 or by starting from series (21.33) and (21.33a), expanding them subsequently in powers of $\sqrt{\mu}$ (see no. 108) and then treating them as I did in the preceding number.

The functions T_i, for y close to 2π, can be expanded in powers of ϵ and of $\cot(y/4)$, while the functions T'_i, for y close to zero, can be expanded in powers of ϵ and of $\tan(y/4)$. This property is characteristic. The function S is actually the only function expandable in powers of ϵ and of $\cot(y/4)$ and satisfying Eq. (21.18). Similarly, S' is the only function expandable in powers of ϵ and of $\tan(y/4)$ and satisfying Eq. (21.18).

On the other hand, we have shown in nos. 207–210 that the functions T_i can be given the form of series arranged in sines and cosines of multiples of $y/2$. Consequently, these can be expanded simultaneously in powers of ϵ and $\cot(y/4)$ for y close to 2π in powers of ϵ and $\tan(y/4)$ for y close to zero.

Hence, we have

$$T_i = T'_i.$$

Thus, if series (21.34) were convergent, we would have

$$S = S'.$$

This means that the series (21.34) diverge.

Therefore, the series in no. 108, from which these can be derived, also will not converge.

(See Vol. I, no. 109, paragraph 4ff., and Vol. II, no. 212, paragraph 30.)

232. Above we have always assumed that $\varphi(y)$ vanishes for $y = 0$. This restriction has no essential significance. If $\varphi(0)$ did not vanish and were equal, for example, to A_0, it would be sufficient to add to the series (21.30) and (21.31) a term

$$\sqrt{\frac{\mu}{8}\,\frac{A_0}{2\alpha}},$$

and to add the same constant to the integrals (21.27) that define ψ and ψ'.

We have discussed this example at great length; however, the example not only made it possible to demonstrate the divergence of the series of nos. 108 and 207 but offered also other advantages.

First, the example shows the mechanism by which one can pass from series analogous to those in no. 225 to series analogous to those given in no. 104, by using intermediary as the series in nos. 226 and 228.

Second, the singularities mentioned in the above discussion represent a first indication of the existence of periodic solutions that are of the second kind and doubly asymptotic, which will be discussed in a later volume.

Poincaré's Footnotes

1. H. Poincaré, "Sur les intégrales irregulières des equations linéaires," Acta Mathematica, Vol. VIII, p. 295 (1886).
2. A. Lindstedt, "Beiträge zur Integration der Differential-Gleichungen der Störungstheorie," Mémoires de l'Académie de Saint-Pétersbourg, Vol. XXXI (1883).
3. H. Poincaré, "Sur une methode de M. Lindstedt," Bulletin Astronomique, Vol. III, p. 57 (February 1886).
4. H. Poincaré, "Sur les series de M. Lindstedt," Comptes Rendus, Vol. CVIII, p. 21 (1889).
5. A. Lindstedt, "Sur la forme des expressions des distances mutuelles...," Comptes Rendus, Vol. XCVII, pp. 1276, 1353 (1883).
6. S. Newcomb, $D°$ of Uranus, Smithsonian Contributions to Knowledge, Vol. XIX, Dec. 1874.
7. P. S. Laplace, *Traité de mécanique céleste*, Vol. II, Chap. 7, Secs. 55 and 59, pp. 321, 334 (Gauthier-Villars, 1878).
8. H. Poincaré, "Sur le convergence des séries trigonométriques," Bulletin Astronomique, Vol. I, p. 324 (July 1884).
9. J. A. H. Gyldén, "Convergenz der Reihen, Welche zur Darstellung...," Acta Mathematica, Vol. IX (1887).
10. H. Poincaré, "Sur les groupes des équations linéaires," Acta Mathematica, Vol. IV, p. 212.
11. I. L. Fuchs, "Lineare Differential-Gleichungen mit veränderlichen Coefficienten," Crelles Journal für die reine und angewandte Mathemathik, Vol. LXVI (1866).
12. T. Tannery, Gauthier-Villars, Paris, 1873.
13. H. Poincaré, "Sur les déterminants d'ordre infini," Bulletin de la Société Mathématique de France, Vol. XIV, p. 77 (1886).
14. G. Hill, "On the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon," Acta Mathematica, Vol. VIII, p. 1 (1886).
15. H. Poincaré, "Sur les fonctions entières," Bulletin de la Société Mathématique de France, Vol. XI, p. 136 (1883).
16. J. Hadamard, "Sur les fonctions entières de la forme...," Comptes Rendus, Vol. CXIV, p. 1053 (1892).
17. H. Poincaré, "Sur le problème des frois corps et les équations de la dynamique," Acta Mathematica, Vol. XIII (1889).

Russian Endnotes

The following notes are excerpts from the comments on the Russian translation of Poincaré's *Les Méthodes nouvelles de la Méchanique céleste*. They appear in volumes I and II of the Collected Works of Poincaré published by Nauka in Moscow, in 1971 and 1972.

Comments for the first volume of the *Méthodes nouvelles* are by V. I. Arnol'd, those for the second volume are by M. V. Alekseev, and those for the third volume are by G. A. Merman; they were translated and excerpted into English by Julian V. Barbour.

Notes on Part 2

R1. This argument is not entirely accurate. In fact, the construction of the functions S_p by the method presented above is possible only if there is no commensurability between the frequencies $n_i^0 = -\partial F_0/\partial x_i^0$. Since x_i^0 for which the frequencies are commensurable are everywhere densely distributed, the use of the derivatives $\partial S_p/\partial x_i^0$ in the change of variables (9.8) is incorrect. This difficulty can be avoided as follows.

Let x_i^* be fixed in such a way that the corresponding frequencies $n_i^* = n_i(x_i^*)$ are incommensurable. We set $x_i = x_i^* + \xi_1$ and instead of (9.5) consider the equation

$$F\left(x_1^* + \frac{\partial S^*}{\partial y_1}, \cdots, x_n^* + \frac{\partial S^*}{\partial y_n}; y_1, \ldots, y_n\right) = C^*. \qquad (9.5^*)$$

We seek the functions S^* and C^* in the form

$$S^* = \xi^0 y_1 + \cdots + \xi_m^0 y_n + \mu S_1^{**} + \sum_{p=2}^{\infty} S_p^*,$$

$$C^* = F_0(x_i^*) + \sum_{p=1}^{\infty} C_p^*, \qquad (9.4^*)$$

where S_1^{**} is periodic in y_i and S_p^* and C_p^* are forms of degree p in μ and ξ_i^0 with coefficients periodic in y_i. For their determination, we obtain the equtions

$$n_1^* \frac{\partial S_1^{**}}{\partial y_1} + \cdots + n_n^* \frac{\partial S_1^{**}}{\partial y_n} = F_1(x_j^*, y_j) - C_1^*,$$

$$n_1^* \frac{\partial S_p^*}{\partial y_1} + \cdots + n_n^* \frac{\partial S_p^*}{\partial y_n} = \Phi_p^* - C_p^*, \qquad (9.7^*)$$

which can be solved in the same way as Eqs. (9.7). The divisors $m_1 n_1^* + \cdots + m_n n_n^*$ that appear in the coefficients of the Fourier series are by assumption nonzero. Thus, the terms of the series (9.4*) are properly defined (for a suitable choice of the n_i^*, the Fourier series converge; see Chap. 13), but the series themselves are still formal in nature.

Since the formal series can be differentiated with respect to ξ_i^0, the change of variables

$$\xi_i = \frac{\partial S^*}{\partial y_i}, \quad W_i^* = \frac{\partial S^*}{\partial \xi_i^0} \qquad (9.9^*)$$

is defined from the point of view of formal analysis and gives us the series

$$x_i = x_i^* + \xi_i^0 + \sum_{p=1}^{\infty} x_i^{*(p)}(\xi_j^0, W_j^*, \mu),$$

$$y_i = W_i^* + \sum_{p=1}^{\infty} y_i^{*(p)}(\xi_j^0, W_j^*, \mu), \qquad (9.3^*)$$

where $x_i^{*(p)}$ and $y_i^{*(p)}$ are forms of degree p in μ, $\xi_1^{(0)},\ldots,\xi_1^{(0)}$ with coefficients periodic in μ. The series (9.3*) determine the solution of the system of equations (9.2), but this solution has a formal nature not only in the sense of Chap. 8, i.e., with respect to the small parameter μ, but also with respect to ξ_i^0.

R2. What is meant is for all right-hand sides. On the other hand, Poincaré's conclusion that this system is solvable is based on the remark made in §126, and does not require verification of the algebraic conditions of compatibility. Such verification would be rather cumbersome, and this is the difficulty in the direct proof of applicability of Lindstedt's method that Poincaré discusses above.

R3. We recall that, in accordance with the formulas of §8, G and θ are even functions of the eccentricities and inclinations.

R4. One can show that in the three-body problem the constant A_4 is identically equal to zero, a result due to the existence of the area integral. Thus, all the constructions of §132 (like all the constructions of the following chapters based on the method of this section) can be applied only for the planar problem.

R5. Here and below, it must be borne in mind that the function F is expanded in powers of $\sqrt{\rho_i}$. Therefore, the Hamilton–Jacobi equation contains not only integral powers of the derivatives $\partial F/\partial\omega_i$ but also half-integral powers (although, as Poincaré emphasizes several times, the lowest terms are linear with respect to these derivaties). The same is true of the expansions in powers of Ω and V. This circumstance makes the method unsuitable for small ρ, i.e., for small eccentricities. Poincaré gives an analysis of this case in Chap. 12.

This difficulty does not arise if to solve the original system of §131 one uses Birkhoff's method (*Dynamical Systems*). This method reduces to the construction of a formal canonical change of variables by which the Hamiltonian R is reduced to the form $R(\xi^2 + (\xi')^2,\ldots,q^2 + (q')^2)$. A system with such a Hamiltonian can be readily integrated.

R6. The method proposed by Poincaré in Chap. 12 cannot be directly extended to the n-body problem with $n > 3$. However, the difficulty that it overcomes can be avoided in problems of a very general type by the following device.

Let $L = (L_1,...,L_n)$, $\xi = (\xi_1,...,\xi_m)$, $\eta = (\eta_1,...,\eta_n)$, $\varphi = (\varphi_1,...,\varphi_m)$ be canonical variables and the Hamiltonian have the form

$$F = F_0(L) + \mu f(L,\xi,\eta) + \mu F_1(L,\xi,\eta,\varphi),$$

$$f = \sum_{k=1}^{m} f_k^{(2)}(L)(\xi_k^2 + \eta_k^3) + f^{(3)}(L,\xi,\eta),$$

where F_1 is 2π-periodic in φ with zero mean value. To such a form the planar n-body problem can, for example, be reduced.

We make the canonical transformation

$$(L,\xi,\eta,\varphi) \to (L',\xi',\eta',\varphi')$$

in accordance with the formulas

$$L = L' + \mu \frac{\partial S}{\partial \varphi}, \quad \varphi' = \varphi + \mu \frac{\partial S}{\partial L},$$

$$\xi = \xi' + \mu \frac{\partial S}{\partial \eta}, \quad \eta' = \eta + \frac{\partial S}{\partial \xi},$$

where S satisfies the equation

$$\sum_{i=1}^{n} n_i \frac{\partial S}{\partial \varphi_i} + F_1(L,\xi,\eta,\varphi) = 0,$$

$$n_i = -\frac{\partial F_0}{\partial L_i}.$$

It is readily noted that in the new variables the Hamiltonian takes the form

$$F(L,\xi,\eta,\varphi) = F'(L',\xi',\eta',\varphi')$$
$$= F_0'(L') + \mu f'(L',\xi',\eta') + \mu^2 F_1'(L',\xi',\eta',\varphi').$$

Here, f' contains in general terms of first degree in ξ' and η'. Similarly, after l steps we arrive at a Hamiltonian in which the term that depends on $\varphi^{(l)}$ has order μ^{l+1}. Ignoring the terms of this order, we obtain an "averaged" system that differs from the systems of §131 only by the presence of terms of first degree in $\xi^{(l)}$ and $\eta^{(l)}$. These terms can be readily eliminated by a change of variables linear in $\xi^{(l)}$ and $\eta^{(l)}$. After this, the averaged system can be solved by Birkhoff's method. Modifying as in Note R13 the method of successive approximation, one can here too avoid the appearance of everywhere discontinuous functions.

R7. Series with small denominators of the type of the series (13.3) that Poincaré considers arise in many problems of mathematics. As an example, we mention the problem of the reduction to normal form of a system of ordinary differential equations in the neighborhood of a singular point. The difficulties that are associated here with the presence of small divisors were overcome for the first time by C. L. Siegel, "Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung," Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 21–30 (1952). Another example is the study of the behavior of solutions of differential equations on a torus [A. M. Kolmogorov, "On dynamical systems with integral invariant on a torus," Dokl. Akad. Nauk SSSR **93** (5) (1953); V. I. Arnol'd, "On mappings of the circle onto itself," Izd. Akad. Nauk SSSR, Mat. **25** (1) (1961); see also Kolmogorov's paper at the Mathematical Congress at Amsterdam].

A common feature of all these investigations is the use of arithmetic arguments. Roughly speaking, the bulk of the irrational numbers are not well approximated by rationals, and therefore the small denominators are "not too small." On the other hand, the irrationals that can be unusually well approximated by rationals enable one to construct examples of divergent series of the type (13.3) even for analytic perturbations.

R8. If Lindstedt's series (13.3) or (13.8) were to converge uniformly for x_i^0 varying in a certain interval, Eqs. (13.8) would determine a foliation of the phase space into n-dimensional tori, and the equations $w_i = n_i t + \tilde{\omega}_i$ would determine a uniform rotation of the angle coordinates on these tori. For independent frequencies n_i, the corresponding motion would be conditionally periodic. This is indeed the situation for $\mu = 0$. The "resonance tori," on which $\Sigma m_i n_i = 0$ for certain integers m_i, then decompose into tori of a lower number of dimensions. It is found that for $\mu \neq 0$ the resonance tori cease to exist, and the greater part of the tori of lower dimension of which they are made up breaks up completely, and only a finite number of such tori continue to exist in the perturbed system, forming an invariant manifold of dimension less than n that may be either stable or unstable. In the case of two degrees of freedom, these manifolds are periodic solutions. Merman (G. A. Merman, "Almost periodic solutions and divergence of the Lindstedt series in the restricted planar circular three-body problem," Proceedings of the Institute of Theoretical Astronomy, No. 8, 1961) has given the corresponding arguments for the case of the Lindstedt series in the restricted three-body problem.

The proof of the existence of "zones of instability" and the analysis of their structure in the general case has been considered in the papers of Moser [J. Moser, "On the theory of quasi-periodic motion," SIAM Review (1966); J. Moser, "Convergent series expansions for quasi-periodic motions," Math. Ann. **169**, 136–176 (1967)] and independently by V. K. Mel'nikov ["On some cases of persistance of conditionally periodic motions after a small change of the Hamiltonian," Dokl. Akad. Nauk SSSR

165 (6), 1245–1248 (1965); "On a family of conditionally periodic solutions of a Hamiltonian system," *ibid.* **181** (3), 546–549 (1965)]. In one model problem Arnol'd [V. I. Arnol'd, "On the instability of dynamical systems with many degrees of freedom," Dokl. Akad. Nauk SSSR **156** (1), 9–12 (1964)] has shown that for more than 2 degrees of freedom instability zones not only exist but that the trajectories can travel along them arbitrarily far.

R9. At this point, Poincaré's arguments contain a lacuna that was pointed out by Merman in his article on "Almost periodic solutions..." cited in the previous note. Let $\omega(\mu) = n_1/n_2$ and $\omega(0)$ be irrational. In accordance with Poincaré's argument, we mus take a μ_0 sufficiently small for $\omega(\mu_0) = p/q$ to be rational. The function $\omega(\mu)$ is analytic and, without loss of generality, we can assume that $\omega'(0) \neq 0$. But then

$$|\mu_0| \geqslant G\,|\omega(\mu_0) - \omega(0)| = C\left|\frac{p}{q} - \omega(0)\right| \geqslant \frac{\alpha}{q^2}$$

for almost all irrational $\omega(0)$. At the same time, the periodic T of the doubly infinite family of periodic solutions obtained from the Lindstedt series will be equal to $2\pi q$. Thus, for the validity of the results of §42 we must know that the solution

$$x_i(t;x_i^0,\widetilde{\omega}_i,\mu), \quad y_i(t;x_i^0,\widetilde{\omega}_i,\mu)$$

can be expanded in powers of μ, and, moreover, when t varies over an interval of length of order q the radius of convergence of these series must be not less than $1/q^2$. For estimating the radius of convergence, Poincaré has at his disposal only the majorizing estimate $\mu_0 \sim e^{-LT} = e^{-2L\pi q}$, but this is certainly inadequate.

R10. Thus, Poincaré does not in fact have a proof of divergence of the Lindstedt series for fixed x_i^0. In the desire to fill this lacuna, two quite different cases must be borne in mind. In analyzing the Lindstedt series, Poincaré devotes particular attention to the possibility of an arbitrary choice of the coefficients in the expansion of the frequencies $n_i(\mu)$ in powers of μ (beginning with the second term). If the Lindstedt series converge and it is assumed for simplicity that the number of degrees of freedom is 2, then the frequency ratio $\alpha = n_1(\mu)/n_2(\mu)$ is an analytic function of μ, in general nonconstant.

Thus, we obtain the following picture. The two-dimensional torus with cyclic coordinates w_1 and w_2 can be embedded by means of Eqs. (13.8) in the phase space and is thus a two-dimensional invariant manifold. The fundamental system of differential equations induces on our torus a system that depends analytically on the parameter μ. The corresponding rotation number α (equal to the ratio of the frequencies) is also an analytic function of μ. Such a situation is very exceptional, since it follows from the work of Arnol'd cited above that the dependence on the coefficients is not, in general, analytic. Therefore, in this case too the Lindstedt series cannot be assumed to converge.

Quite different is the case of constant frequencies, and considerable progess in these problems has been achieved in the work of Kolmogorov, Arnol'd, and Moser. To understand the essence of the methods that they have employed, we return to the beginning of §148 and pose this question: Do there exist functions of the form (13.8), not necessarily analytic in μ, that, the substitution $w_i = n_i t + \widetilde{\omega}_i$ having been made, determine a solution of the Hamilton equations for at least a fairly large set (if not all) of the x_i^0?

To answer this question, Kolmogorov proposed the use of a method of the type of Newton's method (method of tangents). This involves giving up the use of the expansions in powers of μ characteristic of the small-parameter method and the construction of the functions (13.8) by successive approximations. Note that such an idea distinguishes Newcomb's method from Lindstedt's method. However, the convergence of expansions "from the point of view of a geometer," to use Poincaré's words, was of little interest to Newcomb, and Poincare himself assessed Newcomb's method as equivalent to Lindstedt's, in the divergence of whose series he was convinced.

Suppose the Hamiltonian has the form

$$F = F_0(x_i) + F_1(x_i, y_i),$$

where the perturbation F_1 has the order μ, and suppose frequencies $n_i^0 = (\partial F_0/\partial x_i)(x_i^0)$ are incommensurable in a sufficiently strong sense. More precisely, suppose that for all integers m_i not simultaneously equal to zero

$$|m_1 n_1^0 + \cdots + m_n n_n^0| \geqslant \frac{K}{(|m_1| + \cdots + |m_n|)^{n+1}}.$$

It is helpful to note that the points $(n_1^0,\dots,n_n^0)$ in the n-dimensional space for which such an inequality holds for at least one K form a set whose complement has a Lebesgue measure equal to zero.

By a suitable canonical transformation defined in the neighborhood of the n-dimensional torus $x_i = x_i^0$, the Hamiltonian can be transformed to the form

$$F = F_0'(x_i) + F_1'(x_i, y_i),$$

in which the perturbation F_1' now has the order μ^2. After n steps, this perturbation will have the order μ^{2^n}. The resulting "superconvergence," which is characteristic of methods of successive approximation of the

Newton type, make it possible to overcome the influence of the small divisors that arise in each step when the necessary canonical transformation is chosen.

Applying this method, Kolmogorov (A. N. Kolmogorov, "On the invariance of conditionally periodic motions when there is a small change of the Hamiltonian," Dokl. Akad. Nauk SSSR **98** (4) (1954); V. I. Arnol'd, "Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian," Usp. Mat. Nauk **18** (5) (1963) [Russian Math. Surveys **18**, 9 (1963)]) proved that in the absence of inherent degeneracy, i.e., if the Hessian of the Hamiltonian is nonzero, the answer to the question posed above is in the affirmative. For fixed x_i^0 and μ, functions of the form (13.8) determine in the phase space an invariant torus provided the frequencies are "strongly incommensurable," and the motion on this invariant torus is conditionally periodic.

The invariant tori do not fill the complete phase space, but the measure of the set complementary to them tends to zero with μ. The complementary set is filled with the remains of the disrupted "resonance tori," on which the frequencies satisfied a relation of the form $\Sigma n_i^0 m_i = 0$ for $\mu = 0$.

The procedure of the method of successive approximation of Newton type is not related to the dependence of the Hamiltonian on the small parameter μ. But if F depends on μ analytically, this dependence does persist in the limit. Therefore, the functions of the form (13.8) by means of which the invariant tori with conditionally periodic motion are determined must also depend on μ analytically. This means that for a suitable choice of x_i^0, for which the frequencies are "strongly incommensurable," i.e., satisfy the inequality written down above, the Lindstedt series must converge if we choose the coefficients in the expansions of $n_i(\mu)$ equal to zero beginning with the second. In the circle mapping problem, convergence of the series analogous to the Lindstedt series was proved by Arnol'd [Izv. Akad. Nauk SSSR, Mat. **25** (1) (1961)]. Later, Moser proved the same thing for the Lindstedt series themselves [Math. Ann. **169**, 136–176 (1967)]. These results were obtained by comparing the perturbation theory series with what Newton's method gives. As yet, a direct proof of convergence of the Lindstedt series based on estimation of the coefficients has not been found.

In applications of Kolmogorov's theorem, the cases in which the number of degrees of freedom is equal to two or is greater than two differ significantly. In the first case, the isoenergy manifold has dimension 3, and the two-dimensional invariant tori separate it. Therefore, paths that begin in a "gap" between two invariant tori remain forever in a bounded region of the phase space, i.e., the corresponding motion will be stable in the sense of Lagrange. For periodic solutions belonging to the stable (elliptic in the sense of Birkhoff) type, the same arguments make it possible to establish Lyapunov stability. See, for example, A. M. Leontovich, "On the stability of Lagrangian periodic solutions of the restricted three-body problem," Dokl. Akad. Nauk SSSR **143** (3) (1962).

Such arguments do not apply if the number of degrees of freedom is greater than 2, since in this case the n-dimensional tori do not separate the $(2n-1)$-dimensional isoenergy manifold. The instability zones which arise at the places where the frequencies $n_i = \partial F_0/\partial x_i$ are connected by linear relations with integral coefficients are joined to each other, and paths can pass through them arbitrarily far. This was shown by Arnol'd in one model example (see Note R8), and the mechanism used by Arnol'd for the proof has a general nature. The presence of the conditionally periodic motions on the invariant tori is probably complemented by unstable motions in the gaps between them. One cannot rule out, for example, the existence of trajectories that are everywhere dense in a certain region of the phase space, etc.

Kolmogorov's theorem is also invalid when applied to many problems of celestial mechanics. The problem is that when $\mu = 0$ one usually has degeneracy, i.e., the Hamiltonian $F_0(x_i)$ depends on only some of the variables x_i, and therefore there is always a resonance relation between the frequencies. Such a difficulty is also encountered in the construction of the Lindstedt series (see §134), and Poincaré overcomes it by adding to F_0 the result of averaging F_1 over the "fast" variables.

A similar device was used by Arnol'd. In his paper "Small denominators and the problem of the stability of motion in classical and celestial mechanics" [Usp. Mat. Nauk **18** (6) (1963)] he showed that results and analogous to Kolmogorov's theorem can also be obtained in the presence of degeneracy. In particular, Arnol'd showed that for the three-body and many-body problems the measure of the set of initial conditions generating conditionally periodic motions is positive. Although this result does not signify stability of the solar system (it follows from what was said above that such stability is evidently impossible in the Lyapunov sense), it does make stability "fairly probable."

Both Kolmogorov and Arnol'd were studying analytic Hamiltonian systems. In a series of studies, Moser ["A new technique for the construction of solutions of nonlinear differential equations," Proc. Natl. Acad. Sci. U.S.A. **47**, 1824–1831 (1961); "On invariant curves of area-preserving mappings of an annulus," Nachr. Akad. Wiss. Göttinger Math.-Phys. Kl. II, 1–20 (1962); "A rapidly convergent iteration method and nonlinear differential equation," Ann. Scuola Norm. Sup. Pisa (3) **20**, 265–315, 499–535] showed that analogous results can also be obtained for sufficiently smooth systems and perturbations.

Numerous applications of the Kolmogorov–Arnol'd–Moser methods can be found in the already quoted studies, and also in a paper of Arnol'd ["Problems of the Motion of Artificial Celestial Bodies," Izd. Akad. Nauk SSSR, Moscow (1963)].

We also mention some results relating to investigation of the neighborhood of an equilibrium position. Siegel [C. L. Siegel, "Über die Existenz einer Normalform analytischer Hamiltonscher Differentialgleichunge in der Nähe einer Gleichgewichtslösung," Math. Ann. **128**, 144–170 (1954)] showed that for the convergence of the Birkhoff series, which are related to the Lindstedt series, the Hamiltonian must satisfy a countable number of conditions on its coefficients. See also: S. Miyahara, "On the existence of the normal form in the neighborhood of an equilibrium point of analytical Hamiltonian differential equations," Publ. Astr. Soc. Jpn. **14** (3) (1960); H. Rüssmann, "Über die Normalform analytischer Hamiltonscher Differential-Gleichungen in der Nähe einer Gleichgewichtslösung," Math. Ann. **169**, 55–72 (1967).

The problem of the convergence and divergence of the formal series that reduce an analytic system of differential equations to the (not necessarily Hamiltonian) normal form was considered by Bryuno [A. D. Bryuno, "Normal form of differential equations," Dokl. Akad. Nauk SSSR **157** (6) (1964); "On the convergence of transformations of differential equations to normal form," Dokl. Akad. Nauk **165** (5) (1965); "On the divergence of transformations of differential equations to normal form," Dokl. Akad. Nauk SSSR **174** (5) (1967); "On the formal stability of Hamiltonian systems," Mat. Zametki **1** (3) (1967)].

Littlewood considered asymptotic series by means of which one can represent the solution of the three-body problem in the neighborhood of the Lagrangian equilateral triangle, and obtained for their Nth term an estimate of the order $N^2\log^{3/2}(N+1)^N\epsilon^N$ for N of order $1/\sqrt{\epsilon}|\ln|\epsilon||^{3/4}$. This enabled him to conclude that by perturbing the initial conditions by an amount of order ϵ we obtain perturbations of the solutions having order ϵ over a time interval of order $\exp[(1/\sqrt{\epsilon})|\ln\epsilon|^{3/4}]$.

For the considered problem, the number of degrees of freedom is 2 and a similar result over an infinite interval can be obtained by topological arguments. However, Littlewood's method is apparently also valid in higher dimensions, and this could be helpful.

R11. It is not clear what Poincaré has in mind when speaking about an infinitely large number of arguments. In the expansion for μt written down above, all arguments will be multiples of one argument, whereas in the Lindstedt series the arguments are integral combinations of at least n fundamental arguments.

R12. Note that in these formulas the sine and cosine have been interchanged compared with the formulas in §131 and §137.

R13. This equation is now usually known as the Mathieu equation. Its solutions have been well studied [see, for example, M. J. O. Strutt, Lamésche-Mathieusche-und verwandte Funktionen in Physik and Technik, Berlin (1932)]. The instability of solutions of differential equations due to a periodic variation of their parameters is called "parametric resonance." See the monograph of Bogolyubov and Mitropol'skiĭ quoted above and also: J. J. Stoker, Nonlinear Vibrations in Mechanical and Electrical Systems, New York (1950). A deep investigation of parametric resonance phenomena has been made by Kreĭn [M. G. Kreĭn, "The basic propostions of the theory of λ-zones of instability of a canonical system of linear differential equations with periodic coefficients," in: Collection in Memory of A. A. Andronov [in Russian], Moscow (1955), pp. 113–498].

R14. Indeed, using the periodicity of ψ and ψ_1, we obtain

$$\psi_1(t) \equiv \frac{1}{2\pi}\,[F(t+2\pi) - F(t)]$$

and

$$\psi_1(t) \equiv \frac{1}{2\pi}\,[F(t) - F(t-2\pi)].$$

The coefficients of Eq. (17.1) are periodic, so that if $F(t)$ is a solution $F(t+2\pi)$ will be too and, therefore, $\psi_1(t)$ as well. Replacing in the second identity t by $-t$ and using the parity of F, we find that ψ_1 is odd.

R15. Poincaré here evidently gives an incorrect reference, since he has in mind Hadamard's paper: "Sur les fonctions entières de la forme $e^{\mathscr{G}(z)}$," C. R. Acad. Sci. **114**, 1053 (1892). Hadamard proves there the following theorem: If an entire function is to have order zero, the coefficients c_n of its Taylor expansion must satisfy the condition

$$|c_n| < k\Gamma\left(\frac{n}{2}+1\right)^{-\mu},$$

where k is a constant, $\mu > 1$.

R16. The study of systems of differential equations in the neighborhood of an invariant manifold of a more complicated nature than a stationary point or a periodic solution has only just begun. We mention here the papers: É. G. Belaga, "On the reducibility of a system of ordinary differential equations in the neighborhood of a conditionally periodic motion," Dokl. Akad. Nauk SSSR **143** (2) (1962); N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, "The method of integral manifolds in nonlinear mechanics," in: Proc. International Symposium on Nonlinear Oscillations, Izd. Akad. Nauk USSR, Kiev (1963); J. Moser [Proc. Natl. Acad. Sci. U.S.A. **47**, 1824–1831 (1961)]; A. M. Samoĭlenko, "On the reducibility of a system of ordinary differential equations in the neighborhood of a smooth toroidal manifold," Izv. Akad. Nauk SSSR, Ser. Mat. **30**, 1047–1072 (1966); "On the reducibility of a system of ordinary differential equations in the neighborhood of a smooth integral manifold," Ukr. Mat. Zh. **18** (6), 41–65 (1966).

I should like to express my gratitude to V. I. Arnol'd, V. A. Brumberg, I. A. Krasinskiĭ, G. A. Merman, M. S. Petrovskaya, and the Russian translator of Vol. 2 of the *Les Méthods nouvelles de la Mécanique céleste,* Yu. A. Danilov, for pointing out numerous misprints in the French original.

I am also indebted to G. A. Krasnikov for his notes Nos. R1, R4, R5, R6

V. M. Alekseev

Index

T

U

V

W

ISBN 1-56396-115-6 (Part 2)
1-56396-117-2 (Set)